Measurement[1]

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I

The occasion and conditions for measurement. Measurement has been defined as the correlation with numbers of entities which are not numbers[2]). As practised in the developed sciences this is a sufficiently comprehensive, though cryptic, statement of the object of measurement. But in a larger sense, in a sense to include most of those acts of identification, delimitation, comparison, present in every day thought and practise, numerical measurement is only infrequently used. "*This* is the missing book", or "He had a *good* sleep", or "The cake is *too* sweet", are judgments making no explicit reference to number. From this larger point of view, measurement can be regarded as the delimitation and fixation of our ideas of things, so that the determination of what it is to be a man or to be a circle is a case of measurement. The problems of measurement merge, at one end, with the problems of predication.

There are indeed vast domains of reflection and practise where numbers have taken little hold. Prior to Descartes, geometry was not established on a thoroughgoing numerical basis, and many branches of mathematics, like symbolic logic or projective geometry, may be pursued without introducing numbers. In arts like cookery, measurement is not primarily numerical, and the operations used are very often controlled by disciplined judgments on the qualitative alterations of the subject matter.

But the difficulty and uncertainty often experienced in obtaining desired consequences when vaguely defined ideas and crude methods of applying them are used, soon lead wherever possible to the introduction of more or less refined mathematical processes. The immediate, direct evaluation of subject matter secures too little uniformity to

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1) This essay is the second chapter of a larger work on the logic of measurement.

2) Spaier, *La Pensée et la Qualité*, p. 34.

be of much value; and direct judgments, e. g. of lengths by the eye, are consequently replaced by more complicated and indirect operations, such as the transporting of unit lengths along the lines to be compared. It is, indeed, because less error, that is, greater uniformity, is obtained in judgments of spatial coincidences than, e. g. in judgments of differences of length or color, that spatial congruence plays so large a role in laboratory practise. The *raison d'être* of numbers in measurement, is the elimination of ambiguity in classification, and the achievement of uniformity in practise.

It is generally only after numerical measurements have been established and standardized, that references to the "real" properities of things begin to appear: those properities, that is, which appear in circumstances allowing for most facility in their measurement. The "real" shape of the penny is round, because from the point of view from which the penny is round, measurements and correlations of other shapes can be carried on most easily. Nevertheless, it must not be overlooked that a numerical evaluation of things, is only one way of making evaluations of certain selected characters, although it is so far the best. It is preeminently the best, because in addition to the obvious advantage they have as a universally recognized language, numbers make possible a refinement of analysis without loss of clarity; and their emotionally neutral character permits a symbolic rendering of invariant relations in a manifold of changing qualities. Mathematics expresses the recognition of a necessity which is not human.

From this last point of view, therefore, the search for a unified body of principles in terms of which the abrupt, the transitory, the unexpected, are to exhibited and in a measure controlled, is the most conscious guiding principle in the application of numerical science. If the search for mathematical equations is the aim of physics, other activities such as experiment, classification, or measurement are subservient to this aim, and are to be understood only in relation to it. Consequently, if we inquire why we measure in physics, the answer will be that if we do measure, and measure in certain ways, then it will be possible to establish the equations and theories which are the goal of inquiry.

It is relevant, therefore, to demand the logical foundations of the mathematical operations which physics constantly uses. For if mathematics is applicable to the natural world, the formal properties of the symbolic operations of mathematics must also be predicable of many segments of that world. And if we can discover what these formal properties are, since mathematics *is* relevant to the exploration of nature, a physical interpretation *must* be found for them. That physical interpretation will constitute, whenever it can be found, the conditions for the measurement of that subject matter. Consequently, if we ask why in measurement we attend to certain characters of objects to the exclusion of others, the answer will be that the selected characters are precisely those with which applied mathematics can cope. It is only by a reference to the function which the numerical measures or magnitudes of things have in equations, that one can remove the apparent arbitrariness in selecting one rather than another set of conditions as fundamental for measurement.

In recent years the formal conditions of their science have been much discussed by mathematicians. The properties which magnitudes must have in order to be capable of the kind of elaboration which the mathematics of physics requires, have been variously formulated as axioms of quantity[1]). The following set, with some modifications, is taken from Hoelder[2]).

1. Either $a > b$, or $a < b$, or $a = b$.
2. If $a > b$, and $b > c$, then $a > c$.
3. For every a there is an a' such that $a = a'$.
4. If $a > b$, and $b = b'$, then $a > b'$.
5. If $a = b$, then $b = a$.
6. For every a there is a b such that $a > b$ (within limits).
7. For every a and b there is a c such that $c = a + b$.
8. $a + b > a'$.
9. $a + b = a' + b'$.
10. $a + b = b + a$.
11. $(a + b) + c = a + (b + c)$.
12. If $a < b$, there is a number n such that $na > b$ (also within limits)[3]).

[1]) It need be mentioned only in passing that mathematics cannot be regarded as exclusively the science of quantity. Its essence is the study of types of order, of which the quantitative one is a single instance.

[2]) „Die Axiome der Quantität", *Ber. d. Sächs. Gesellsch. d. Wiss., math.-phys. Klasse,* 1901.

[3]) On the importance of the Archimedean axiom, see Hilbert, „Axiomatisches Denken", *Math. Annalen,* Vol. 78, p. 408. The fact that one can estimate stellar distances by adjoining earthly ones, is not a logical consequence of theorems on congruence, but is an independent experimental conclusion.

How much more perspicacious such an axiomatic or functional analysis of magnitudes can be than definitions in terms of private, intrinsic properties (e. g. "quantity is the relation between the existence and the non-existence of a certain kind of being")[1]); how much more successfully it satisfies the demand for a formulation which, true for *every* instance of quantity, should not be the exhaustive statement of *any one* instance, will be evident from the sequel. Even Meinong's definition of magnitude as "whatever is capable of being limited toward zero" or as "that which is capable of having interpolations between itself and its contradictory", has little clarity to recommend it[2]). Later on a distinction will be made between magnitudes which satisfy all twelve axioms and those which satisfy only the first six. A magnitude in the most complete sense, however, is whatever is capable of verifying the whole set.

