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PLANE VISCOPLASTIC FLOW IN NARROW CHANNELS WITH DEFORMABLE WALLS

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The equations of motion of a continuum in a thin layer are derived for a given functional dependence of the stress tensor on the strain rate tensor. The general problem of viscoplastic flow is considered in the thin-layer approximation for boundary surface material points travelling in the lateral direction in a predetermined fashion.

The projections of the continuum point velocity, pressure, flow rate through a cross-section of the channel, and the power of external forces are expressed as functions of the boundary deformation law. The problem of determining the channel boundary deformation law is formulated for a given boundary pressure distribution. The expressions for the continuum flow rate and pressure and the power of external forces written as functionals of the channel width allow formulation of the problems of controlling viscoplastic flows in thin layers and optimizing the processes.

A method for analyzing viscoplastic flows subject to complex boundary conditions was discussed in [1]. It is based on the notion of equivalent viscosity whose analytic determination justifies passage from the Hencky to the Navier-Stokes equations in analyzing viscoplastic flows at relatively low Saint-Venant numbers. Plane viscoplastic flows subject to various boundary conditions have been analyzed in a number of papers [3, 4].

In this paper we consider an alternative method for solving the problems of viscoplastic flow in thin layers under complex boundary conditions. The method is based on the replacement of the accurate Hencky equations [2] by approximate equations at an arbitrary Saint-Venant number.

1. Let us consider a thin viscoplastic layer bounded by a deformable surface and single out an element whose characteristic dimensions $l \sim l_1 \sim l_2$, $l_3 << l$, Fig. 1. We introduce the following characteristic parameters: flow velocity V in planes parallel to the plane xy which touches the layer surface; the flow velocity along the z-axis, U, which is perpendicular to the layer surface; the stress at continuum points Θ, pressure drop Δp, and time T.
The flow equations written in stresses are rewritten in nondimensional variables. For this we use the formulas

$$x = lx',\ y = ly',\ z = l_3 z',\ t = Tt',\ v_x = Vu_x,$$
$$v_y = Vu_y,\ v_z = Uu_z,\ \tau_{ij} = -\Delta p p'\delta_{ij} + \Theta\tau_{ij}'\ (i, j = 1, 2, 3), \tag{1.1}$$
$$\delta_{ij} = 0\ (i \neq j),\ \delta_{ii} = 1$$

where the primes mark the nondimensional coordinates, time, pressure, and stresses, and u_x, u_y, and u_z are the nondimensional projections of the continuum point velocity.

Assuming that the continuum is incompressible and the form of the nondimensional compressibility equation is the same as that of the dimensional one, from the latter we get:

$$V/l \sim U/l_3,\quad U \ll V \tag{1.2}$$

Let us substitute relations (1.1) and (1.2) in the equations of motion written in stresses, assuming that

$$\frac{l_3}{l} \ll 1,\ \frac{\rho U^2 l}{l_3\Theta} \ll 1,\ \frac{\rho U^2}{\Theta} \ll 1,\ \frac{\Delta p}{\Theta} \gg 1,\ \frac{\Delta p}{\Theta} \gg \frac{\rho U l_3}{T\Theta}$$

Discarding the terms whose coefficients are small quantities, we arrive at the following equations of continuum motion in a thin layer:

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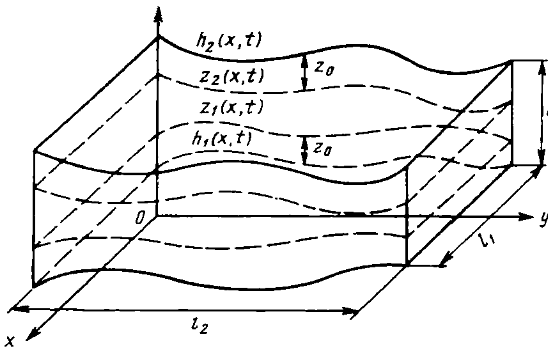

