II. SOLVING p-MEDIAN PROBLEMS

DISCRETE STOCHASTIC LOCATION MODELS

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Abstract

In this paper, we study how the two classical location models, the simple plant location problem and the p-median problem, are transformed in a two-stage stochastic program with recourse when uncertainty on demands, variable production and transportation costs, and selling prices is introduced. We also discuss the relation between the stochastic version of the SPLP and the stochastic version of the p-median.

Keywords and phrases

Stochastic programming with recourse, p-median, simple plant location.

1. Introduction

The classical facility location problem consists in finding the optimal location and size of facilities to be established among a given set of possible sites in order to meet supposedly known demands specified at a given set of locations with the objective of minimizing total costs consisting of fixed costs for establishing facilities and variable production and transportation costs.

In the static uncapacitated case, extensively studied since Kuehn and Hamburger [10] and for which we can refer to the recent surveys by Krarup and Pruzan [11] and Cornuejols et al. [3], the main issue is on the location choice since sizes are obtained as the sum of the demands served from each open location. Consequently, the established capacities are fully utilized.

In the dynamic context discussed by Manne [14], the time-phasing of the decisions becomes important. The dynamic uncapacitated facility location problems were introduced by Roodman and Schwarz [16] and in a slightly different form by Wesolowsky and Truscott [18]. Van Roy and Erlenkotter [17] propose a dual-based procedure that extends approaches developed by Bilde and Krarup [2] and Erlenkotter [4] for static uncapacitated problems. Their method assumes that capacities are fully used in each period. They propose to solve capacitated problems by extensions similar to those introduced by Guignard and Spielberg [7] in the static case.

This paper addresses the stochastic facility location problem in which demands, variable production and transportation costs as well as selling prices can be random. Uncertainty in demands induces that full utilization of capacities becomes infeasible and the requirement that demands should be met in all circumstances becomes unrealistic. This explains why a selling price is introduced, since optimal decisions on the size of facilities will result from a trade-off between the cost of increasing the capacity, the net profit of selling goods and the probability of the various demand levels.

Other work on stochastic location problems is mainly concerned with optimal location on networks, see e.g. Handler and Mirchandani [8], including reallocation decisions, see e.g. Berman and Leblanc [1] or Louveaux and Thisse [13], or is based on dominance assumptions that transform the problem into two simpler problems, see e.g. Jucker and Carlson [9].

Franca and Luna [6] propose to apply Bender's decomposition to the stochastic transportation problem introduced by Williams [19] in which the shipments are decided before the random events are observed.

In this paper, we present a stochastic model for the simple plant location problem and for the p-median problem in terms of a two-stage stochastic program with recourse, and we study the relations existing between the two models.

2. A private sector model

The deterministic model of the uncapacitated facility location problem, also known as the simple plant location problem, is the following program:

(SPLP) minimize $$z_p = \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij} \tag{1}$$

subject to $$\sum_{j \in J} y_{ij} = 1 \qquad i \in I \tag{2}$$

$$y_{ij} - x_j \leqslant 0 \qquad i \in I,\ j \in J \tag{3}$$

$$y_{ij} \geqslant 0 \qquad i \in I,\ j \in J \tag{4}$$

$$x_j \in \{0, 1\} \qquad j \in J, \tag{5}$$

where I is the set of customer locations or demand points, J the set of potential facility locations, x_j is 1 if facility j is open and 0 otherwise, y_{ij} is the fraction of location's i demand supplied from facility j, c_{ij} is the total of the variable capacity, production and distribution costs for supplying all of location's i demand from facility j, and $f_j \geqslant 0$ is the fixed cost for establishing facility j.

The solution of the SPLP is a set of facilities to be established. The size of a given facility is obtained as the sum of all demands that it serves.

In the stochastic case where production and distribution costs on one hand, demands on the other hand, become random, it is no longer possible to define the size of a facility as the sum of the demands it serves: this sum is not uniquely defined as demands become random, but also for some realizations of the production and distribution costs, it might be more appropriate not to serve all demands. Therefore, both the choice of the demands to be served as well as the size of the facilities to be established become part of the decision process. Hence, it is necessary to introduce either a profit for meeting demands or a penalty for unmet demands.

The stochastic formulation of the simple plant location problem is best defined in terms of a two-stage stochastic program with recourse, where the first-stage decisions are the location and the size of the facilities to be established and the second-stage decision is the allocation of the available production to the most profitable demand points.