II

Order and equality. The illustration of these relations by indicating some of the empirical procedure involved in measurement, is the next portion of our task. At the very outset it must be pointed out, however, that an adequate interpretation of the axioms must inevitably lead out of the laboratory where measurements are usually made, and lead on to a consideration of the manufacture of the laboratory *instruments*. When a laboratory experiment is studied behavioristically, the measurements performed consist in the observation of the movement of a pointer on a scale, or of the superposition of lengths[3]). How natural, therefore, to suppose that measurement consists only in the observation of space-time coincidences! It is important to remember, however, that the experimenter, working with marked or calibrated instruments, assumes that the calibrations indicate various qualitative continuities not *explicitly* present. The process of measurement has not been fully exhibited until all those operations of calibration have been noted. When a weight is attached to a spring balance, and the position of a marker on the scale read, only a very small fraction of the process actually necessary to estimate the weight as five pounds has been observed; the operations entering into the construction and *correlation* of scale

[1]) So Warrain, a follower of Wronski, in *Quantite, Infini, Continu*, p. 9.

[2]) „Über die Bedeutung d. Weberschen Ges." in *Gesam. Abhandl.*, Vol. 2, p. 219.

[3]) Cf. Duhem, *La Theorie Physique*, p. 219.

and spring must be included. It is of the essence of an experiment that it be repeatable. Therefore it is not the particular instrument used any more than it is the unique experiment which has such an overwhelming importance in science; it is rather the repeatable process capable of producing the markings on the instrument which is. Every marked instrument implies the construction and existence of some standard series of magnitudes, correlation with which constitutes the calibration. A whole-hearted recognition of this reference of instruments to something beyond themselves, is a recognition that other characters of existence besides the spatial are capable of, and are involved in, the process of measurement[1]).

The relation $>$, or its converse $<$, is a transitive assymetrical relation. It finds its exemplification in some discovered qualitative domain which is sufficiently homogeneous to allow identification as a well defined range of a single quality[2]). Within this domain the character studied must be capable of such a serial gradation that a transitive assymetrical relation can be discovered to hold between the discriminated elements. So we find the character of density which liquids manifest in relation to one another to be such a relation. This character may be defined, with somewhat more care than is shown here, as the capacity of a liquid to float upon other liquids. Liquid a will be said to be more dense than b if b can float on a but a cannot float on b. And it can and *must* be shown *experimentally*, that, for a set of liquids distinguishable from each other by all sorts of physical and chemical properties, the relation $>$ (more dense) is a transitive assymetrical relation: if liquid a is more dense than b (i. e. b floats on a) and if b is more dense than c, then a is more dense than c (i. e. c floats on a).

The relation of equality ($=$) can be defined in terms of $>$. We say that $a = b$, if a is not $> b$, and a is not $< b$, and if $a \gtreqless c$, then also $b \gtreqless c$. Equality of density may therefore, be defined thus: c has a density equal to that of d if c does not float on d and d does not float on c; and if c floats on e so does d[3]).

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[1]) Dingler, in *Das Experiment*, p. 51 ff., has an important discussion in this connection, whatever one may think of his characterization of theoretical physics as without a physical meaning.

[2]) Runge, „Maß u. Messen" in *Enzyk. d. math. Wiss.*, Bd. V, p. 4.

[3]) This definition of equality in terms of the relation generating the series is due to Campbell, *Measurement and Calculation*, p. 5. The many debts of this paper to Campbell are evident to all his readers, and cannot be made explicit even by repeated citations. The discussion in Duhem, *Theorie Physique*,

The experimental establishment of series of this kind involving the first six axioms is the first step in the introduction of number. The function of the experiments is the careful exhibition of physical relations symbolized by these axioms; without such an experimental exploration of the subject matter, it may turn out that the relation between the objects under survey will not generate the transitive assymetric sequence. If, to take an absurd example, we compare the lengths of elastic rubber bands by superposition, without specifying further the manner in which this is to be done, and if we define x as longer than y when x extends beyond y, it may well be that axiom two will not be satisfied.

Numbers may of course be introduced merely for purposes of identification, and many numerical designations have no more arithmetical significance than do the names of individuals. Thus, the policemen of a large city are often known by their number; but the policeman with number 500 is not thereby known to be stronger, or more efficient, or more handsome, or wealthier, or older than the one numbered 475.

This kind of out-and-out arbitrariness can be considerably reduced if a qualitative series is first established. The numbers assigned in the operations of science are more than a conventional tag; it is desirable that an object numbered 50 occupy a position higher or lower in a qualitative series as determined above, than one marked 40[1]). So, for example, the liquids gasoline, alcohol, water, glycerine, hydrochloric acid, carbon bisulphide, and mercury, are arranged in the order of increasing density as defined above; we can establish a one-to-one correspondence between these liquids and a series of real numbers such that the order of numbers will be symbolic of the order of densities. We may assign any one of the following sets of numbers to the series of liquids: (*a*) 1, 2, 3, 4, 5, 6, 7, or (*b*) 100, 90, 88, 85, 80, 60, 10, or (*c*) 22.5, 20, 19.6, 19.3; 11.2; 10.5; 8.9; or (*d*) .75, .79, 1, 1.26, 1.27, 1.29, 13.6. If, however, we specify

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p. 159 ff., takes similar lines, while Helmholtz's important essay on *Zaehlen u. Maessen*, contains the germ of almost all subsequent works on the subject. The essay is reprinted in his *Schriften zur Erkenntnistheorie*, edited by Hertz and Schlick. "Equality between the comparable properties of two objects is an exceptional occurence, and can be recognized in empirical observations only by this, that the two equal objects make possible the noting under proper conditions of a special effect, not usually present in the interaction of other pairs of similar objects"; p. 85. See also Cournot, *Essai* . . ., p. 286 ff.

