Aspects of the Turbulence Problem

Survey Report

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Second Part[2])

IV. GENERAL DISCUSSION

1. Equations of Motion. Reynolds Equations

We assume that the velocity components u_i, the pressure p, the density ϱ, the temperature T, etc. satisfy the general Navier-Stokes equations of motion, that is, we have

$$\frac{\partial \varrho u_i}{\partial t} + \frac{\partial \varrho u_i u_k}{\partial x_k} = \frac{\partial \tau_{ik}}{\partial x_k}, \tag{IV-1a}$$

$$\frac{\partial \varrho \mathcal{J}}{\partial t} + \frac{\partial \varrho u_k \mathcal{J}}{\partial x_k} = + \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_k} [\tau_{ik} u_i + q_k], \tag{IV-1b}$$

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho u_e}{\partial x_e} = 0 \tag{IV-1c}$$

with

$$\tau_{ik} = -\left[p + \frac{1}{3} (\lambda + 2\mu) \frac{\partial u_e}{\partial x_e}\right] \delta_{ik} + \mu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right), \tag{IV-1d}$$

$$q_k = -\alpha \frac{\partial T}{\partial x_k}, \tag{IV-1e}$$

$$\mathcal{J} = \frac{1}{2} u_i u_i + h. \tag{IV-1f}$$

λ, μ are the two viscosity coefficients, α the heat conductivity, and h the enthalpy per unit mass of the fluid.

A direct approach to the turbulence problem would consist in solving (IV-1) for a given set of boundary or initial values and to compute mean values over the ensemble of solutions. Even for the most restricted problem, turbulence of an incompressible fluid, this appears to be a hopeless undertaking due to the nonlinear terms in the equations. Thus, the standard procedure, following REYNOLDS' classical studies, consists of averaging over the equations rather than over the solutions. In this way relations between averaged quantities are obtained. However, due to the nonlinear transport terms, the system of equations obtainable in this way is undetermined since it is impossible to obtain an equation which involves only correlations of the same order, for example, double correlations.

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[2]) First Part see ZAMP *3*, 321 (1952).

Indeed, averaging (IV-1a) for a field of flow consisting of a mean flow and turbulent fluctuations

$$u_i = U_i + u'_i\,, \quad \overline{u_i} = U_i$$

yields Reynolds equations which couple the mean flow U_i to the quadratic mean values of the fluctuations. For example, for incompressible flow we have

$$\frac{\partial \varrho\, U_i}{\partial t} + \frac{\partial \varrho\, U_i\, U_k}{\partial x_k} = \frac{\partial}{\partial x_k}\,(\tau_{ik} - \varrho\, \overline{u'_i\, u'_k}) \qquad \text{(IV-2a)}$$

in which the "apparent" or Reynolds stresses now appear on the right hand side. Multiplying (IV-1a) by u'_j, for example, and averaging we obtain an expression connecting the stress tensor $\overline{u'_i\, u'_j}$ with the expression containing the triple correlations $\overline{u'_i\, u'_j\, u'_k}$, etc. The same procedure evidently applies to the general compressible case except that more terms and equations are involved.

Hence, the method of averaging the equations can not lead to a final determined solution since it evidently does not exhaust the content of the original equations. The averaged equations require additional relations which have to come from statistical or similarity considerations or from general physical considerations which allow a choice of the possible solutions to be made from the number of solutions compatible with the averaged equations.

2. The Analogy with Kinetic Theory

From Reynolds equations, for example, in the simple form (IV-2a), an immediate analogy between turbulence in a fluid and molecular motion can be drawn. The viscous stress tensor τ_{ik} expressed in terms of the molecular velocities c_j is given by $\tau_{ik} = -\varrho\, \overline{c_i\, c_k}$, thus (IV-2a) can be written

$$\frac{\partial \varrho\, U_i}{\partial t} + \frac{\partial \varrho\, U_i\, U_k}{\partial x_k} = -\frac{\partial}{\partial x_k}\,[\varrho\, \overline{c_i\, c_k} + \varrho\, \overline{u'_i\, u'_k}]\,. \qquad \text{(IV-2b)}$$

