CONVERGENCE OF SPECTRAL METHOD IN TIME FOR BURGERS' EQUATION

WU SHENGCHANG (吴声昌) LIU XIAOQING (刘小清)

(*Institute of Applied Mathematics, the Chinese Academy of Sciences, Beijing* 100080, *China*)

Abstract

For solving Burgers' equation with periodic boundary conditions, this paper presents a fully spectral discretization method: Fourier Galerkin approximation in the spatial direction and Chebyshev pseudospectral approximation in the time direction. The expansion coefficients are determined by means of minimizing an object functional, and rapid convergence of the method is proved.

Key words. Spectral method, Burgers' equation, Galerkin approximation, pseudospectral approximation

1. Introduction

The classical spectral methods for Burgers' equation

$$U_t + UU_x - \nu U_{xx} = f$$

usually discretize the time direction with finite difference method[1] so that the order of convergence in the t-direction is lower than that in the x-direction which is discretized with spectral method. Therefore, to obtain integral high order of convergence, we may also apply spectral method to the t-direction. Such a trial can be found in [2], where some numerical results were given by using the tau method in time, but convergence of the method is not proved theoretically. As we shall find in paragraph 2, if we just approximate the equation and its initial condition with standard spectral discretization, the derived system of algebraic equations of the expansion coefficients will be overdetermined, or say, unsolvable. To overcome the difficulty, we shall construct an object functional and determine the expansion coefficients by minimizing it. In paragraph 3, we prove the convergence of this method.

2. Algorithm

Let

$$I = \{x : 0 < x < 2\pi\}, \qquad T = \{t : -1 < t < 1\}, \qquad \Omega = I \times T.$$

Received December 29, 1993.

We consider the following problem:

$$\begin{cases} DU \equiv U_t + UU_x - \nu U_{xx} = f, & \text{in } \Omega, \\ U(x,-1) = \varphi(x), & \text{in } I, \end{cases} \tag{2.1}$$

where f, φ are of period 2π with respect to x. B-L. Guo proved the existence and uniqueness of the solution $U(x,t)$ of (2.1) in [3]. Obviously we have

$$U(x,t) = U(x+2\pi, t), \qquad \forall t \in [-1,1].$$

Since Chebyshev polynomials are orthogonal with respect to the weight function

$$\omega(t) = 1/\sqrt{1-t^2}, \qquad t \in (-1,1),$$

we introduce the weight norms for a function $\xi(t)$ as:

$$\|\xi\|_T^2 = \int_{-1}^{1} \omega(t)\xi^2(t)\,dt, \qquad \|\xi\|_{q,T}^2 = \|\xi\|_T^2 + \left\|\frac{d\xi}{dt}\right\|_T^2 + \cdots + \left\|\frac{d^q\xi}{dt^q}\right\|_T^2.$$

We define two finite dimensional spaces:

$$F_M = \mathrm{Span}\,\{e^{ikx} : |k| \le M\}, \qquad C_N = \mathrm{Span}\,\{T_j(t) : 0 \le j \le N\},$$

where $T_j(t)$ is Chebyshev polynomial of degree j. And we denote by P_M the orthogonal projection from $L^2(I)$ to F_M, and by Π_N, the interpolation from $C(T)$ to C_N, i.e. for all $\eta(t) \in C(T)$, $\Pi_N\eta \in C_N$ satisfies

$$\Pi_N\eta(t_j^N) = \eta(t_j^N), \qquad j = 0, 1, \cdots, N,$$

where t_j^N, $j = 0, \cdots, N$ are the extreme points of Chebyshev polynomial $T_N(t)$ within the interval $[-1,1]$, i.e.

$$t_j^N = \cos(j\pi/N), \qquad j = 0, 1, \cdots, N.$$

According to the idea of spectral method, we naturally hope to find

$$u(x,t) = \sum_{\substack{0\le j\le N \\ |k|\le M}} u_{jk}T_j(t)e^{ikx} \in C_N \otimes F_M$$

such that

$$\big(u_t(t_j^N) + u(t_j^N)u_x(t_j^N) - \nu u_{xx}(t_j^N),\, v\big) = \big(f(t_j^N),\, v\big), \qquad \forall v \in F_M,\ 0 \le j \le N,$$

and

$$\big(u(-1),\, v\big) = (\varphi, v), \qquad \forall v \in F_M.$$

Unfortunately, this brings us $(2M+1)\times(N+2)$ independent equations for merely $(2M+1)\times(N+1)$ unknowns $\{u_{j,k}\}$. Such an overdetermined system of equations is usually unsolvable.