[1]) Cf. Runge, *op. cit.*, p. 5 ff.

further that the order of increasing numerical magnitude must correspond with the order of increasing density, the sets of numbers (*b*) and (*c*) are no longer available.

Nevertheless, there is still very much that is arbitrary and often misleading in assigning the numbers, say, 1 to 7 to the above sets of liquids. In choosing the set (*a*) instead of (*d*) or other sets having the same order, we exhibit the arbitrariness. The choice becomes positively misleading, if, without further definition and experimental confirmation, we suppose that because alcohol in this scheme has a density of two and gasoline that of one, there is involved a physical meaning (a physical operation defined in terms of density) in talking of alcohol as *twice* as dense as gasoline, or of *adding* densities of gasoline to obtain the density of alcohol. The operational interpretation of the axioms assigns, so far, physical meaning only to the *order* in which numbers are employed; there is no such meaning as yet for numerical *differences.*

It is just such misunderstandings, however, that are at the basis of many of the confusions in psychological and social measurements. Bogoslovsky's attempt to measure the proportion of "mental activity elements") in various tasks, is one instance of such confusions[1]). He presented to several people twenty descriptions of different situations involving mental and physical activity, and obtained their estimate of the proportion of mental activity "elements" to physical activity "elements" in each. The judges were asked first, to arrange the descriptions in order of increasing mental activity, (an attempt to generate a qualitative series) and then to express the numerical percentage of mental activity "elements" in each. But "percentage of mental activity elements" has a well defined meaning only on the assumption that the situation can be regarded as the *summed* total of elements; and that assumption must be justified both by an adequate criterion for an "element", and by the exhibition of a process of *addition* for them. Without that assumption and its justification, the results obtained indicate nothing. How little the nature of measurement is understood by the author is clear from the defense he makes for his procedure: "All our scales are arbitrary. There are no special intrinsic reasons for dividing an hour into sixty minutes." An hour may, of course, be divided into any number of intervals. But it is the possibility of definite operations defining more or less

[1]) *Technique of Controversy,* pp. 162—74.

time and the addition of intervals, that makes the division of an hour into sixty parts valuable, and the non-existence of such operations which makes valueless the computation of mental activity percentages.

When, therefore, numbers are correlated with some qualitative spectrum, it is not obvious and not always true, that the differences between the numbers represent differences between qualities definable otherwise than ordinally. Since none of the operations exhibited so far in connection with density have defined anything besides "greater than" and "equality", it is meaningless to talk, at this stage, of addition.

III

Addition. "At this stage" must be amended in the case of some properties to read "ever". For it is a well established character of existence that, although many qualities can be serially ordered and so numbered in accordance with an arbitrary plan, a few of these qualities (and a limited few they are) possess also a capacity for "addition" which the rest do not. This cleavage between additive or extensive qualities, and non-additive or intensive qualities, is of fundamental importance in the philosophy of physical measurement.

Consequently, new operations and experiments must be introduced to define addition. And for density, defined as above, such operations are not obtainable, since there is no clear sense in which two liquids equally dense could be added to produce a liquid twice as dense, and so obtain a "sum" which would possess the formal properties listed in the last six axioms. Other physical properties must be used to exhibit these new operations[1]).

"Illumination" is a very important photometric property which is capable of addition when defined as follows. It will be assumed, in the first place, that the immediate but disciplined judgments, with respect to perceived inequality or equality of the brightness of cer-

[1]) For the following illustration, see Campbell and Dudding, "Measurement of Light", *Philos. Magazine,* 6 Ser. Vol. 44. The discussion in Lambert's *Photometrie* (translated in Ostwald's Klassiker) is the foundation for the whole science. He shows, incidentally, the self-corrective nature of scientific procedure. "If one wanted to investigate the validity of eye judgments, one must remember these psychological illusions in order to take cognizance of the remaining principles of photometry. But it is these principles that are presupposed in any investigation of the errors of eye judgments. Hence I do not see how a logical circle can be avoided if a rigorously proved photometry is desired. But if this rigor is abated a little, one can obtain the propositions of photometry with some degree of certainty." Vol. 1. p. 7.

tain surfaces, can be obtained with sufficient uniformity. Brightness thus forms a qualitative domain within which more or less bright can be distinguished; equality of brightness may therefore be defined in conformity to the formal characters already specified. The surfaces used will have the same shape, same color, and the same reflection and diffusion coefficients. Surfaces will be said to form a "pair" if, when their positions are interchanged and everything else remains the same, no disturbance can be noted in the equality of the brightness of the surfaces.

A pair of surfaces will be said to have equal *illumination*, when they are judged to be equally bright under specified circumstances: e. g. the lines which join the two surfaces to the eye must make equal angles with the perpendiculars to the surfaces. Many more specified conditions must be introduced in practise. When some body can be found such that, by altering its physical state or of the medium between it and the surfaces, the brightness of the surfaces is changed, the body is defined as the *source* of the illumination. It can now be shown experimentally that if a pair of surfaces form an equally illuminated pair under a certain source, a second pair will also be equally illuminated if substituted for the first pair, even though a member of the first set does not form a "pair" with a member of the second set. This is the reason why illumination comes to be regarded as a quality not of the pair of surfaces but of the conditions under which they are placed; these conditions include, as a minimum, the nature of the source and the position of the surfaces with respect to it.

Addition of illumination can now be defined. The illumination of a surface S from sources A *and* B is defined to be equal to the *sum* of the illumination of S from A alone and the illumination from B alone, if A and B remain in the same physical state and relative position to S. If we use sources of the same color, (strictly, only monochromatic light can be used) it can be shown that axioms 8, 9, and 11, receive an experimental confirmation; but for heterochromatic light, addition in this complete sense no longer exists[1]).

