CONVERGENCE AND DIVERGENCE OF FOURIER INTEGRALS IN THE SOBOLEV-SPACES PAIR

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Let $L_p^\alpha(R^n)$ denote the completion of the $C_0^\infty(R^n)$ space of all infinitely differentiable functions in R^n with compact carriers in the norm $\|f \mid L_p^\alpha(R^n)\| = \|g \mid L_p(R^n)\|$, where $\tilde{g} = (1 + |s|^2)^{\alpha/2}\tilde{f}$, $\tilde{f}$ is the Fourier transform of the function f, $|s|^2 = \sum_1^n |s_k|^2$. For an elliptic polynomial A(s) one sets $E_\lambda^0 f = \int_{A(s)\leqslant\lambda} \tilde{f}(s)\, e^{-i\langle s,x\rangle}\, ds$.

For which parameters $\{\alpha, p; \beta, q;$ n and A$\}$ for any functions $\varphi, \psi \in C_0^\infty(R^n)$ are the operators

$$\varphi E_\lambda^0(\psi\cdot) : L_p^\alpha(R^n) \to L_q^\beta(R^n) \qquad (*)$$

bounded uniformly in λ?

A simple necessary condition: $\alpha \geq \beta$. Since for all p one has $1 < p < \infty$, the same operator $(1-\Delta)^{\lambda/2}$, which can be regarded as a fractional power of the Laplace operator, realizes the shift in the superscript in the biscale of L_p^α (see [1], Chap. 9); to answer the question only the difference $\alpha - \beta$ is essential; thus the problem has to be solved only for $\beta = 0$.

THEOREM 1. Let one of the following three relations be satisfied:

a) $$\frac{\alpha}{n} > \max\left\{\frac{1}{p} - \frac{1}{q};\ \frac{1}{2} - \frac{1}{2n} - \frac{1}{q}\right\},\quad \frac{2n}{n-1} \leqslant q \leqslant \infty;$$

b) $$\frac{\alpha}{n} > \max\left\{\frac{1}{p} - \frac{1}{q};\ 0\right\},\quad \frac{2n}{n+1} \leqslant q \leqslant \frac{2n}{n-1};$$

c) $$\frac{\alpha}{n} > \max\left\{\frac{1}{p} - \frac{1}{2} - \frac{1}{2n};\ 0\right\},\quad 1 \leqslant q \leqslant \frac{2n}{n+1},$$

also n = 2 or n > 2 but $\alpha > 1/n^2$. Then for any $\varphi, \psi \in C_0^\infty(R^n)$ the operators (*) from $L_p^\alpha(R^n)$ in $L_q(R^n)$ generated by an elliptic polynomial A(s) are uniformly bounded in λ.

This proposition is a direct corollary (using the interpolation theorem of Stein [2]) of the results of A. G. Kostyuchenko and of the author [3] referring to the case $q = \infty$ as well as of the results of Carleson and Sjölin [4] which refer to the case $A(s) = s_1^2 + s_2^2$, $q = p$, $n = 2$, $4/3 \leq p \leq 4$ and their extensions to the case of any elliptic polynomial A (P. Sjölin, n = 2; V. Z. Meshkov, n > 2 and $\alpha > 1/n^2$).

THEOREM 2. Let one of the following three relations be satisfied:

a) $$\frac{\alpha}{n} < \max\left\{\frac{1}{p} - \frac{1}{q};\ \frac{1}{2} - \frac{1}{2n} - \frac{1}{q}\right\};\quad \frac{2n}{n-1} \leqslant q \leqslant \infty;$$

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b) $$\frac{\alpha}{n} < \frac{1}{p} - \frac{1}{q}; \quad \frac{2n}{n+1} \leqslant q \leqslant \frac{2n}{n-1}; \quad 1 \leqslant p < \frac{2n}{n-1};$$

c) $$\frac{\alpha}{n} < \frac{1}{p} - \frac{1}{2} - \frac{1}{2n}; \quad 1 \leqslant q \leqslant \frac{2n}{n+1}; \quad 1 \leqslant p < \frac{2n}{n+1}.$$

Then for any function $\varphi \in C_0^\infty(R^n)$ not vanishing identically with $\psi = \varphi$ the operators (*) from $L_p^\alpha(R^n)$ into $L_q(R^n)$ are not uniformly bounded in the parameter λ.

These negative results have previously been mentioned for separate limiting cases ($q = \infty$ [3]; $q = p$ [5] and the improved result of C. Fefferman [6, 7]; $p \neq q$, $\alpha = 0$ [8]). The theorem of Stein and Nikishin on maximal (invariant) operators ([2] Theorem; [9], Theorem 2) as well as the negative results for the limiting case $p = q$ are taken as a basis for Theorem 2, similarly as in [8] (Theorem 2).

Theorems 1 and 2 can be extended to multiple Fourier series which correspond to spectral expansions generated by elliptic differential operators with periodic boundary conditions. For the cases of arbitrary spectral expansions, however, (compare [3] and [8]) additional problems would have to be solved (the analysis of the difference $E_\lambda - E_\lambda^0$ in the corresponding pairs of Sobolev spaces).

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