In order to determine $\{u_{jk}\}$, let us first construct an object functional $E_{M,N} : C_N \otimes F_M \to \mathbb{R}^+$. For all

$$w = \sum_{\substack{0\le j\le N \\ |k|\le M}} w_{jk}T_j(t)e^{ikx},$$

let

$$E_{M,N}(w) = \|w(-1) - P_M\varphi\|_I^2 + \frac{\pi}{2N}\sum_{j=0}^{2N}\frac{1}{d_j}\|Dw(t_j^{2N}) - P_{2M}f(t_j^{2N})\|_I^2, \tag{2.2}$$

where $\{t_j^{2N}\}$ are the extreme points of $T_{2N}(t)$,

$$d_j = \begin{cases} 1, & 1 \le j \le 2N-1; \\ 2, & j = 0, 2N. \end{cases}$$

Proposition 2.1. There exists a minimum for $E_{M,N}(w)$ in $C_N \otimes F_M$.

Proof. In fact, $E_{M,N}(w)$ can be regarded as a nonnegative multivariate function mapping $\{w_{jk}\}$ to $E_{MN}(w)$. For any j and k, we have

$$E_{M,N}(w) \to +\infty, \qquad \text{as } |w_{jk}| \to +\infty.$$

Therefore, there exists a positive number R, such that for all w satisfying

$$\sum_{\substack{0\le j\le N\\ |k|\le M}} w_{jk}^2 > R^2,$$

the following inequality holds:

$$E_{M,N}(w) > E_{M,N}(o).$$

On the other hand, $E_{M,N}(w)$ can reach its minimum in $\overline{B}_R$ because the closed ball $\overline{B}_R$ is compact in Euclidean space $\mathbb{R}^{(2M+1)\times(N+1)}$. Thus the proposition is proved.

Numerical methods for minimizing $E_{M,N}(w)$ have been well discussed in the literature. See [4].

3. Convergence Theorem

Theorem 3.1. Suppose the solution $U(x,t)$ of (2.1) is sufficiently smooth and let $u \in C_N \otimes F_M$ be a minimal solution of $E_{M,N}(w)$ in $C_N \otimes F_M$. For any $r > 0$ and M, N sufficiently large, we have the following estimation:

$$\|u(t) - U(t)\|_I = O(M^{-r} + N^{-r}), \qquad \forall t \in [-1,1].$$

To prove the convergence theorem, we need two results from approximation theory.

Lemma 3.2[5]**.** Suppose function $\xi(t)$ is sufficiently smooth. For $0 \le q \le \frac{1}{2}s$, there exists a positive constant c independent of N and ξ, such that

$$\|\Pi_N\xi - \xi\|_{q,T} \le cN^{2q-s}\|\xi\|_{s,T}.$$

Lemma 3.3[6]**.** Suppose periodic function $\eta(x)$ is sufficiently smooth. For $0 \le q \le p$, there exists a positive constant c independent of M and η, such that

$$\|P_M\eta - \eta\|_{q,I} \le cM^{q-p}|\eta|_{p,I}.$$

From the lemmas above, it is not difficult to obtain the following corollaries.

Corollary 3.4. Let $v(x,t)$ be fixed and sufficiently smooth. For any $r>0$ and M,N sufficiently large, we have

$$\int_{-1}^{1}\omega(t)\left\|D^{\alpha}\left(\Pi_N P_M v(t)-v(t)\right)\right\|_I^2 dt=O(M^{-2r}+N^{-2r}),$$

where $|\alpha|\le 2$.

Corollary 3.5. Let $v(x,t)$ be fixed and sufficiently smooth. The following estimate holds:

$$\sup_{(x,t)\in\Omega}\left|\Pi_N P_M v(x,t)-v(x,t)\right|\le c,$$

where c is a positive constant.

Proof. By Sobolev's embedding theorem, $H^2(\Omega)\to C^0(\Omega)$. It follows that

$$\sup_{(x,t)\in\Omega}\left|\Pi_N P_M v(x,t)-v(x,t)\right|\le\|\Pi_N P_M v-v\|_{2,\Omega}$$
$$\le\left(\int_{-1}^{1}\omega(t)\sum_{|\alpha|\le 2}\left|D^{\alpha}\left(\Pi_N P_M v(t)-v(t)\right)\right|_I^2 dt\right)^{1/2}.$$

By Lemma 3.2 and Lemma 3.3, the proof can be finished.