With physical addition defined, numbers can now be introduced, so that all arbitrariness in assigning them, except in the choice of a unit, is removed. Some constant source B_1 is chosen and placed at

[1]) Walsh, *Photometry*, p. 7. Campbell points out that heat is capable of addition in a very restricted sense only, because no definition of addition will make the commutative and distributive axioms verifiable. *Physics.*, p. 287.

a fixed, convenient position with respect to the surface S_2, one of the pair S_1 and S_2. The illumination thus obtained is taken as unity. Next, two sources A_1 and A_1' are found and placed on a cone whose axis is perpendicular to S_1, so that either alone makes the illumination on S_1 equal to that on S_2 under B_1, and gives therefore unit illumination on S_1. Now letting A_1 and A_1' illuminate S_1 together, and extinguishing B_1, a source B_2 is found such that, placed with respect to S_2, it makes the illumination on S_2 equal to that on S_1 under both A_1 and A_1'. A source A_2 is found and placed on the cone so that the illumination of S_1 under A_2 is equal to that of S_2 under B_2. This illumination on S_1 is two, and it now has a clear physical meaning to speak of this illumination as *twice* that under A_1. This method of assigning numbers can be extended indefinitely for integral as well as fractional values, so illustrating the last axiom. To define fractional values we must find n sources placed in such positions on the cone, that any one of them makes the illumination on S_1 equal to that on S_2 under a constant source, and such that all n together make the illumination on S_1 equal to a unit illumination. Each of these n sources gives an illumination equal to $\frac{1}{n}$ of the unite illumination.

The construction of such a series of standards, both integral and fractional, in terms of explicit physical operations which take note of qualitative homogeneities and differences, is the logical prius to any other mode of physical measurement. Only now can the theorem, that illumination is inversely proportional to the square of the distance from the source, be given experimental confirmation. Only after the physical meaning of numerical operations has been thus fixed, may mathematical variables be introduced to denote unambiguously portions of subject matter. Only then may the movements of pointers be taken as signs for qualitative differences.

Clearly, the most important of the operations used in the definition of magnitude, are those fixing the meaning of addition. Operations for defining physical addition can be found for mass, length, period, electrical resistance, area, volume, force, and about a dozen others; for these, processes of fundamental measurement can be found, while other properties studied in physics are measurable only in terms of them, that is, derivatively. The example chosen makes clear, as Helmholtz pointed out long ago[1]), how experimental a con-

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[1]) Helmholtz, *Op. cit.*, p. 89 ff.

cept addition is, valid only in so far as axioms 9, 10 and 11 are verified. It should, moreover, help expose the dogma that measurement consists in the observation of pointer coincidences, as well as the belief that addition is exclusively spatial juxtaposition and division. Undoubtedly, spatial juxtaposition is the primitive meaning in the addition of lengths; but in the example used addition involves *conjoint activity* of sources; in measurement of weights, it means the establishment of *rigid connections* between solids; in the estimation of time periods, it requires the *temporal repetition* of certain rhythmus; in the evaluation of volumes it signifies the discovery of liquids which *fill containers* without implying spatial contiguity. The presence of spatial relations is not a sufficient condition for addition; there is necessary a distinctive qualitative context, inclusion in which identifies different instances of addition, as addition of the *same* characters.

When once the standard sets of different magnitudes are constructed, the measurement of even those properties whose standards these sets are, is indirect, and consists in the comparison of the properties with their standards. The comparisons are very often circuitous, because advantage is taken of the relations between variations in the fundamentally measured magnitudes and the variations in the position of a pointer with which they are in some physical connection. It is easier to make the correlation between pointers and weights once for all, than to engage in fundamental measurement whenever one is estimating the tonnage of an elephant.

IV

Some objections examined. The preceding analysis and exposition (1) assumed without much question that measurement was the evaluation of the empirical relations between physical objects diversely qualified; and (2) stressed the importance of the distinctions between extensive and intensive qualities. Both of these doctrines have met some opposition.

(1) Impressed, no doubt, by the profound difference in cognitive status between the unmeasured and measured qualitative world, Russell converts that difference, achieved in terms of the processes already examined, into a difference between a concrete actuality and a realm of essences: this latter realm is understood not as the ordered relations of and between existences, but as a domain of immaterial entities having no necessary reference to existence. For Russell, therefore, actual footrules are quantity, their lengths

are magnitudes. It is only by an ellipsis that two quantities can be said to be equal: they are equal because they possess the same magnitude; and it is improper to say that one of two quantities is greater than the other: what is meant is that the magnitude which the first quantity possesses is greater than the second magnitude. Only quantities can be said to be equal, by having the same magnitude; two magnitudes cannot be equal, since there is only one of each kind[1]).

Russell's first objection to the relative view of quantity (the view that >, <, and =, are relations holding directly between physical things), and his espousal of the absolute theory (for which things must first be referred to an otherwise undefined realm of magnitudes in order to be measured), is based upon the observation that in any proposition asserting >, <, or =, an *equal* quantity may always be substituted anywhere, without altering the truth value of the proposition. It is not the *actual* quantity, but some character which it has with other quantities that is of importance.

It will be granted that this point is well taken, if possession of relations like equality is interpreted without reference to the specific process which defines equality. When two weights are equal, that relation holds by virtue of the way the two weights enter into a complicated evaluating process. Two weigths are not equal "in themselves", they are equal as a consequence of the manipulation to which they are subjected. Of course, once having specified the defining operation, whether it is actually performed or not, the things measured have a nature prior to the actual performance which conditions their behavior in it. This observation may be verbal only: if equality is defined in terms of the process, quantities can be called equal prior to the process only proleptically; unless at *some* time the process eventuates, we cannot know that there is such a property as equality.

But it is one thing to say that relations like equality hold between two objects only in specified contexts, and another thing to convert those relations into possession of some third entity or common essence, incapable of every empirical verification. That third entity is the reification of a relation. The absolute space which haunted physics is just such a hypostatization of relations; space and time may be construed more simply as pervasive relations between events, rather than as containers, extrinsic and outside the changing qualities. So when magnitudes, which are always found to be relations

[1]) *Principles of Mathematics*, p. 164 ff.

exhibited in the physical operations of things, are invoked as the locus of those operations, it seems legitimate to ask what empirical difference their existence or nonexistence as "common essences" would make.