Ideas such as BOUSSINESQ's "exchange coefficient", PRANDTL'S mixing length, etc. make use of the analogy between molecular and turbulent shearing stresses. It should be emphasized, however, that the computation of $\overline{c_i\, c_k}$ from kinetic theory is reasonably simple only in the case of a perfect gas, that is, for weak molecular interaction. The interaction between turbulent elements, on the other hand, is not weak in all interesting cases. Indeed, if an analogy of turbulent mixing with molecular kinetics is drawn at all it has to be drawn with the liquid state rather than with the gaseous state. Furthermore, to consider the fluid incompressible implies that dissipated energy is small compared to the heat content of the fluid. That is, $\overline{c_i\, c_k} \ll U_i\, U_k$ and the Mach number $U^2/c^2 \sim U^2/\overline{c^2}$ is small compared to unity. For the turbulent fluid $\overline{u'_i\, u'_k}$ is not necessarily small compared to $U_i U_k$ and consequently the proper analogy refers to "compressible fluid flow" of this turbulent fluid. Indeed, the triple correlation terms of the turbulent fluctuations correspond to the heat flux tensor in the same way as the double correlation corresponds to the viscous stress tensor. For compressible flow the relation between heat flux and shear can not be neglected, in analogy to the relation between double and triple correlations in some problems in turbulence.

One of the difficulties in dealing statistically with turbulence is evidently the fact that we have to deal with strongly interacting elements. This precludes, in most cases, the use of the general asymptotic laws of probability theory such as the central limit theorem which form the background to many of the results of statistical mechanics, for example, of perfect gases, black body radiation, etc. all of which are problems of independent or nearly independent systems.

Furthermore, the fact that turbulence is a "secondary" structure superimposed upon the molecular one means that we deal with dissipative systems throughout and not with a fluid in thermal equilibrium. The very simplest state conceivable is a stationary one in which energy is fed to the system at the same rate as it is removed from the system and transformed into heat.

3. Some Characteristics of the Equations Mathematical Models of Turbulence

Turbulence is a phenomenon typical for large Reynolds numbers. It is thus important to ask what the general characteristics of the equation are for large Reynolds numbers. The typical phenomenon at large Reynolds numbers is the existence of boundary layers in a general sense, that is, of thin layers where most of the dissipation takes place. In this sense shock waves are included in the boundary layer phenomena. The reason for the existence of boundary layers is the fact that the stress terms in the equation of motion are the terms of highest order. These terms are multiplied by a factor inversely proportional to the Reynolds number. Physically this means the existence of regions of dimension δ, say, within the fluid, in which the dissipation takes place and where δ tends toward zero as some power of ν, the kinematic viscosity, in such a fashion that the dissipated energy remains finite. This implies, for example, in the case of the shock wave $\delta \sim \nu$, for the two-dimensional boundary layer on a solid wall $\delta \sim \nu^{1/2}$, in a plane jet $\delta \sim \nu^{2/3}$, etc. (Similar considerations can be made for temperature layers.) The general tendency of the contribution of the quasi-linear terms in the equation of motion and the higher order stress term thus apparently consists at high Reynolds numbers in the production of narrow zones of intense shear. The narrow shear zones caused by transversal waves of finite amplitude are vortex lines and vortex sheets. The random field of these transversal disturbances is called turbulence. In an incompressible fluid longitudinal waves do not exist. In a compressible fluid both types exist and if the Mach number is sufficiently high both the longitudinal and transversal phenomena, that is, sound and shock waves on one hand and turbulence on the other, are coupled, and energy is transferred between both fields.

While shock waves exist in flow with one space coordinate, turbulence, at least in the case of incompressible flow and in the sense outlined above, is a strictly three-dimensional phenomenon. In two dimensions the balance between the regrouping or stretching of vortex filaments and dissipation is not possible. (Random fields made up of transversal shear waves are, of course, possible in two dimensions but they do not represent turbulence in the proper sense.)

Expressed in terms of a Fourier representation of the velocity field the interplay of the nonlinear term with the stress term appears as a flux of energy from the lower frequencies to higher frequencies and the eventual dissipation at higher frequencies. In the course of time one effect of the nonlinear term consists in increasing the coefficients of the highest frequencies. The nonlinear terms thus contribute indirectly to the dissipation.