The formulation of the stochastic simple plant location problem is as follows:

(SSPLP)
$$\underset{x,z}{\text{maximize}}\ E_\xi U\Bigg[-\sum_{j\in J} f_j x_j - \sum_{j\in J} g_j z_j + \underset{y}{\text{maximize}} \sum_{i\in I}\sum_{j\in J} d_i(\xi)\cdot(p_i(\xi) - c_{ij}(\xi))\cdot y_{ij}(\xi)\Bigg] \tag{6}$$

subject to
$$x_j \in \{0,1\} \qquad j\in J \tag{7}$$

$$z_j \geqslant 0 \qquad j\in J \tag{8}$$

$$\sum_{j\in J} y_{ij}(\xi) \leqslant 1 \qquad i\in I,\ \ \xi\in\Xi \tag{9}$$

$$\sum_{i\in I} d_i(\xi)\cdot y_{ij}(\xi) - z_j \leqslant 0 \qquad j\in J,\ \ \xi\in\Xi \tag{10}$$

$$y_{ij}(\xi) - x_j \leqslant 0 \qquad i\in I,\ \ j\in J,\ \ \xi\in\Xi \tag{11}$$

$$y_{ij}(\xi) \geqslant 0 \qquad i\in I,\ \ j\in J,\ \ \xi\in\Xi,$$

where z_j is the size of plant j, f_j, x_j and y_{ij} are as before, g_j is the variable capacity cost, d_i the demand at location i, p_i the unit profit for meeting demand in i, c_{ij} the unit production plus distribution cost from j to i, E_ξ denotes the mathematical expectation with respect to the random variable ξ, and U is some utility function. As indicated before, the model is a recourse model in the sense that the second-stage decision variables y_{ij} depend on the particular realizations of the random event ξ.

Several extensions of the SPLP have been proposed in the deterministic case. Most of them can also be dealt with in the stochastic case. As an example, if the plants have known capacity a_j (see Guignard and Spielberg [7]), the variables z_j disappear from this formulation, the fixed cost f_j is replaced by $f_j + a_j g_j$, and the constraints (10) are replaced by

$$\sum_{i \in I} d_i(\xi) \cdot y_{ij}(\xi) - a_j x_j \leqslant 0 \qquad j \in J, \;\; \xi \in \Xi . \tag{13}$$

Another natural extension is to consider a multi-stage model with the possibility of building additional capacity in the later stages (see Van Roy and Erlenkotter [17]) in the deterministic case.

Finally, as far as computational aspects are concerned, a dual-based procedure for solving the above SSPLP problem (6)–(12) is proposed by Louveaux and Peeters [12].

3. A public sector model

The deterministic p-median problem

$$(p\text{-M}) \qquad \underset{x,y}{\text{minimize}} \quad \sum_{i \in I} \sum_{j \in J} a_{ij} y_{ij} \tag{14}$$

$$\text{subject to} \quad \sum_{j \in J} y_{ij} = 1 \qquad i \in I \tag{15}$$

$$y_{ij} \leqslant x_j \qquad i \in I, \;\; j \in J \tag{16}$$

$$\sum_{j \in J} x_j = p \tag{17}$$

$$y_{ij} \geqslant 0 \tag{18}$$

$$x_j \in \{0, 1\} \tag{19}$$

consists in finding the optimal location for exactly p facilities in order to meet a specified demand at the lowest possible transportation cost, where I is the set of client locations, J the set of facility locations, among which p will be open, x_j is 1 if facility j is open and 0 otherwise, y_{ij} is the fraction of location i's demand served at facility j, and a_{ij} is the variable transportation cost for client i to be served in j.

The stochastic version of the p-median model is defined as follows:

$$\text{(S}p\text{-M)} \quad \underset{x}{\text{maximize}} \; E_\xi U \Big[\underset{y}{\text{maximize}} - \sum_{i \in I} d_i(\xi) \cdot \sum_{j \in J} a_{ij}(\xi) \cdot y_{ij}(\xi) - \sum_{i \in I} p_i(\xi) \cdot d_i(\xi) \cdot \Big(1 - \sum_{j \in J} y_{ij}(\xi)\Big)\Big] \tag{20}$$

$$\text{subject to} \quad x_j \in \{0, 1\} \qquad j \in J \tag{21}$$

$$z_j \geqslant 0 \qquad j \in J \tag{22}$$

$$\sum_{j \in J} y_{ij}(\xi) \leqslant 1 \qquad i \in I, \;\; \xi \in \Xi \tag{23}$$