Russell's second point is that every transitive symmetrical relation is analyzable into a complex of two assymetric relations and a third term, and that since equality is such a relation, it should be analyzable into the possession of a common magnitude by two quantities. "The decision between the absolute and relative theories can be made at once by appealing to a certain general principle, the principle of abstraction. Whenever a relation of which there are instances is symmetric and transitive, then the relation is not primitive, but analyzable into sameness of relation to some other term, and this common relation it is such that there is only one term at most to which a given term can be so related, though many terms may be se related to a given term[1])."

If the analysis given above is sound, all measurement which is not fundamental in the sense defined, or which is not surrogative in the manner to be explained below, consists in the direct or indirect correlation of quantities to be measured with the standard series. It is correct to say, therefore, that two equal footlengths are equal because they possess a common magnitude, *if* this means that they are compared, ultimately, with the same member of a standard series. But in establishing equal magnitudes *within* the standard series itself, no reference to a "common term" was necessary. Equality, it is true, was not taken as a fundamental character, since it was defined in terms of two transitive assymetrical relations. But those relations held or did not hold between *qualities*, and did not relate to some third term *outside* the qualitative series. The weights *a* and *b* are equal, because with respect to the relation "greater than" which defines the series of weights (it may be defined in terms of the sinking or rising of the arms of a lever), *a* is neither greater than nor less than *b*, and if *c* is any other *weight* (a member of the *series*, not *outside* of it) then if *a* is greater or less than *c* so also is *b*. No unique third term outside the physical domain is involved in defining equality; if Occam's razor still can cut, the magnitudes demanded by the absolute theory may be eliminated[2]).

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[1]) *Op. cit.*, p. 166.

[2]) Couturat declares that sound common sense sides for the absolute theory. "It is reasonable to suppose that equal magnitudes have something more in common than those magnitudes of the same kind which are not equal; and it is

(2) The doctrine that magnitudes are essences, and therefore not divisible or additive even though it is only between them that the relation "greater than" can hold, leads very naturally to the view that the distinction between extensive and intensive magnitudes is purely conventional. Additiveness belongs, on the absolute theory, only indirectly to magnitudes, as a manner of speaking of the addition of quantities whose magnitudes they are. For "addition" of two magnitudes yields two magnitudes, not a new magnitude, while the addition of two quantities does give a new single whole, "provided the addition is of the kind which results from logical addition by regarding classes as the wholes formed by their terms[1])." Addition, for Russell, thus always refers to the conjunction of collections, and to the enumeration of the number of parts in the new whole. And even in the logical addition of classes there is no clear warrant, he believes, for affirming that the "divisibility" of a sum of *n* units is *n*-fold that of one unit. "We can only mean that the sum of two units contains twice as many parts, which is an arithmetical, not a quantitative judgment, and is adequate only in the case when the number of parts is finite, since in other cases the double of a number is in general equal to it. Thus even the measurement of divisibility by numbers contains an element of convention."

Now, indeed, if magnitudes express the results of physical measurement, it is not the magnitudes which are addeed, just as the measure of anything measured it not itself measured. If magnitudes have a logical status, then surely it is only logical addition of which they are capable. For *magnitudes,* therefore, the distinction between extensive and intensive has no *meaning.* Nevertheless, on Russell's theory, it is magnitudes which are measured, and it is because

contradictory to suppose that two equal magnitudes are differentiated from two unequal ones only by the fact of the unessential relation which connects them: equality in one case, inequality in the others." *Rev. de Met. et de Morale,* 1904, p. 677. The reified concept is very evident in the "something in common". Moreover, on Russell's theory there is a magnitude for every specific kind: a specific magnitude of pleasure for each grade of pleasure, a specific magnitude of density for each grade of density. Is the magnitude of density of two substances, whose densities are the same but are otherwise very dissimilark, *two* magnitudes or *one*? On the absolutee theory, since two magnitudes are of the same kind if one can be greater or less than the other, the answer is: one magnitude. But what determines whether two magnitudes *can* be compared? Is not the decision made, not by appealing to the magnitudes, but to the physical operations on quantities?

[1]) *Op. cit.,* p. 178; cf. also *Analysis of Matter,* p. 116.

he takes "logical sum" to be the primry sense of addition that he can find only a convention in the addition of spatial distances or time intervals. How the transition from the conceptual to the existential order is effected, how "logical addition" may receive an interpretation in terms of physical operations, is a consideration omitted from his analysis. Is it not, however, more perspicacious to think of mathematical "addition" as a *universal*, whose variable empirical content will be *cases* of addition, but which will require further specific definition and experimental proof of the presence of those formal characters which make those empirical contents instances of that universal?

Nevertheless, the unusual sense in which addition is sometimes used should not be overlooked. The order generated by the relation "male ancester" may be measured in the following fashion: Suppose *a* is the father of *b*, *b* the father of *c*, *c* the father of *d*; then the "relation-distance" father-of is "equal" in all the pairs *a b*, *b c* and *cd*, and *a* has the relation of "great-grand-father-of" to *d*. By an obvious convention, we could express the ancestral relation of *a* to *d* as three times the relation of *a* to *b*; and we could say that the relation of *a* to *c* is the sum of the relations of *a* to *b* and *b* to *c*. It is clear, however, that the "addition" here defined does not possess *all* the formal properties demanded by the axioms; it does not obey, for example, the commutative rule. The addition here defined is not very much more than the ordinal arrangement of relations. It is the failure to recognize the necessity of obtaining all the formal characters in fundamental measurement, which makes so unsatisfactory the attempt of Spaier to defend the measurement of non-spatial properties, and which enables him so easily to minimize the distinction between intensive and extensive qualities[1]). The introduction of numbers has a function more inclusive than the *identification* of quality.

V

Surrogative measurement. In the light of what has been said, the dichotomy between primary and secondary qualities becomes more illuminating if we view this distinction not as between objective and subjective, efficient and otiose, pervasive and local, permanent and evanescent, but as between those qualities which are capable of fundamental measurement and those which cannot.