Fig. 1

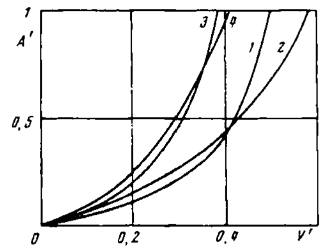

Fig. 2

$$
\begin{gathered}
\frac{\rho Ul}{T\Theta}\frac{\partial u_x}{\partial t'} = -\frac{\Delta p l_3}{\Theta}\frac{l_3}{l}\frac{\partial p'}{\partial x'} + \frac{\partial \tau_{xz}'}{\partial z'}, \quad \frac{\rho Ul}{T\Theta}\frac{\partial u_y}{\partial t'} = -\frac{\Delta p}{\Theta}\frac{l_3}{l}\frac{\partial p'}{\partial y'} + \frac{\partial \tau_{yz}'}{\partial z'} \\
0 = \frac{\partial p'}{\partial z'} \\
\frac{\partial u_x}{\partial x'} + \frac{\partial u_y}{\partial y'} + \frac{\partial u_z}{\partial z'} = 0
\end{gathered} \tag{1.3}
$$

Putting $\rho Ul/T\Theta << 1$ in the first two equations (1.3), we get the equations of stationary/quasi-stationary flow in a thin layer:

$$
\begin{gathered}
\frac{\Delta p}{\Theta}\frac{l_3}{l}\frac{\partial p'}{\partial x'} = \frac{\partial \tau_{xz}'}{\partial z'}, \quad \frac{\Delta p}{\Theta}\frac{l_3}{l}\frac{\partial p'}{\partial y'} = \frac{\partial \tau_{yz}'}{\partial z'} \\
\frac{\partial p'}{\partial z'} = 0 \\
\frac{\partial u_x}{\partial x'} + \frac{\partial u_y}{\partial y'} + \frac{\partial u_z}{\partial z'} = 0
\end{gathered} \tag{1.4}
$$

If an isotropic continuum flows, for instance, parallel to a plane, then $\varepsilon_{yx}' = 0$ and $\tau_{yz}' = 0$ and Eqs. (1.4) become

$$
\frac{\Delta P}{\Theta}\frac{\partial p'}{\partial x'} = \frac{\partial \tau_{xz}'}{\partial z'}, \quad \frac{\partial p'}{\partial y'} = \frac{\partial p'}{\partial z'} = 0, \quad \frac{\partial u_x}{\partial x'} + \frac{\partial u_z}{\partial z'} = 0 \tag{1.5}
$$

Equations (1.3), (1.4), and (1.5) are analogous to the Reynolds equations [5, 6] for a viscous fluid.

2. Consider viscoplastic flow in a thin layer between two material deformable shells described by the equations $z = h_1(x, t)$ and $z = h_2(x, t)$ for $z \in [h_1, h_2]$, Fig. 1. If the conditions

$$
\frac{\partial h_i}{\partial x} \sim \frac{l_3}{l} < 1, \quad h_2 - h_1 \sim l_3 < l \quad (i = 1, 2)
$$

hold, then, after returning to the dimensional variables with the help of (1.1), we can use Eqs. (1.5) in the form

$$
\frac{\partial p}{\partial x} = \frac{\partial \tau_{xz}}{\partial z}, \quad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0, \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \tag{2.1}
$$

Since $l_3 << 1$, the tangential stress component

$$
\tau_{xz} = \mu\frac{\partial v_x}{\partial x} + \tau_0 \operatorname{sign}\frac{\partial v_x}{\partial z} \tag{2.2}
$$

Let us assume that the boundary material points determined by equations $z=h_1(x, t)$ and $z=h_2(x, t)$ travel exactly along the z-axis, i.e. $v_x=0$ on the boundary.