BURGERS [38][1]) has extensively studied a mathematical model of turbulence. This model accounts for both the nonlinear term and the dissipation term. It is simplified mostly in the space dimensions since only one and two dimensions are considered. Hence, the model does not apply directly to turbulence. In the one-dimensional case it can be applied to a random arrangement of shock waves rather than to turbulence. It does, however, allow the study of the interplay between inertia and viscous terms. COLE [42], [43] and independently E. HOPF [49] have recently found a general solution of BURGER'S one-dimensional, nonlinear equation which facilitates an analytical discussion of the behavior of the solutions with respect to initial and boundary values.

Recently an electrical model nearly corresponding to BURGER'S one-dimensional equation has been studied experimentally by BETCHOV [37]. BETCHOV feeds noise with a white spectrum and Gaussian distribution into his circuit and observes the resulting output. The probability distribution of the response is skew and the joint probability of the output voltage and its derivative shows a striking resemblance to similar observations of turbulent fluctuations.

4. Local Isotropy

KOLMOGOROFF'S concept of local isotropy [51] can be considered as one of the most important general ideas in turbulence.

The exchange of energy between eddies of various size or the flux of energy through a wave number space represents a cascade process. The energy is supplied essentially in the low wave number region and passed on and dissipated in the region of large wave numbers. The question arises: is it possible to make

[1]) Numbers in brackets refer to the Bibliography, page 423.

an asymptotic statement concerning the form of the energy spectrum of the large wave numbers? This spectrum is due to a large number of random events, and we may hope therefore to be able to establish a general trend. The events are not independent and consequently the central limit theorem, or a similar statement intimately connected with independence, can not apply and it is to be expected that any asymptotic statement made will be much weaker than, for example, the central limit theorem.

KOLMOGOROFF introduces essentially three hypotheses; the first of which much broader than the other two.

(1) The small scale motion is always isotropic. This is the concept of "local isotropy".
(2) The energy spectrum of the small scale motion can not depend upon details of the flow; the energy spectrum $E(k)$ or the space correlation function $\varphi(r)$ will depend upon the dissipated energy per unit volume and time ε and the viscosity ν.
(3) If the Reynolds number of the problem is sufficiently large, the zone of dissipation and the zone of production of turbulent energy will be widely separated in the wave number space. Then there may exist a range of wave numbers which are in a state of local isotropic equilibrium but which are still not in the dissipation region. In this case $E(k)$ will become independent of ν.

(2) and (3) imply a transfer mechanism which acts between neighboring wave number ranges. The contribution of a direct transfer of energy from one wave band into another far removed is assumed negligible.

The simplest consequences of (1), (2) and (3) for the spectrum or correlation function are: due to (1) the pattern is isotropic, and one function suffices to describe correlation or spectrum; dimensional analysis can be applied to $E(k)$. Characteristic length and velocity made up of ν and ε are:

$$l = \nu^{3/4}\, \varepsilon^{-1/4}\,, \tag{IV-3a}$$

$$c = \nu^{1/4}\, \varepsilon^{1/4}\,. \tag{IV-3b}$$

l, c can be interpreted as wave length and phase velocity of shear waves in a viscous fluid. (2) yields:

$$E(k) = \nu^{5/4}\, \varepsilon^{1/4}\, \mathfrak{E}(k\, \nu^{3/4}\, \varepsilon^{-1/4})\,. \tag{IV-4}$$

For the subrange (3) ν has to vanish from this expression and it follows that

$$E(k) = \text{const}\, \varepsilon^{2/3}\, k^{-5/3}\,. \tag{IV-5}$$

This is the famous $k^{-5/3}$ law, first given by KOLMOGOROFF[1]) and independently

[1]) KOLMOGOROFF, ONSAGER, and VON WEIZSÄCKER did not actually use the spectrum concept. They gave a result equivalent to (IV-6) for the correlation function, the result for the spectrum is actually due to HEISENBERG [83].

discovered by ONSAGER [56] and VON WEIZSÄCKER [66]). The corresponding double correlation function has the form

$$\varphi(r) = (1 - \text{const}\,\varepsilon^{-1/3}\,r^{-2/3})\,\overline{u^2}\,. \tag{IV-6}$$

Clearly, (2) implies that all quantities associated with eddies within the local isotropic zone are expressible in terms of (IV-3) and that they are similar.