$$\sum_{i \in I} d_i(\xi) \cdot y_{ij}(\xi) - z_j \leqslant 0 \qquad j \in J, \;\; \xi \in \Xi \tag{24}$$

$$\sum_{j \in J} f_j x_j + \sum_{j \in J} g_j z_j + \sum_{j \in J} s_j(\xi) \cdot \Big(\sum_{i \in I} d_i(\xi) \cdot y_{ij}(\xi) \Big) \leqslant B, \;\; \xi \in \Xi \tag{25}$$

$$y_{ij}(\xi) - x_j \leqslant 0 \qquad i \in I, \;\; j \in J, \;\; \xi \in \Xi \tag{26}$$

$$y_{ij}(\xi) \geqslant 0 \qquad i \in I, \;\; j \in J, \;\; \xi \in \Xi, \tag{27}$$

where a_{ij} is the variable transportation cost for client i being served at location $j \in J$, p_i is a penalty for unmet demand, B is an upper bound on the budget, and s_j is the service cost at facility j.

As in the stochastic version of the SPLP, the location and the size of the facilities are the first-stage decisions, while the allocation of the available service to the clients is done in the second stage.

The elegance and simplicity of the deterministic p-median model are certainly not maintained in the stochastic version. This is mainly due to two factors: the possibility of unmet demand and the explicit representation of the budget constraints. We discuss these factors in the next two sections.

4. Penalty for unmet demand

Since the sizes may not be sufficient to cover the demands for all states of nature ξ, the usual constraint $\Sigma_{j \in J} y_{ij} = 1$ is replaced by the weaker one (23). A penalty p_i for unmet demand is introduced in the objective function (20), since otherwise the optimal solution would trivially become $y_{ij} = 0$, $i \in I$, $j \in J$. In other words, randomness in the demands makes it necessary to specify explicitly the penalty for unmet demand, or equivalently the value of meeting demand which in the deterministic case is in fact assumed to be infinite.

The necessity of this penalty is clearly an undesirable feature. In many cases, it seems rather difficult to evaluate the social cost of not being able to serve a client. In the case of emergencies involving human beings for instance, this amounts to assessing the value of human life, a problem which is at present unsettled. There exist, however, cases where this evaluation seems possible, such as the damage costs for insufficient cleaning response to oil spillings discussed by Psaraftis et al. [15].

The existence of this penalty is however a necessity. Formally, as pointed out above, since otherwise the optimal solution would trivially be to serve no client at all, and more fundamentally, since the problem in fact involves two different criteria: to serve 'as many' clients as possible on the one hand, and to minimize service, transportation and investment costs on the other hand. This issue is resolved in the deterministic setting by replacing 'as many' by 'all', a solution which becomes impracticable when demands become uncertain, except maybe when the range of possible demand values is narrow.

5. The budget constraint

In this section, we discuss three different approaches for handling the budget constraint: a conservative approach by upper bounding, a separability approach, and the introduction of a system of prices. This last approach will also provide a relation between the stochastic versions of the p-median and the SPLP.

5.1. A CONSERVATIVE APPROACH

The budget constraint (25) is satisfied only if the maximum possible value of the left-hand side is less than B. If uncertainties on s_j and d_i are independent, this suggests replacing the $\Sigma_{i \in I} d_i y_{ij}$ term by its upper bound z_j, as follows:

$$\sum_{j \in J} f_j x_j + \sum_{j \in J} g_j z_j + \sum_{j \in J} s_j(\xi) \cdot z_j \leqslant B .$$

By making assumptions similar to those made in the deterministic case, namely $f_j = f$, and $b = g_j + \sup_{\xi \in \Xi} s_j(\xi)$ for all $j \in J$, the budget constraint becomes

$$\sum_{j \in J} x_j \leqslant \frac{B - b \sum_{j \in J} z_j}{f} .$$

Now, replacing $\Sigma_{j \in J} z_j$ by the upper bound $\sup_{\xi \in \Xi} \Sigma_i d_i(\xi)$, one obtains the p-median constraint

$$\sum_{j \in J} x_j \leqslant \frac{B - b \cdot \sup_{\xi \in \Xi} \sum_{i \in I} d_i(\xi)}{f} ,$$

with p being the integer part of the right-hand side.