This boundary condition may be validated, if the boundary shell $z=h_2(x, t)$ is inextensible and the coordinate $x=0$ is fixed. If the above thin-layer conditions are met, then the length of the upper-boundary shell arc is

$$s=\int_0^{x(t)}\sqrt{1+(\partial h_2/\partial x)^2}\,dx \approx x(t)+\int_0^{x(t)}\frac{1}{2}\left(\frac{\partial h_2}{\partial x}\right)^2 dx$$

Taking into account the inextensibility condition $ds/dt=0$, the estimate $\partial h_2/\partial x \sim l_3/l << 1$, and the mean-value theorem, we can write:

$$z=h_i(x,\ t):\ v_x=\frac{\partial\psi}{\partial z}=0,\quad v_z=-\frac{\partial\psi}{\partial x}=\frac{\partial h_i}{\partial t} \tag{2.3}$$

$$\begin{aligned} &x=x_1:\ p=p_1 \text{ or } Q(x_1,\ t)=Q_0(t)\\ &x=x_2:\ p=p_2 \end{aligned} \tag{2.4}$$

Here, $\psi(x, z, t)$ is the stream function, $i=1, 2$; p_1 and p_2 are the pressures in the cross-sections $x=x_1$ and x_2 respectively, and $Q(x_1, t)$ is the continuum flow rate in cross-section $x=x_1$.

From the second boundary condition (2.3) we get the expression for stream functions ψ_1 and ψ_2 on the boundaries $z=h_1(x, t)$ and $z=h_2(x, t)$ respectively, and for the flow rate $Q(x, t)=\psi_2 - \psi_1$ through the cross-section at point x –

$$\psi_1=-\int_{x_1}^{x}\dot{h}_1\,dx,\quad \psi_2=Q(x_1,\ t)-\int_{x_1}^{x}\dot{h}_2\,dx \tag{2.5}$$

$$Q=Q(x_1,\ t)-2\int_{x_1}^{x}\dot{h}\,dx \tag{2.6}$$

$$h=\frac{1}{2}(h_2-h_1) \tag{2.7}$$

Here, $Q(x_1, t)$ is the flow rate in the cross-section $x=x_1$ and the arbitrary constant is chosen subject to the condition $\psi(x_1, h_1)=0$. Then, the tangential stress τ_{xz}, defined by expression (2.2), becomes

$$\tau_{xz}=\mu\frac{\partial^2\psi}{\partial z^2}+\tau_0\,\mathrm{sign}\,\frac{\partial^2\psi}{\partial z^2} \tag{2.8}$$

According to Eqs. (2.1), $\partial p/\partial x$ is independent of z and hence τ_{xz} is a linear function of z. Introducing the quasirigid region boundaries $z=h_1(x, t)$ and $z=h_2(x, t)$, where the function $\tau_{xz}(x, z, t)$ assumes the limiting absolute value τ_0, we get

$$\tau_{xz}=\mathrm{sign}\,Q\frac{\tau_0}{z_1-z_2}(2z-z_1-z_2) \tag{2.9}$$

From the first equation (2.1) and Eq. (2.9) we find the pressure gradient

$$\frac{\partial p}{\partial x}=\mathrm{sign}\,Q\frac{2\tau_0}{z_1-z_2} \tag{2.10}$$

Since the function $\psi(x, z)$ is continuous in the layer $z \in [h_1, h_2]$ and the strain rate intensity is zero inside the quasirigid region, Eqs. (2.8), (2.9), and boundary conditions (2.5) yield for the stream function

$$\psi = \psi^\circ + \frac{\tau_0 \operatorname{sign} Q}{2\mu(h - z_0)} \begin{cases} -\frac{1}{3}(x - z_2)^3 - (h^\circ - z)z_0^2, & z \in [z_2, h] \\ -(h^\circ - z)z_0^2, & z \in [z_1, z_2] \\ -\frac{1}{3}(z - z_1)^3 - (h^\circ - z)z_0^2, & z \in [h_1, z_1] \end{cases} \tag{2.11}$$

$$\psi^\circ = 1/2(\psi_1 + \psi_2), \quad h^\circ = 1/2(h_1 + h_2), \quad z_1 - h_1 = h_2 - z_2 = z_0 \tag{2.12}$$