KOLMOGOROFF'S results have an asymptotic nature. Hence for a comparison of, say, (IV-5) or (IV-6) with experiments the range of Reynolds numbers in which the asymptotic laws can be applied has to be established. Only after this has been done can the adequacy of the general ideals underlying the hypothesis be checked with experimental results. The status of an experimental verification appears to be as follows:

(i) *Local Isotropy:* The existence of local isotropy in shear flow appears rather well established. The most convincing results are the measurements of CORRSIN [46] and of LAUFER [52] in a jet and channel respectively. CORRSIN and LAUFER show that the contribution of small scale eddies to the apparent shear vanishes faster than their energy. Extensive results concerning the dissipation, skewness, etc. are due to TOWNSEND [61], [62].

(ii) *Similarity in the Viscous Region:* Results of STEWART and TOWNSEND [99] in isotropic turbulence show that the high frequency components of the spectrum are similar in the sense of (IV-4).

(iii) *Nonviscous Subrange:* Measurements of the spectrum or correlation function in isotropic turbulence behind grids have been carried out to establish whether or not a subrange of the form (IV-5) or (IV-6) exists within the range of wind tunnel experiments. At the Reynolds numbers obtained so far in wind tunnels the results show that no extensive subrange of this form exists [61], [91], [99]. This has been most clearly demonstrated by STEWART and TOWNSEND, who estimate that the Reynolds number based on the grid mesh required to obtain a nonviscous subrange should be of the order of three million or roughly ten times larger than hitherto investigated.

Hence, wind tunnel measurements have not yet lead to a decision concerning KOLMOGOROFF'S nonviscous subrange and the form of the corresponding spectrum and correlation function.

Recent investigations by MACCREADY [54] of the spectrum of atmospheric turbulence near the ground give support to the $k^{-5/3}$ law. MACCREADY'S measurements indicate that local similarity exists for wave lengths up to 160 cm. Reasonable agreement with the $k^{-5/3}$ law or the corresponding $r^{-2/3}$ law was found for even larger eddy dimensions up to an order of a hundred meters. While the results of MACCREADY are very suggestive, the accuracy of measurement for atmospheric turbulence is naturally less than for wind tunnel investigations and hence MACCREADY'S result can not yet be taken as entirely conclusive evidence.

Some astrophysical results concerning the existence of the $k^{-5/3}$ law have been given by VON WEIZSÄCKER [67] and VON HORNER [48], but these results are rather inconclusive due to the difficulty of proper interpretation of the observations.

The concept of local isotropy by its very nature does not contribute directly to an understanding of the transport phenomena. The Kolmogoroff region contains little energy but, at high Reynolds numbers the whole dissipation should occur within the region and hence the dissipation terms are identical with the simple expressions for isotropic turbulence. However, there are physical situations in which one is mainly interested in the small scale motion even if it does contain comparatively little of the total turbulent energy. Typical examples are the gust problem for the motion of an airplane in atmospheric turbulence, scattering of waves due to turbulence, etc. For problems of this type where the general turbulent motion is either unknown or not even interesting, a general asymptotic statement is of paramount importance.

BATCHELOR'S work on axisymmetrical and general nonisotropic but homogeneous turbulence should be mentioned here [36]. BATCHELOR shows that the pressure-velocity correlation terms in the equation transfer energy at one wave number from one velocity component to another and thus contribute a trend toward isotropy. In isotropic turbulence these terms vanish.

5. Turbulent Flows with Secondary Structure

In recent years the importance of the existence of a secondary, large scale structure in turbulent shear flow has become apparent. CORRSIN [44] and TOWNSEND [60] found that the flow near the outer edge of a jet or wake is only intermittently turbulent. TOWNSEND has shown the importance of this motion, which is of a scale comparable to the width of the wake, upon the momentum and heat transfer in the wake. Recent experiments by TOWNSEND [65] and KLEBANOFF and DIEHL [50] demonstrated the same intermittency near the edge of a boundary layer. Photographs of the wake of high-speed projectiles show that the turbulent wake has a large scale structure superimposed upon the fine scale turbulence quite similar to the wake structure at low speeds.