The next lemma will show that the functional $E_{M,N}(w)$ is essentially equivalent to

$$\|w(-1)-P_M\varphi\|_I^2+\int_{-1}^{1}\omega(t)\left\|Dw(t)-\Pi_{2N}P_{2M}f(t)\right\|_I^2 dt.$$

Lemma 3.6[5]**.** Let $\xi(t)$ be a polynomial of degree N. We have

$$\int_{-1}^{1}\omega(t)\xi^2(t)\,dt\le\frac{\pi}{N}\times\sum_{j=0}^{N}\frac{1}{c_j}\xi^2(t_j^N)\le 2\int_{-1}^{1}\omega(t)\xi^2(t)\,dt,$$

where

$$c_j=\begin{cases}1, & 1\le j\le N-1,\\ 2, & j=0,N.\end{cases}$$

Lemma 3.7 (Gronwall's inequality[7]). Let $v(t)\ge 0$ be continuous on $[t_0,T]$. If there are constants c and L, such that

$$v(t)\le c+L\int_{t_0}^{t}v(s)\,ds,\qquad\forall t\in[t_0,T],$$

then the following Gronwall inequality holds:

$$v(t)\le ce^{L(t-t_0)}.$$

Now we prove the convergence theorem. Let $\triangle u=u-U$. We have, by (2.1), that

$$\triangle u_t+(uu_x-UU_x)-\nu\triangle u_{xx}=\boldsymbol{D}u-f,$$
$$\triangle u_t-\nu\triangle u_{xx}=\boldsymbol{D}u-f-(uu_x-UU_x)=\boldsymbol{D}u-f-(U\triangle u_x+U_x\triangle u+\triangle u\triangle u_x).$$

Forming inner products of $L^2(I)$ with Δu on both sides, we obtain

$$\begin{aligned}&\big(\Delta u_t(t),\,\Delta u(t)\big)-\nu\big(\Delta u_{xx}(t),\,\Delta u(t)\big)\\=&\big(\boldsymbol{D}u(t)-f(t),\,\Delta u(t)\big)-\big[\big(U(t)\Delta u_x(t),\,\Delta u(t)\big)\\&+\big(U_x(t)\Delta u(t),\,\Delta u(t)\big)+\big(\Delta u(t)\Delta u_x(t),\,\Delta u(t)\big)\big].\end{aligned}\tag{3.1}$$

Since $U(t)$ and $\Delta u(t)$ are of period 2π, it is easy to prove, by means of integration by parts, that

$$\begin{aligned}&\big(\Delta u_{xx}(t),\,\Delta u(t)\big)=-|\Delta u(t)|_{1,I}^2,\qquad \big(\Delta u(t)\Delta u_x(t),\,\Delta u(t)\big)=0,\\&\big(U_x(t)\Delta u(t),\,\Delta u(t)\big)=-2\big(U(t)\Delta u_x(t),\,\Delta u(t)\big).\end{aligned}$$

Thus (3.1) is simplified as

$$\big(\Delta u_t(t),\,\Delta u(t)\big)+\nu|\Delta u(t)|_{1,I}^2=\big(\boldsymbol{D}u(t)-f(t),\,\Delta u(t)\big)+\big(U(t)\Delta u_x(t),\,\Delta u(t)\big).$$

We suppose $\sup\limits_{(x,t)\in\Omega}\big|U(x,t)\big|=B_1$. It follows from above that

$$\begin{aligned}&\big(\Delta u_t(t),\,\Delta u(t)\big)+\nu\big|\Delta u(t)\big|_{1,I}^2\\\le&\frac{1}{2}\big(\|\boldsymbol{D}u(t)-f(t)\|_I^2+\|\Delta u(t)\|_I^2\big)+\frac{B_1}{2}\Big(\frac{\nu}{B_1}\big|\Delta u(t)\big|_{1,I}^2+\frac{B_1}{\nu}\|\Delta u(t)\|_I^2\Big).\end{aligned}$$