According to (2.9), quasirigid boundaries z_1 and z_2 lie at equal distances from the domain boundaries, Fig. 1. Calculating the stream function ψ on the flow boundaries, $\psi_2 = \psi(x, h_2)$ and $\psi_1 = \psi(x, h_1)$, from (2.11), taking (2.5) into account, and subtracting the second value from the first, we arrive at the equation for the quasirigid region boundary z_0, which may be written in the nondimensional form:

$$Z_0^2\left(1 - \frac{1}{3}Z_0\right) = a(1 - Z_0) \tag{2.13}$$

$$a = \frac{\mu |Q|}{\tau_0 h^2}, \quad Z_0 = \frac{z_0}{h} \tag{2.14}$$

Thus, the nondimensional distance Z_0 of the quasirigid region boundary from the flow domain boundary depends only on the nondimensional parameter a, which is the reciprocal of the Saint-Venant number. For any $a \geq 0$ Eq. (2.15) has a single root that satisfies the condition $0 \leq Z_0 \leq 1$, for which the following asymptotic expansions can be obtained:

$$Z_0(a) = \begin{cases} 1 - \frac{1}{3}(a + 1)^{-1} + O\left(\frac{1}{a}\right)^4, & a > 1 \\ \sqrt{a} - \frac{1}{3}a + O(a^{3/2}), & a < 1 \end{cases} \tag{2.15}$$

Expressing the pressure gradient in (2.10) via $Z_0(a)$ with the help of Eqs. (2.14) and (2.12), we get

$$\frac{\partial p}{\partial x} = -\frac{\operatorname{sign} Q \tau_0}{h(1 - Z_0(a))} \tag{2.16}$$

Then, integrating between the limits $[x_1, x]$, we arrive at the following expression for the pressure drop Δp on the interval $[x_1, x]$:

$$\frac{\partial p}{\tau_0} = -\int_{x_1}^{x} \frac{\operatorname{sign} Q}{h} \frac{dx}{1 - Z_0(a)} \tag{2.17}$$

In the thin-layer approximation the projection of the total stress p_{zz} onto the z-axis reduces to $p_{zz} \approx -p$. Therefore, relation (2.19) can also be used for calculating at any moment of time t the x-distribution of the external pressure acting on the channel walls, which ensures that the wall deforms according to a given law. In order to derive the dependence of the pressure gradient on the wall deformation law, we express $Z_0(a)$ in terms of $\partial p/\partial x$ from (2.18) and substitute it in Eq. (2.15):

$$\left(h\frac{\partial p}{\partial x} + \tau_0\right)^2\left(2h\frac{\partial p}{\partial x} - \tau_0\right) = -3\mu|Q|\left(\frac{\partial p}{\partial x}\right)^2 \tag{2.18}$$

$$\frac{2}{3}\Pi + \frac{1}{3\Pi^2} = a + 1, \quad \Pi = -\frac{h}{\tau_0 \operatorname{sign} Q}\frac{\partial p}{\partial x} \tag{2.19}$$

For any $a \geq 0$ Eq. (2.21) has three roots: $\Pi_1 < \Pi_2 < \Pi_3$. The first root is negative, the other two are positive. The greatest root corresponds to asymptotic solution (2.17). The asymptotics for the greatest root $\Pi(a)$ can be obtained by substituting (2.17) in (2.18):

$$\Pi(a) = \begin{cases} \frac{3}{2}(a+1) - \frac{2}{9(a+1)^2} + O\left(\frac{1}{a+1}\right)^5, & a > 1 \\ 1 + \sqrt{a} + \frac{2}{3}a + O(a^{3/2}), & a < 1 \end{cases} \tag{2.20}$$

Thus, for a known deformation law $h(x, t)$, the function Π and pressure gradient $\partial p/\partial x$ are found either from simple algebraic equations (2.21) and (2.20) or, approximately, from asymptotic expansions (2.22).