Intermittency and thus the existence of elements of a very large scale seem to be typical for turbulent flows with free boundaries.

While the large eddies ordinarily found in intermittently turbulent flow appear statistically distributed, there do exist a number of cases in which a regular or nearly regular motion of large scale superimposed upon turbulent flow has been observed. PAI [57] and MACPHAIL [55] found that the three-dimensional vortices which initiate the laminar instability in the flow between rotating cylinders (TAYLOR [100]) persist even if the flow has become fully turbulent. In recent measurements on a vortex street behind a cylinder ROSHKO

[58] found a similar result: above a cylinder Reynolds number of about 150 the wake flow is essentially turbulent with superimposed, nearly equally spaced vortices. It is very likely that similar patterns exist in other situations, for example, in zones of large thermal instability.

The importance of these large eddies lies in their bulk transfer of turbulent fluid and their consequent influence on transfer properties and on the energy balance. Due to the size of these elements, which is comparable to the physical dimension of the flow, for example, the width of the wake, it is not possible to account for their behavior in terms of local quantities. For the case of the wake TOWNSEND [63] has discussed the possibility of a quasi-equilibrium taking into account this secondary structure. While there may be doubt about the details of this secondary structure there is no doubt that the large scale motion can not be ignored for a great many problems including problems in sound production from jets, combustion, etc.

If we consider turbulence as a sort of secondary structure superimposed upon the flow of an ordinary viscous fluid we can look at the large scale motion again as a superstructure on the turbulent motion. The interesting and significant feature of this hierarchy is the fact that the length scales are strictly separated. This feature is probably intimately connected with the form of the equation of motion and again related to the boundary layer phenomenon.

V. ISOTROPIC TURBULENCE

In dealing with turbulence in nonuniform mean flow we are faced with a typical transport problem. The intensity of the turbulence varies in space and possibly in time and the gradient of the mean velocity introduces a preferred direction into the problem. We have to deal with a nonhomogeneous and nonisotropic state of turbulence. The most fruitful results in turbulence resulted from G. I. TAYLOR'S introduction of the concept of isotropic and homogeneous fields of turbulence. Consider, for example, a box filled with a fluid. At the time $t = 0$ the fluid is stirred up in a suitable way, say, by dropping a grid through the box. The turbulence created in this fashion can be assumed to become isotropic and homogeneous in space, but it decays, so that the field is not stationary. It is possible, of course, to conceive a setup such that energy is continuously fed into the system by the stirring motion and removed again in the form of heat created by viscous dissipation. In considering this latter problem both the similarity and the essential difference between a box "filled with isotropic, homogeneous turbulence" and a gas or radiation in thermal equilibrium become evident. The similarity consists in the lack of over-all transport of momentum, heat, etc.; only the interactions between the elements making up the fluid are important. The difference is that in the turbulent system there is an energy flux which must be maintained by an external agent, that is, the system is dissipative. To be sure, we may well question the possibility of such a setup. For example, the initial motion set up by a stirring mechanism can not be isotropic and homogeneous. Hence, we assume a trend toward isotropy rapid enough that between the production of turbulence and the dissipation into heat there exists a sufficiently long time to set up an isotropic homogeneous state. The reasoning here is evidently based upon ideas similar to those used in the concept of local isotropy. It may be expected that an isotropic state is reached more rapidly the more uniform the stirring mechanism.

Granted that a state of isotropic and homogeneous turbulence is possible, we can set up the kinematics of the field and the equations interrelating the mean values. The general scheme to introduce correlation tensors and to set up the equations from there is due to VON KÁRMÁN and HOWARTH [85]. In the following we will restrict the discussion to the incompressible case.