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[1]) Spaier, *op. cit.*, pp. 242—55.

No science, certainly not physics, can dispense with qualities that are incapable of addition in the fullest sense, and the progress of modern science has consisted very largely in bringing non-additive qualities like density, temperature, hardness, viscosity, compressibility, under the sway of numerical determination[1]). There is of course one obvious way, already suggested, how this could be done. That way is to place qualities like density into a serial order, and to assign numbers to points of this qualitative spectrum. It is characteristic of modern science, however, that such is *not* the method which has been adopted, just as in zoology it is with bats and not with fish that whales are classified. For it is the particular virtue of modern science not to be concerned with the grouping of the most obvious qualities, thereby treating them as isolated from and unconnected with other groups; that virtue resides in the persistent attempt to obtain well defined connections, expressed mathematically wherever possible, between qualities measured or measurable fundamentally and those incapable of such measurement.

Unfortunately, the correlation of qualities has been often interpreted as the production of secondary qualities by the primary ones: the latter alone have been endowed with causal efficacy, the former degraded as otiose and epiphenomenal. Thus a distinction which in *operation* is a practical and logical one, has been converted into a distinction between grades of reality, on the ground that causes are more real than their supposed effects. Mathematical physics has been understood to make nonsense of the poet's cry — „Natur hat weder Kern noch Schale.“ — None the less, all that the equations of physics and the method of establishing them do imply, is that non-additive qualities are inextricably interwoven with additive ones. This dependence is existentially mutual; from the point of view of the *logic* of measurement, it is assymetrical. It is because of this dependence, expressible in the form of numerical laws, that numbers may be correlated unambiguously with non-additive qualities; it is because of such laws for density, that of the four sets of numbers entertained on a previous page, only the last set is adopted[2]).

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[1]) Duhem, *L'Evolution de la Mécanique*, p. 199 ff.

[2]) "The ascertainment of qualitative features and relations is called measuring. We measure e. g. the length of different chords that have been put into a state of vibration, with an eye to the qualitative difference of the tones caused by their vibration, corresponding to this difference of length." Hegel's *Logic* (Wallace tr.), p. 200.

How non-additive qualities may be unambiguously denoted, will be clear from an example. In the case of density it is discovered that the weight of an object is intimately connected with its volume, a connection exhibited in the uniform association of these characters. Weight and volume are measurable fundamentally, so that independently of the numerical relations which affirm their uniform association, numbers can be assigned to them. There is no special theoretical difficulty, although there may be many practical ones, of determining the value of the constant which expresses this uniform association, and which mathematically is the ratio of the numerical value of the mass to the numerical value of the volume. The *form* which the mathematical equation will take is, of course, dictated only partly by the measurements on the properties studied in any *one* instance: more general considerations will come into play arising from the desire to make the many numerical equations themsevles interconnected and parts of a unified doctrine.

In most cases, the order of the constants or ratios which are determined for several objects can be shown to be the same order as the serial order of some non-additive quality which the objects possess in addition to those already measured fundamentally. So the order of the ratios of mass to volume is identically the order of the density of liquids as defined by their floating capacities relative to one another. Consequently, the same set of numbers may be used to denote both the uniform association of mass and volume, and the relative buoyancies. It goes without saying that the numbers thus obtained for qualities from numerical equations, are not always amenable to a physical interpretation of addition. When one body is said to be thirteen times as heavy as another, a different meaning must be given to such a statement from the meaning of the statement that mercury is thirteen times as dense as water; only in terms of the numerical law connecting mass and volume has the latter proposition significance.

All equations which define a constant, to be identified perhaps with some property capable of definition independently of the equation, require therefore that the other terms of the equation be measurable without reference to the defining equations of the constant. In the sense that the constants are defined, not *all* magnitudes can be defined without leading to a circle; there must be an ultimate reference to magnitudes obtained by fundamental measurement. The equation $pv = RT$ has meaning only if p, v, T, have a meaning outside of this equation; only then may R be determined experimentally. If p, v, T, are not measurable fundamentally there must be a chain of equations connecting them with magnitudes which are. It is a testimony to the endless *complexity* of nature, not to her poverty of qualities, that only six independent fundamentally measured magnitudes are required for the investigations of physics. "There are only a few independent magnitudes in physics. But between these and the countless number of independent magnitudes appearing in human life, there is no sharp separation." Strength of wind would be a genuine aspect of some events, even if strength of wind were completely definable, which it is not yet, in terms of the velocity and force of air particles[1]).

Moreover, the power of all symbols, and especially of numerical symbols, to refer simultaneously to several contexts must be recognized if mystification is to be avoided. The unification and identification in statement that follows from the introduction of mathematical methods is due to the pervasiveness of certain formal characters in situations qualitatively different, which are as a consequence capable of a unified treatment. In the interpretation of equations as the literal identification of different qualitative continua, and as the attribution of intrinsic, non-relational common characters to diverse subject matter, lies the force of most of the petulant criticism of science. Let us, for example, study the equilibrium conditions of a beam balance. If x, y, z, represent certain bodies A, B, C, then the equation $x + y = z$ will mean that A and B on the same pan will balance C on the other, and it will also mean that the numerical measure of the weight of C is equal to the sum of the measures of A and B. The perhaps less relevant case of a chemical equation like $2H_2 + O_2 = 2H_2O$ represents theoretical conceptions like atom, valence, physical operations like passing a spark through a mixture of two gases, and numerical relations between the measures of weights and volumes.

The confusions which arise from the failure to note how complex the functions of symbols may be, is illustrated in a recent criticism of the achievements of science. The principle of the lever, as expressed in the form of a proportion, is the climax of an extended critique, but this special discussion is reproduced in full.

"It will be best to point out that it is the result of two apparently

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[1]) *Cf. Handbuch d. Physik*, Bd. II, p. 5.

unjustified leaps of imagination... A double leap it is, and the reader can supply whatever theory of revelation, reincarnation, or conventional fiction he prefers, to account for it.