As compared to the situation in the deterministic case, the value of p will be much smaller, since the values of b and $\Sigma_{i \in I} d_i(\xi)$ are replaced by an upper bound instead of the mean value in the deterministic case. In other words, due to uncertainties on service costs and on demands, and also due to the willingness to serve the maximal possible demand, the effect would be to reduce the number of facilities to be open and to increase the size of these facilities. This would result in an inappropriate balance in the usage of the budget. Although some other solutions could be considered, such as to base the value of p on expected demand level, the next two approaches give more adequate and balanced responses to the budget usage.

5.2. THE SEPARABLE CASE

Assuming the budget consists of two different parts, an annuity B_i for investments and a yearly service budget B_s, the constraint (25) can be replaced by

$$\sum_{j \in J} f_j x_j + \sum_{j \in J} g_j z_j \leqslant B_\mathrm{i}$$

and

$$\sum_{j \in J} s_j(\xi) \cdot \left(\sum_{i \in I} d_i(\xi) \cdot y_{ij}(\xi) \right) \leqslant B_\mathrm{s} \qquad \xi \in \Xi .$$

Assuming, as usual in the deterministic case, identical fixed sizes and identical fixed and variable costs for all facilities ($a_j = a$, $f_j = f$ and $g_j = g$ for all j), the investment budget constraint becomes

$$\sum_{j \in J} x_j \leqslant (B_\mathrm{i} - pga)/f .$$

Since the right-hand side is equal to p, one obtains the p-value as

$$p = B_\mathrm{i}/(ga+f) . \tag{28}$$

Similarly, assuming identical service costs in all facilities, $s_j(\xi) = s(\xi)$ for all j, one obtains the service constraint

$$\sum_{j \in J} \sum_{i \in I} d_i(\xi) \cdot y_{ij}(\xi) \leqslant B_\mathrm{s}/s \qquad \xi \in \Xi ,$$

where the right-hand side gives an upper limit on the number of clients who can be served.

There should clearly exist some relation between the two budgets B_i and B_s such that a sufficient number of clients can be served given the available service capacities. In particular, since $\Sigma_j d_j(\xi) \cdot y_{ij}(\xi)$ is bounded from above by a, if the relation $a \cdot p < B_\mathrm{s}/s$ is satisfied, then all open facilities can provide full service for every possible demand. In that case, one obtains a simplified version of the stochastic p-median as follows:

$$\underset{x}{\text{maximize}} \quad E_\xi U \left[\underset{y}{\text{maximize}} - \sum_{i \in I} d_i(\xi) \cdot \sum_{j \in J} a_{ij}(\xi) \cdot y_{ij}(\xi) \right.$$

$$\left. - \sum_{i \in I} p_i(\xi) \cdot d_i(\xi) \cdot \left(1 - \sum_{j \in J} y_{ij}(\xi)\right)\right] \tag{20}$$

$$\text{subject to} \quad x_j \in \{0, 1\} \qquad j \in J \tag{21}$$

$$\sum_{j \in J} y_{ij}(\xi) \leqslant 1 \qquad i \in I, \ \xi \in \Xi \tag{23}$$

$$y_{ij}(\xi) - x_j \leqslant 0 \qquad i \in I, \ j \in J, \ \xi \in \Xi \tag{26}$$

$$y_{ij}(\xi) \geqslant 0 \qquad i \in I, \ j \in J, \ \xi \in \Xi \tag{27}$$

$$\sum_{i \in I} d_i(\xi) \cdot y_{ij}(\xi) \leqslant a_j \qquad j \in J, \ \xi \in \Xi \tag{29}$$

$$\sum_{j \in J} x_j \leqslant p , \tag{30}$$

where the value of p in (30) is given by (28).

5.3. A PRICE SYSTEM

In this section, following the approach proposed by Erlenkotter [5] in the deterministic case, we introduce the possibility of charging some price for each unit of service and we show how that price system introduces both a natural formulation for the budget constraint and the equivalence between the stochastic p-median and the stochastic SPLP.

For a detailed discussion of the validity of the assumptions on the price system, we refer the reader to Erlenkotter [5].

First, we assume that the quantity of delivered service and the associated costs are well defined. Second, service demands must be met inside the limits of the available capacity. Potential demand is exogeneous and inelastic to travel cost. The penalty for unmet demand is large enough so that every 'reasonable' demand is satisfied.