Therefore

$$\big(\Delta u_t(t),\,\Delta u(t)\big)\le\frac{1}{2}\big\|\boldsymbol{D}u(t)-f(t)\big\|_I^2+\frac{\alpha}{2}\big\|\Delta u(t)\big\|_I^2,\tag{3.2}$$

where $\alpha=1+B_1^2/\nu$. Integrating (3.2), we obtain

$$\frac{1}{2}\big(\|\Delta u(t)\|_I^2-\|u(-1)\|_I^2\big)\le\frac{1}{2}\int_{-1}^{t}\big\|\boldsymbol{D}u(t)-f(t)\big\|_I^2\,dt+\frac{\alpha}{2}\int_{-1}^{t}\big\|\Delta u(t)\big\|_I^2\,dt,$$

or

$$\|\Delta u(t)\|_I^2\le\|\Delta u(-1)\|_I^1+\int_{-1}^{1}\omega(t)\big\|\boldsymbol{D}u(t)-f(t)\big\|_I^2\,dt+\alpha\int_{-1}^{t}\|\Delta u(t)\|_I^2\,dt.$$

By Lemma 3.7, we have the following estimate:

$$\|\Delta u(t)\|_I^2\le\Big(\|\Delta u(-1)\|_I^2+\int_{-1}^{1}\omega(t)\big\|\boldsymbol{D}u(t)-f(t)\big\|_I^2\,dt\Big)e^{2\alpha}.\tag{3.3}$$

Since

$$\begin{aligned}&\big\|\Delta u(-1)\big\|_I^2=\|u(-1)-\varphi\|_I^2\le2\big(\|u(-1)-P_M\varphi\|_I^2+\|P_M\varphi-\varphi\|_I^2\big),\\&\int_{-1}^{1}\omega(t)\big\|\boldsymbol{D}u(t)-f(t)\big\|_I^2\,dt\\\le&2\Big(\int_{-1}^{1}\omega(t)\,\big\|\boldsymbol{D}u(t)-\Pi_{2N}P_{2M}f(t)\big\|_I^2\,dt+\int_{-1}^{1}\omega(t)\,\big\|\Pi_{2N}P_{2M}f(t)-f(t)\big\|_I^2\,dt\Big),\end{aligned}$$

it follows, by Lemma 3.2 and Corollary 3.4, that

$$
\begin{aligned}
&\left\|\Delta u(-1)\right\|_I^2+\int_{-1}^1 \omega(t)\left\|Du(t)-f(t)\right\|_I^2 dt \\
\le& 2\Big(\left\|u(-1)-P_M\varphi\right\|_I^2+\int_{-1}^1 \omega(t)\left\|Du(t)-\Pi_{2N}P_{2M}f(t)\right\|_I^2 dt\Big) \\
&+O(M^{-2r}+N^{-2r}).
\end{aligned} \tag{3.4}
$$

Let $u^*=\Pi_N P_M U$. We can make a series of estimations as follows:

$$
\begin{aligned}
&\left\|u(-1)-P_M\varphi\right\|_I^2+\int_{-1}^1 \omega(t)\left\|Du(t)-\Pi_{2N}P_{2M}f(t)\right\|_I^2 dt \\
\le& E_{M,N}(u)\le E_{M,N}(u^*) \\
=&\|P_MU(-1)-P_M\varphi\|_I^2-\frac{\pi}{2N}\times\sum_{j=0}^{2N}\frac{1}{d_j}\left\|Du^*(t_j^{2N})-P_{2M}f(t_j^{2N})\right\|_I^2 \\
\le& 2\int_{-1}^1 \omega(t)\left\|Du^*(t)-\Pi_{2N}P_{2M}f(t)\right\|_I^2 dt \\
\le& 4\Big(\int_{-1}^1 \omega(t)\left\|Du^*(t)-f(t)\right\|_I^2 dt+\int_{-1}^1 \omega(t)\left\|\Pi_{2N}P_{2M}f(t)-f(t)\right\|_I^2 dt\Big) \\
=& 4\int_{-1}^1 \omega(t)\left\|Du^*(t)-f(t)\right\|_I^2 dt+O(M^{-2r}+N^{-2r}).
\end{aligned} \tag{3.5}
$$