"To say that $W_1 : W_2 :: D_2 : D_1$, is by itself ambiguous. Perhaps it only means that certain numbers stand to each other this way: $2 : 4 :: 3 : 6$. In that case it is merely a happy discovery in arithmetic. But W stands for weight and D stands for distance. It may therefore mean that the relation between two weights is the same as the relation between two distances. But this is not true for many relations; for instance 'heavier than' is a relation betweech two weights, but not between two distances. The only relation that works is a hybrid combination of these two.

"The combination is evidently derived from two previous proportions, namely: $W_1 : W_2 :: 2 : 4$ and $D_2 : D_1 :: 3 : 6$. Then because we already know in arithmetic that $2 : 4 :: 3 : 6$ we can finally see how it is that $W_1; W_2 :: D_2 : D_1$. But why weights and distances are like numbers, to use the simile, has still to be explained. The only answer that I know is that some poet of the commonplace was playing with words, and somebody took him literally[1])."

It cannot be pretended that the actual history of the principle of the lever is being reported. As a deduction or validation of an important numerical law, however, it merits a conspicuous place in some future Budget of Paradoxes. The difficulties which are raised seem to arise partly from a dogma that numbers cannot be the numbers of anything without losing caste, and partly from the failure to realize that in the statement of the law at least two relations are symbolized. In the first place, the relation $W_1 : W_2 :: 2 : 4$ can be intelligibly interpreted as meaning that four weigths each equal to W_1 are equal to two weigths each equal to W_2, where "equal" is defined in some unambiguous way. Secondly, it must be observed that the *symbol* :: stands for both a *numerical* relation between numbers assigned in the way suggested, and a *physical* relation into which the lever enters in a very specific way. The proportion as it stands expresses *two* sets of relations, and we can use the same *symbol* to represent both because the relations have certain formal properties which are *identical.* Numbers are not *like* weights, indeed, but numbers are, and numerals express, definite properties of weights. It is a great sin to *compare* the statement of

[1]) Buchanan, *Poetry and Mathematics,* p. 90.

a relation with the relation itself. If discourse cannot be literally compared with what it is about, is it not wisdom to recognize that discourse expresses it? The equation, in every case, is a symbolic statement, pointing to several aspects of the subject matter. When once the plural referents of the symbols are made explicit, and when numbers are not regarded either as common qualities or as chaste platonic beings, but rather as the expression of relations or operations between qualities, belief in the power and validity of mathematical physics need not be superstition.

As there are critics who find the application of numbers to additive properties a puzzle, so there are other critics who challenge the validity of the application of numbers to non-additive ones. It is never the non-additive qualities that are measured, it is said; velocity has a unique existential quale as a velocity, and to express it as the ratio of space and time is to measure space and to measure time, but it is not to measure velocity. A twofold reply may be made.

(1) There are many qualities which, as a matter of practise, are measured as derived magnitudes by means of numerical equations, but which can be measured fundamentally. Bridgeman has shown how this may be done for velocity. Areas, volumes, or electric charges, are in the same position: they are usually measured as derived magnitudes, but are capable of fundamental evaluation. Nevertheless, even when they enter numerical equations derivatively and measured therefore in their relation to other characters, *they* are measured.

(2) It is true that most qualities, being incapable of addition, must be measured in terms of their "surrogates" in numerical equations. If the term "measurement" is restricted only to such qualities which are fundamentally measurable, it must be acknowledged that density and acceleration are incapable of measurement. But by calling the process of assigning numerical values to density some other name, the *significance* of what is done is not destroyed. What is beyond much doubt, is that the measure numbers of those characters incapable of addition, or even incapable of sensuous intuition like magnetism, are not mere numerals or formulae for nominal combinations of additive characters; they represent rather certain coordinated qualities or certain relational properties of the systems studied[1]). If it is the expansion of mercury that we actually measure in the more restricted sense, the measurement is performed because there is a

[1]) Meinong, *op. cit.*, p. 228, 275. The expression "surrogative measurement" is, of course, due to Meinong.

uniform association between this expansion and qualitative temperature changes[1]).

Several modes of surrogative measurement may be obtainable for the same property. So we may define the temperatures of black bodies by using the well known Stefan-Boltzmann law of energy radiation. Two temperatures will then be equal, if the energy of radiation which the surfaces of black bodies send out in a given time into a given space, is the same for the two temperatures; two temperature-differences will be equal, if the differences of energies radiated are the same. Except for the zero point and unity, the temperature scale is defined. Comparison of this scale with the gas thermometer scale would show that the scales cannot be made directly congruent. However, by a proper choice of the zero point, the numbers of the gas thermometer are found proportional to the fourth roots of the numbers of the energy scale; if the fourth roots of the energy scale are used to define the temperature of the black body, the two scales can be used interchangeably[2]). Similary, velocity may be measured not in terms of space and time, but in terms of the resistance which a given body meets when moving through a specified medium.

It is by discovering the recurrence of certain constants in different numerical laws, that the ideal of a unified science is progressively realized.

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[1]) But one need not therefore confuse, as does Hegel, the distinction between temperature as intensive and the correlated expansion, which is extensive. Hegel, *op. cit.*, p. 194.

[2]) Runge, *op. cit.*, p. 8.

Inhaltangabe zu Nagel:
Measurement

Der Verfasser untersucht die formalen und materialen Voraussetzungen der Messung in der Physik. Insbesondere sucht er zu zeigen, daß die üblichen Formulierungen, die etwa besagen, daß eine Messung eine Korrelation zwischen Zahlen und nichtnumerischen Gebilden bedeute, oder daß eine Messung im Grunde auf die Feststellung einer raumzeitlichen Koinzidenz hinauslaufe, zum mindesten recht unvollständige Charakterisierungen des physikalischen Verfahrens darstellen; in der Arbeit wird eine genauere Analyse versucht.