It is interesting to observe that this penalty has been explicitly introduced in the stochastic model of the p-median and that it is precisely the role of that penalty to make the balance between budgetary issues which would tend to limit the available service capacity and the desire to meet the largest possible demands. In other words, it is the role of that penalty to decide what 'reasonable' means in terms of demands to be met, especially when demands are random.

Since potential demand is inelastic, it will not be affected if some price $\Psi_j(\xi)$ is charged for each unit of service provided at location j. The level of the price is a second-stage or recourse decision, taken after observing the value of the random variable. These service charges are added to the client costs in the objective function, and at the same time are also added as revenues to the right-hand side of the budget constraint.

The stochastic p-median problem becomes

$$\underset{x}{\text{maximize}} \quad E_\xi U \left[\underset{y}{\text{maximize}} - \sum_{i \in I} d_i(\xi) \cdot \left(\sum_{j \in J} a_{ij}(\xi) + \Psi_j(\xi) \right) \cdot y_{ij}(\xi) - \sum_{i \in I} p_i(\xi) \cdot d_i(\xi) \cdot \left(1 - \sum_{j \in J} y_{ij}(\xi) \right) \right] \tag{31}$$

subject to (21), (22), (23), (24), (26), (27)

$$\sum_{j \in J} f_j x_j + \sum_{j \in J} g_j z_j + \sum_{j \in J} s_j(\xi) \cdot \left(\sum_{i \in I} d_i(\xi) \cdot y_{ij}(\xi) \right) \leqslant B + \sum_{i \in I} d_i(\xi) \cdot \sum_{j \in J} \Psi_j(\xi) \cdot y_{ij}(\xi) \qquad \xi \in \Xi . \tag{32}$$

Note that the original p-median problem (20)–(27) is a restriction of the present model, where the variables $\Psi_j(\xi)$ are forced to take the value zero for all $j \in J$, $\xi \in \Xi$. Therefore, the extended model with service charges is preferable to the original one and will yield an increased expected utility.

Now we observe that the constraint (32) must hold with equality for all $\xi \in \Xi$ in an optimal solution, since otherwise decreasing some $\Psi_j(\xi)$ corresponding to one $y_{ij}(\xi) > 0$ would reduce (31). We may therefore substitute from (32) into (31) to obtain

$$\begin{aligned}
&\underset{x,z}{\text{maximize}}\ E_\xi U\Bigg[-\sum_{j\in J} f_j x_j - \sum_{j\in J} g_j z_j \\
&+ \underset{y}{\text{maximize}}\ -\sum_{i\in I} d_i(\xi)\cdot \sum_{j\in J}(a_{ij}(\xi) + s_j(\xi))\cdot y_{ij}(\xi) \\
&\qquad - \sum_{i\in I} p_i(\xi)\cdot d_i(\xi)\cdot\Big(1 - \sum_{j\in J} y_{ij}(\xi)\Big) + B\Bigg],
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\underset{x,z}{\text{maximize}}\ E_\xi U\Bigg[-\sum_{j\in J} f_j x_j - \sum_{j\in J} g_j z_j \\
&+ \underset{y}{\text{maximize}}\ \sum_{i\in I}\sum_{j\in J} d_i(\xi)\cdot(p_i(\xi) - a_{ij}(\xi) - s_j(\xi))\cdot y_{ij}(\xi) \\
&\qquad + B - \sum_{i\in I} p_i(\xi)\cdot d_i(\xi)\Bigg]. \qquad (33)
\end{aligned}$$

Two interpretations can be given to the result (33). First, that the problem (33) subject to (21)–(24), (26), (27) is exactly the same as the private-sector model except for the constant term $B - \Sigma_{i\in I} p_i(\xi)\cdot d_i(\xi)$. In particular, if the utility is linear, then the decisions taken in the public- and in the private-sector model will be exactly the same. This constant term represents the difference between the allocated budget and the value of demand.

A second interpretation is obtained by considering that (33) consists of two parts to be minimized: the difference between the expenses and the budget,

$$\Big(\sum_{j\in J} f_j x_j + \sum_{j\in J} g_j z_j - B\Big),$$

and the difference between the value of the demand and the net value effectively obtained by the public

$$\left(\sum_{i \in I} p_i(\xi) \cdot d_i(\xi) - \sum_{i \in I} \sum_{j \in J} d_i(\xi) \cdot (p_i(\xi) - a_{ij}(\xi) - s_j(\xi) \right) \cdot y_{ij}(\xi).$$

Note that the balance between these two terms fully depends on the choice of the scaling factor used to define the value of $p_i(\xi)$.