The flow rate Q is readily found from Eq. (2.20) as a function of h and $\partial p/\partial x$. Then, for a given pressure distribution the deformation law is found from the equation

$$\frac{\partial h}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial x}\left[Q\left(h, \frac{\partial p}{\partial x}\right)\right]$$

The external force power spent on taking the continuum from point x_1 to point x is determined by the expression

$$W = \int_{x_1}^{x} \frac{\tau_0 |Q|}{h} \frac{dx}{1 - Z_0(a)} \tag{2.21}$$

Analytic expressions (2.6), (2.15), (2.19)—(2.23) yield the mathematical formulation of the optimal control problem. The function $h(x, t)$ may be viewed as a control function, and the flow rate Q, pressure drop Δp, and power W as the functionals to be optimized. Using Eqs. (2.20) and (2.21), we can optimize functionals Q and W, viewing function $p(x, t)$ as the control function.

3. Let us determine the work done by the external forces in forcing out a given volume of viscoplastic continuum through a slot with an impermeable left wall.

Suppose that, at the start, we have a thin rectangular layer $0 \le x \le l$, $0 \le z \le h_0$ filled with a viscoplastic continuum. The left-hand wall $x=0$ is rigid and impermeable and the right-hand slot $x=l$ is open. The viscoplastic continuum is forced out through the right-hand slot with the layer boundary deforming continuously according to the law

$$\frac{h}{h_0} = f(x', t'), \qquad x' = \frac{x}{l}, \qquad t' = \frac{t}{T} \tag{3.1}$$

where T is the deformation time. The volume V forced out in time T is given by the formula

$$V' = \frac{V}{V_0} = 1 - \int_0^1 f(x', t')\, dt', \qquad V_0 = 2h_0 l \tag{3.2}$$

The work A done in time T is found by integrating (2.23) with respect to time:

$$A' = \frac{A}{A_0} = \int_0^1\int_0^1 \frac{|\dot{f}_1|\, dx'\, dt'}{f(1 - Z_0(a))}, \qquad \dot{f}_1 = \frac{\partial}{\partial t'}\int_0^x f(x', t')\, dx', \qquad A_0 = 2V_0 l^2 \tag{3.3}$$

Equations (2.6), (2.16), (3.1), and (3.3) allow the parameter a and flow rate Q in the cross-section x' to be expressed via nondimensional functions f and $\dot{f}_1$:

$$Q = -\frac{2h_0 l}{T}\dot{f}_1(x', t'), \qquad a = \frac{2\dot{f}_2}{f^2 S}, \qquad S = \frac{\tau_0 h_0 T}{\mu l} \tag{3.4}$$

Here, S is the Saint-Venant number.

Let us consider two examples of the extrusion of a viscoplastic continuum.

Example 1. Extrusion according to an exponential law

$$f(x', t') = e^{-\alpha x' t'} \tag{3.5}$$

Example 2. Extrusion with the help of parallel plates moving according to the law

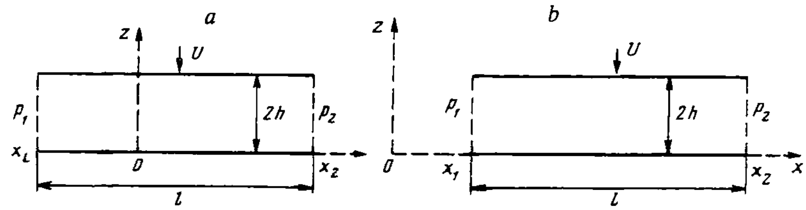

Fig. 3

$$f(t') = \frac{(1 - e^{-\alpha t'})}{\alpha t'} \tag{3.6}$$

It can readily be shown (see Eq. (3.2)) that in both cases the same volume

$$V' = \frac{1}{\alpha}(e^{-\alpha} - 1 + \alpha) \tag{3.7}$$

of viscoplastic continuum is forced out in time T.