Diese geht aus von den formalen Eigenschaften der Größenordnung, die jeder Messung zugrunde liegt. Eine geeignete Formulierung dieser Eigenschaften bietet sich dem Verfasser in den zwölf „Axiomen der Quantität", die in der S. 315 angegebenen Gestalt von Hölder aufgestellt worden sind.

Dann wird die Frage nach dem physikalischen Inhalt untersucht, den die somit charakterisierten formalen Forderungen an die Grundbeziehungen der Größenordnung gewinnen, wenn jene Beziehungen zur Grundlage der „Messung", d. h. der „Auswertung empirischer Beziehungen zwischen verschiedenartigen physikalischen Gegenständen" (S. 323) gemacht werden. Um etwa zu einer Ordnung der Flüssigkeiten nach ihrer Dichte zu gelangen, kann man die Grundrelationen zunächst der ersten sechs Axiome des Hölderschen Systems wie folgt inhaltlich belegen: Die Dichte einer Flüssigkeit heiße *größer* als die einer zweiten, wenn die zweite Flüssigkeit auf der ersten schwimmen kann, aber nicht umgekehrt. (Die hier erforderlichen Verfeinerungen werden in der Arbeit nicht näher behandelt.) Zweitens wird $a < b$ mit $b > a$ gleichbedeutend erklärt, und $a = b$ wird äquivalent gesetzt zu folgender Beziehung: Es ist weder $a > b$ noch $b > a$, und wenn $a \gtreqless c$, so auch $b \gtreqless c$.

Die Einführung einer derartigen inhaltlichen Belegung der Grundrelationen auf einem bestimmten physikalischen Messungsgebiet ist dem Verfasser zufolge der erste Schritt auf dem Wege zur zahlenmäßigen Messung der Größen dieses Gebietes; es ist nun Aufgabe der experimentellen Untersuchung, die Gültigkeit der physikalischen Sätze zu prüfen, in die einige der Axiome — zunächst der ersten sechs — durch diese Belegung übergehen.

Ist das Ergebnis der experimentellen Prüfung positiv, so ist auf die angegebene Weise eine Größenordnung der Dichten definiert, aber noch keine eindeutige Zuordnung numerischer Werte zu den Dichten festgelegt. Unter den Gesichtspunkten, die für die Wahl der — logisch freilich willkürlichen — Metrik entscheidend sind, steht Nagel zufolge der Gedanke an erster Stelle, daß die Art der Zuordnung es möglichst gestatten solle, der Addition (und Subtraktion) der geordneten Größen — also der letzten noch in den obigen Axiomen der Quantität auftretenden Grundbeziehung — ebenfalls eine unmittelbare physikalische Deutung zu geben.

Dem Verfasser zufolge zerfallen nun die meßbaren physikalischen Eigenschaften in zwei fundamental verschiedene Gruppen: in solche, die außer einer Größenordnung auch noch eine „Addition" zulassen (im angedeuteten Sinne einer unmittelbaren physikalischen Belegung der Relation) — Nagel nennt sie *additive* oder *extensive* Eigenschaften — und solche, für die das nicht gilt: die *nicht-additiven* oder *intensiven* Eigenschaften. (Dem Verfasser zufolge gibt diese Unterscheidung auch die Möglichkeit einer sinnvollen, metaphysikfreien Deutung der Trennung zwischen primären und sekundären Qualitäten.)

Die Dichte von Flüssigkeiten, deren Größenordnung nach dem angedeuteten Prinzip eingeführt wurde, gehört, Nagel zufolge, zu den intensiven Eigenschaften; ebenso Temperatur, Viskosität, Kompressibilität u. a. — Als Beispiele für additive Eigenschaften führt Nagel u. a. folgende an: Masse, Länge, Flächen- und Rauminhalt, Kraft, elektrischer Widerstand, Beleuchtung. —

An dem zuletzt genannten Beispiel erläutert der Verfasser nach kurzer Besprechung der Größenordnung ausführlich die inhaltliche Deutung der Addition und weist dann auf die Notwendigkeit einer experimentellen Prüfung der somit in physikalische Sätze übergehenden unter den restlichen Axiomen hin. Es wird dann erläutert, daß durch die Festlegung der Addition und weiter durch die Wahl einer Einheit der Beleuchtung die Einführung einer Metrik ermöglicht wird, die im Prinzip alle rationalen Werte der Beleuchtung physikalisch zu deuten gestattet.

Für die vollständige Meßbarkeit, d. h. für die Möglichkeit einer unmittelbaren physikalischen Deutung aller Relationen des Axiomensystems, stellt dem Verfasser zufolge die Reduzierbarkeit aller einschlägigen experimentellen Maßnahmen auf die Messung räumlicher Beziehungen noch keine hinreichende Bedingung dar; insbesondere stütze sich die Einführung der Zahlen in die Messung auf die erwähnte „Additivität", und das mache deutlich, daß den Zahlen im Rahmen der Messung eine umfassendere Funktion zukomme, als nur die einer eindeutigen Bezeichnung der verschiedenen Werte einer Größe.

Das Schlußkapitel ist der Betrachtung der Methoden gewidmet, die die Physik zur Messung der nicht „fundamental meßbaren", d. h. der nicht-additiven, Größen entwickelt hat. Diese bestünden stets darin, die intensiven Größen (oft auf mehreren Wegen, wie z. B. im Falle der Temperatur) durch eine Kette von Gleichungen mit extensiven Größen in Zusammenhang zu setzen und so die Messung der ersteren auf die der letzteren zurückzuführen. Die Dichte z. B. werde zurückgeführt auf die extensiven Größen Masse und Volumen.

In die Untersuchung ist eine Auseinandersetzung mit einigen Einwänden eingeschoben, insbesondere mit der „absoluten Theorie" (S. 324) der Messung, die Russell in „Principles of Mathematics" und in „Analysis of Matter" vertritt, und der der Verfasser u. a. eine metaphysische Verabsolutierung der der Messung zugrunde liegenden Relationsbegriffe zum Vorwurf macht.

C. G. Hempel.