The actual assignment of prices is a difficult problem with many degrees of freedom, since for each $\xi \in \Xi$ there is only one constraint binding the prices $\Psi_j(\xi)$, $j \in J$. Assuming the prices cover the marginal service cost $s_j(\xi)$ plus a uniform charge $\Psi(\xi)$, we deduce from (32) that

$$\Psi(\xi) = \frac{\sum_{j \in J} f_j x_j + \sum_{j \in J} g_j z_j - B}{\sum_{i \in I} \sum_{j \in J} d_i(\xi) \cdot y_{ij}} \qquad \xi \in \Xi .$$

Since y_{ij} must be feasible for all $i \in I$, $j \in J$, (27) is satisfied and therefore

$$\Psi(\xi) \geqslant \frac{\sum_{j \in J} f_j x_j + \sum_{j \in J} g_j z_j - B}{\sum_{j \in J} z_j} \qquad \xi \in \Xi . \tag{34}$$

Since, in the deterministic case, $\Sigma_{j \in J} z_j$ is exactly equal to $\Sigma_{i \in I} d_i$, the right-hand side of (34) corresponds to the price charged in the deterministic case. Hence, (34) indicates that a natural consequence of uncertainty is that for a given budget, the price to be charged is larger than in the deterministic case.

References

[1] O. Berman and B. Leblanc, Location-relocation of N mobile facilities on a stochastic network, Transportation Science 18(1984)315.

[2] O. Bilde and J. Krarup, Sharp lower bounds and efficient algorithms for the simple plant location, Ann. Discr. Math. 1(1977)79.

[3] G. Cornuejols, G.L. Nemhauser and L.A. Wolsey, The uncapacitated facility location problem, in: *Discrete Location Theory*, ed. R.L. Francis and P. Mirchandani (Wiley Interscience, 1986), forthcoming.

[4] D. Erlenkotter, A dual-based procedure for uncapacitated facility location, Oper. Res. 26 (1978)992.

[5] D. Erlenkotter, On the choice of models for public facility location, in: *Locational Analysis of Public Facilities*, ed. J.F. Thisse and H.G. Zoller (North-Holland, Amsterdam, 1983).

[6] P.M. Franca and H.P. Luna, Solving stochastic transportation-location problems by generalized Bender's decomposition, Transportation Science 16(1982)113.

[7] M. Guignard and K. Spielberg, A direct dual method for the mixed plant location problem with some side constraints, Math. Progr. 17(1979)198.

[8] G.Y. Handler and P.B. Mirchandani, *Location on Networks. Theory and Algorithms* (MIT Press, Cambridge, MA, 1979).

[9] J.V. Jucker and R.C. Carlson, Simple plant location problem under uncertainty, Oper. Res. 24(1976)1045.

[10] A.A. Kuehn and M.J. Hamburger, A heuristic program for locating warehouses, Management Science (1963) 643.

[11] J. Krarup and P.M. Pruzan, The simple plant location problem: Survey and synthesis, Eur. J. Oper. Res. 12(1983)36.

[12] F.V. Louveaux and D. Peeters, A dual-based procedure for stochastic facility location, Cahier de Recherche 69, Fac. Sciences Economiques, University of Namur, Belgium (1985).

[13] F.V. Louveaux and J.F. Thisse, Production and location on a network under demand uncertainty, Oper. Res. Lett. 4(1985)145.

[14] A.S. Manne (ed) *Investments for Capacity Expansion: Size, Location and Time-Phasing* (MIT Press, Cambridge, MA, 1967).

[15] H.N. Psaraftis, G.G. Tharakan and A. Ceder, Optimal response to oil spills: The strategic decision case, Working Paper OE-SP-84-1, MIT (1984).

[16] G.M. Roodman and L.B. Schwarz, Extensions of the multi-period facility phase-out model: New procedures and applications to a phase-in phase-out problem, AIIE Trans. 9(1977)103.

[17] T.J. Van Roy and D. Erlenkotter, A dual-based procedure for dynamic facility location, Management Science 28(1982)1091.

[18] G.O. Wesolowsky and W.G. Truscott, The multiperiod location-allocation problem with relocation of facilities, Management Science 22(1975)57.

[19] A.C. Williams, Stochastic transportation problem, Oper. Res. 11(1963)759.