1. Correlation Functions

The field of isotropic, homogeneous turbulence is described by the two-point correlation tensors. Take two points $P_1(\vec{R})$ and $P_2(\vec{R}+\vec{r})$ in the field and form the tensors

Pressure Correlation

$$\pi(\vec{r}) = \overline{p(\vec{R})\, p(\vec{R}+\vec{r})} \equiv \overline{p\, p'}; \qquad \text{(V-1a)}$$

Velocity, Double Correlation

$$R_{ij}(\vec{r}) = \overline{u_i(\vec{R})\, u_j(\vec{R}+\vec{r})} \equiv \overline{u_i\, u_j'}; \qquad \text{(V-1b)}$$

Velocity, Triple Correlation

$$T_{ijk}(\vec{r}) = \overline{u_i(\vec{R})\, u_j(\vec{R})\, u_k(\vec{R}+\vec{r})} \equiv \overline{u_i\, u_j\, u_k'}\,. \qquad \text{(V-1c)}$$

For isotropic and homogeneous fields

$$\left.\begin{aligned} \pi = \pi(r)\,, \quad R_{ik} &= R_1(r)\, r_i\, r_k + R_2(r)\, \delta_{ik}\,, \\ T_{ijk} = T_1(r)\, r_i\, r_j\, r_k + T_2(r)\, r_i\, \delta_{jk} &+ T_3(r)\, r_j\, \delta_{ik} + T_4(r)\, r_k\, \delta_{ij}\,, \quad \text{etc.} \end{aligned}\right\} \qquad \text{(V-2)}^{1)}$$

For an incompressible fluid the continuity equation gives

$$\frac{\partial u_j}{\partial x_j} = 0$$

or

$$\overline{p(\vec{R})\, u_j(\vec{R}+\vec{r})} = 0; \quad \frac{\partial R_{ij}}{\partial r_j} = 0; \quad \frac{\partial T_{ijk}}{\partial r_k} = 0\,. \qquad \text{(V-3)}$$

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1) $f(r)$, $h(r)$, etc. is sometimes written for $f(r, t)$, $h(r, t)$, etc. Similarily f', h', etc. denote derivatives with respect to r.

Hence R_{ij}, T_{ijk} can each be expressed by a single scalar function of r. Following the Kármán-Howarth notation we use the following functions:

$$\left.\begin{aligned} R_{11}\{r, 0, 0\} = f(r); \quad R_{11}\{0, r, 0\} = g(r)\,, \\ T_{111}\{r, 0, 0\} = k(r); \quad T_{112}\{0, r, 0\} = h(r); \quad T_{212}\{r, 0, 0\} = q(r)\,. \end{aligned}\right\} \qquad \text{(V-4)}$$

From continuity

$$g = f + \frac{r}{2} f'; \quad k = 2\,h; \quad q = -\left(h + \frac{r}{2} h'\right). \qquad \text{(V-5)}$$

It is also useful to introduce the trace of R_{ik}

$$R(r) = R_{ii} = f + 2\,g \qquad \text{(V-6)}$$

and

$$T_{iji} = T(r)\,\frac{r_j}{r}; \quad T(r) = \frac{1}{r^3} \cdot \frac{\partial r^4\,k}{\partial r}\,. \qquad \text{(V-7)}$$

Furthermore, the products of the derivatives of the velocity components can be expressed as derivatives of the correlation tensors. This is shown in exactly the same way as in the case of one-dimensional stochastic processes (II-11)

$$\left.\begin{aligned} -\left(\overline{\frac{\partial u_i}{\partial x_l} \cdot \frac{\partial u_j}{\partial z_m}}\right) &= \left(\frac{\partial^2 R_{ij}}{\partial r_l\,\partial r_m}\right)_{r=0} \\ &= f''(0)\left[2\,\delta_{lm}\,\delta_{ij} - \frac{1}{2}\,(\delta_{il}\,\delta_{jm} + \delta_{im}\,\delta_{jl})\right] \end{aligned}\right\} \qquad \text{(V-8)}$$

which leads to the expression for the dissipation function Φ:

$$\Phi = \overline{\tau_{ij}\,\frac{\partial u_i}{\partial x_j}} = -15\,\mu\,f''(0) = -5\,\mu\,R''(0)\,. \qquad \text{(V-9)}$$

The mean square vorticity $\overline{\Omega^2}$ is also easily found from (V-8)

$$\overline{\Omega^2} = -15\,\mu\,f''(0) = \frac{\Phi}{\mu}\,. \qquad \text{(V-10)}$$