The work done on forcing out the volume is calculated numerically using formulas (3.3).

Figure 2 compares the work A' done in forcing out volume V' for deformation laws (3.5) and (3.6) respectively. Curves *1* and *3* correspond to (3.5) with $S = 0.5$ and 2 and curves *2* and *4* to (3.6) with $S = 0.5$ and 2 respectively.

According to Fig. 2, there is a value V_* such that when $V' < V_*(S)$ the exponential method of deformation is more economical. Conversely, when $V' > V_*(S)$ the other method is preferable.

Thus, in the initial stage of extrusion the simplest method involving a pair of plates is less economical. The larger the Saint-Venant number the longer the stage.

4. Consider the problem of asymmetric extrusion of a viscoplastic continuum from a narrow channel with the help of parallel plates for $\Delta p = p_1 - p_2$, Fig. 3. The problem of symmetric extrusion was solved in [7] by assuming that $h/l \ll 1$ and $S \gg 1$, the solution for viscoplastic flow being sought as a correction to the known Prandtl solution for fully plastic flow.

Let the plate be l long, the distance between the plates $2h$, the velocity with which the plates approach each other $U = |2\dot{h}|$, and the pressure at points x_1 and x_2 p_1 and p_2 respectively. To be definite, we assume that $p_1 \geq p_2$. The origin is placed in the zero-flow-rate cross-section. Then, in cross-sections x_1 and x_2 the flow rates $Q(t, x)$ are found from (2.6), the local reciprocal Saint-Venant numbers a_i $(i = 1, 2)$ from (2.17), and the pressures p_1 and p_2 from (2.19). Thus, we get

$$Q(t, x_i) = Q_i = -2\dot{h}x_i, \qquad a_i = a_0\frac{|x_i|}{l}, \qquad a_0 = \frac{\mu U l}{\tau_0 h^2} \tag{4.1}$$

$$p(x_i) = p_i = p_0 - \frac{\tau_0 l}{h a_0}P(a_i), \qquad P(a_i) = \int_0^{a_i} \frac{da}{1 - Z_0(a)} \tag{4.2}$$

The coordinates of the plate ends x_1 and x_2 should be found from the system of equations

$$p(x_1) - p(x_2) = \Delta p, \qquad x_2 - x_1 = l \tag{4.3}$$

Let us solve this system of equations, and then the problem as a whole, for $a_0 \gg 1$ when (4.2) and (2.17) allow the nondimensional function $P(a)$ to be represented in the form of the expansion

$$P(a_i) = \frac{3}{4}a_i^2 + \frac{3}{2}a_i - \frac{2}{9} + O\left(\frac{1}{a}\right) \tag{4.4}$$

Using Eqs. (4.4), (4.1), and (4.2), we can reduce (4.3) to the nondimensional form

$$\frac{3}{4}(a_2^2 - a_1^2) + \frac{3}{2}(a_2 - a_1) = \Pi_0 a_0$$

$$a_1 + a_2 = a_0, \quad \Pi_0 \le \frac{3}{4}a_0 + \frac{3}{2} \tag{4.5}$$

$$a_2 - a_1 = a_0, \quad \Pi_0 \ge \frac{3}{4}a_0 + \frac{3}{2}$$

$$\Pi_0 = \frac{\Delta p h}{\tau_0 l}$$

where the inequalities for Π_0 and a_0 correspond to a zero-flow-rate cross-section $x=0$ lying inside and outside the (x_1, x_2) segment respectively. It may readily be shown that the solution of system (4.5) has the form:

$$a_2 = \begin{cases} \dfrac{a_0}{2} + \dfrac{2\Pi_0 a_0}{3(a_0 + 2)}, & \Pi_0 \le \dfrac{3}{4}a_0 + \dfrac{3}{2} \\ \dfrac{2}{3}\Pi_0 - 1 + \dfrac{1}{2}a_0, & \Pi_0 \ge \dfrac{3}{4}a_0 + \dfrac{3}{2} \end{cases} \tag{4.6}$$