Now we estimate $\int_{-1}^1 \omega(t)\left\|Du^*(t)-f(t)\right\|_I^2 dt$. By (2.1), we have

$$
\begin{aligned}
&\int_{-1}^1 \omega(t)\left\|Du^*(t)-f(t)\right\|_I^2 dt=\int_{-1}^1 \omega(t)\left\|Du^*(t)-DU(t)\right\|_I^2 dt \\
\le& 3\Big(\int_{-1}^1 \omega(t)\left\|\frac{\partial}{\partial t}(u^*(t)-U(t))\right\|_I^2 dt+\nu^2\int_{-1}^1 \omega(t)\left\|\frac{\partial^2}{\partial x^2}(u^*(t)-U(t))\right\|_I^2 dt \\
&+\int_{-1}^1 \omega(t)\left\|u^*(t)u_x^*(t)-U(t)U_x(t)\right\|_I^2 dt\Big) \\
=& 3\int_{-1}^1 \omega(t)\left\|u^*(t)u_x^*(t)-U(t)U_x(t)\right\|_I^2 dt+O(M^{-2r}+N^{-2r}).
\end{aligned} \tag{3.6}
$$

The term $\int_{-1}^1 \omega(t)\left\|u^*(t)u_x^*(t)-U(t)U_x(t)\right\|_I^2 dt$ in (3.6) indicates the error caused by the nonlinear term of Burgers equation. We estimate it as follows:

$$
\begin{aligned}
&\int_{-1}^1 \omega(t)\left\|u^*(t)u_x^*(t)-U(t)U_x(t)\right\|_I^2 dt \\
\le& 3\Big(\int_{-1}^1 \omega(t)\left\|U(t)\big(u_x^*(t)-U_x(t)\big)\right\|_I^2 dt+\int_{-1}^1 \omega(t)\left\|U_x(t)\big(u^*(t)-U(t)\big)\right\|_I^2 dt \\
&+\int_{-1}^1 \omega(t)\left\|\big(u^*(t)-U(t)\big)\big(u_x^*(t)-U_x(t)\big)\right\|_I^2 dt\Big).
\end{aligned}
$$

Suppose $\sup_{(x,t)\in\Omega} |U_x(x,t)| = B_2$, and suppose $\sup_{(x,t)\in\Omega} |u^*(x,t) - U(x,t)| = B_3$ by Corollary 3.5. We eventually have

$$\begin{aligned}&\int_{-1}^{1} \omega(t) \left\|u^*(t)u_x^*(t) - U(t)U_x(t)\right\|_I^2 dt\\ \leq&3\Big(B_1^2 \int_{-1}^{1} \omega(t) \left\|u_x^*(t) - U_x(t)\right\|_I^2 dt + B_2^2 \int_{-1}^{1} \omega(t) \left\|u^*(t) - U(t)\right\|_I^2 dt\\ &+ B_3^2 \int_{-1}^{1} \omega(t) \left\|u_x^*(t) - U_x(t)\right\|_I^2 dt\Big)\\ =&O(M^{-2r} + N^{-2r}). \end{aligned} \tag{3.7}$$

By (3.3)–(3.7), we conclude that

$$\left\|u(t) - U(t)\right\|_I = O(M^{-r} + N^{-r}), \qquad \forall t \in [-1,1].$$

References

[1] B-C. Jiang, et al. Spectral Methods in Computational Physics. Hunan Science and Technology Press, Changsha, 1989 (in Chinese).

[2] G. Ierley, et al. Spectral Method in Time for a Class of Parabolic Partial Differential Equations. *J. Comp. Phys.*, 1992, 102: 88–97.

[3] B-L. Guo, et al. Soliton. Science Press, Beijing, 1987 (in Chinese).

[4] D-R. Wang. Methods for Solving Nonlinear Equations and Optimization Methods. People's Education Press, Beijing, 1979 (in Chinese).

[5] R.G. Voigt, et al. Spectral Method for PDE. SIAM Philadelphia, 1984.

[6] C. Canuto, et al. Analysis of the Combinated Finite Element and Fourier Interpolation. *Numer. Math.*, 1982, 39: 205–220.

[7] D-Q. Li, et al. Nonlinear Evolutional Equations. Science Press, Beijing, 1989.

[8] P. Dutt. Spectral Methods for Initial Boundary Value Problem—an Alternative Approach. *SIAM J. Numer. Anal.*, 1990, 27(4): 885–903.

[9] D. Gottlieb. Numerical Analysis of Spectral Method. CBMS-NSF, Regional Conference Series in Applied Math., 26, 1977.

[10] R.A. Adams. Sobolev Spaces. Science Press, 1975 (in Chinese).