Using (4.1), the coordinates of the slot ends x_1 and x_2 and coordinate x can be represented in the form

$$x_2 = \frac{la_2}{a_0}, \qquad x_1 = \frac{l(a_2 - a_1)}{a_0}, \qquad x = x_1 + d \tag{4.7}$$

where d is the distance from the left-hand edge of the slot. The local reciprocal Saint-Venant number

$$a = \frac{a_0|x|}{l} \tag{4.8}$$

Formulas (2.11), (2.17), and (2.19) permit the stream function, the flow velocity field, the quasirigid-core region, and the pressure to be expressed through the parameter a in cross-section x:

$$p = p_0 - \frac{\tau_0 h}{la_0}P(a) \tag{4.9}$$

Equations (4.6)—(4.9) allow calculation of the force exerted on the plate by the continuum being extruded,

$$F = \int_{x_1}^{x_2} p\,dx \tag{4.10}$$

Integrating by parts and passing to the nondimensional pressure and coordinate a (see Eqs. (4.9) and (4.8) respectively), we get

$$F = p_2 x_2 - p_1 x_1 + F_0 \tag{4.11}$$

$$F_0 = -\int_{x_1}^{x_2} x\frac{\partial p}{\partial x}\,dx = \frac{\mu U l^3}{2h^3}\begin{cases} \dfrac{a_0 + 3}{4a_0} + \dfrac{4\Pi_0^2(a_0 + 1)}{3(a_0 + 2)^2}, & \Pi_0 \le \dfrac{3}{4}(a_0 + 2) \\ 1 + \dfrac{3}{2a_0}, & \Pi_0 \ge \dfrac{3}{4}(a_0 + 2) \end{cases} \tag{4.12}$$

In the case of viscoplastic continuum symmetric outflow $p_1 = p_2$, $\Pi_0 = 0$, and from (4.11) and the upper inequality (4.12) we have

$$F_0 = \frac{\mu U l^3}{8h^3} \frac{a_0 + 3}{a_0} \tag{4.13}$$

As $a_0 \to \infty$ (viscous fluid) Eq. (4.12) becomes

$$F_0 = \frac{\mu U l^3}{2h^3} \begin{cases} \dfrac{1}{4} + \dfrac{4\Delta p^2 h^6}{2\mu^2 U^3 l^4}, & \dfrac{\Delta p h^3}{\mu U l^2} \le \dfrac{3}{4} \\ 1, & \dfrac{\Delta p h^3}{\mu U l^2} \ge \dfrac{3}{4} \end{cases}$$

If inequalities (4.11) and (4.14) are replaced with the exact equalities, then the expressions for F_0 from the upper and lower equalities (4.11) and (4.14) coincide.

The above results enable us to conclude that the proposed method for analyzing plane viscoplastic flows in channels and cavities whose walls deform in an arbitrary fashion is quite efficient. It permits the determination of all the parameters characterizing viscoplastic flow in a thin layer, namely, the flow velocity and pressure fields, the quasirigid region boundary, the flow rate through any cross-section, the pressure drop between cross-sections, the force exerted by the continuum on the wall, the power of the external forces, and certain other parameters.

The method can be extended to solve analogous problems of the thin-layer flow of a wide class of non-Newtonian fluids whose stress and strain tensors are related to each other in a variety of nonlinear ways. All the problems reduce to the solution of one and the same general system of equations of motion (1.5).

Moreover, the above-derived expressions for the continuum flow rate, the pressure, and the power of the external forces, written in the form of functionals of the distance between the channel walls, make it possible to formulate the problems of controlling viscoplastic flows in thin layers and optimizing the processes.

Formulas (3.2) and (3.3) enable the energy spent on extruding a continuum to be compared for various boundary deformation laws.

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