2. Spectrum Functions

Using the same formalism as in Section II we can introduce the Fourier transforms of the correlations functions, the spectra. This was first done by TAYLOR [103] for the so-called "one-dimensional spectrum", that is, the transform of $f(r)$. HEISENBERG [83] introduced the three-dimensional spectrum and KAMPE DE FERIET [15] and BATCHELOR [36] made use of a general spectrum tensor Φ_{ij} obtained by a Fourier transformation of R_{ij}. Thus

$$\Phi_{ij}(\vec{k}) = \frac{1}{8\,\pi^3} \int d\vec{r}\; R_{ij}(\vec{r})\; e^{-i\vec{k}\vec{r}} \qquad \text{(V-11)}$$

where $\vec{k}$ denotes the wave number vector. For isotropic turbulence, Φ_{ij} must be of the form

$$\Phi_{ij}(\vec{k}) = \Phi_1(k)\, k_i\, k_j + \Phi_2(k)\, \delta_{ij}\,. \tag{V-12}$$

Continuity for an incompressible fluid means the existence of transversal waves only, hence it follows that

$$\Phi_{ij}\, k_j = (\Phi_1\, k^2 + \Phi_2)\, k_i = 0$$

and thus

$$\Phi_{ij} = \Phi_1\, k^2 \left[\frac{k_i\, k_j}{k^2} - \delta_{ij}\right] \tag{V-13}$$

from which HEISENBERG's spectrum $E(k)$ is obtained by

$$E(k) = 2\,\pi\, k^2\, \Phi_{ii} = -8\,\pi\, k^4\, \Phi_1\,. \tag{V-14}$$

Thus

$$\Phi_{ij}(\vec{k}) = \frac{E(k)}{8\,\pi\, k^2} \left[\delta_{ij} - \frac{k_i\, k_j}{k^2}\right]. \tag{V-15}$$

$F(k_1)$, TAYLOR's one-dimensional spectrum, results from this by noting from (V-14), (V-11) and (V-7) that

$$E(k) = \frac{1}{\pi} \int_0^\infty R(r)\, k\, r \sin k\, r\, dr \tag{V-16}$$

thus

$$E(k) = \frac{1}{2}\, [k_1^2\, F''(k_1) - k_1\, F'(k_1)]_{k_1 = k}\,. \tag{V-17}$$

The relation between TAYLOR's and HEISENBERG's form of the spectrum is quite analogous to the relation between, say, the Maxwell distribution function for one component of the velocity $\varphi(u_i)$ and the distribution function for the absolute velocity $\psi(c)$. The latter sorts the molecules with respect to their kinetic energies while the former sorts only according to the contribution of one component to the kinetic energy. Thus $\varphi(o) \neq 0$ while $\psi(o) = 0$ and similarly $F(o) \neq 0$ while $E(o) = 0$. $\psi(c)$ and $E(k)$ have more direct physical significance, however $\varphi(u_i)$ and $F(k_1)$ are more easily accessible to measurement.

3. The Equations of Motion

The Kármán-Howarth equation establishes the relation between R_{ij} and T_{ijk} and their derivatives. The equation can be written in the form

$$\frac{\partial f}{\partial t} + \frac{2}{r^4} \cdot \frac{\partial h\, r^4}{\partial r} = \frac{2\,\nu}{r^4} \cdot \frac{\partial}{\partial r}\left(r^4\, \frac{\partial f}{\partial r}\right), \tag{V-18}$$

where $f(r, t)$ and $h(r, t)$ are the correlation functions defined in (V-5). Another useful form is

$$\frac{\partial R}{\partial t} - \frac{2}{r^2} \cdot \frac{\partial}{\partial r} (r^2 T) = \frac{2 \nu}{r^2} \cdot \frac{\partial}{\partial r} (r^2 R), \qquad \text{(V-19)}$$

where $R(r, t)$ and $T(r, t)$ have been previously defined. Multiplying by $k r \sin k r$ and integrating over all r we have, with (V-16) for the function $E(k, t)$

$$\frac{\partial E}{\partial t} - \frac{2}{\pi} k^2 \int (T r^2)' \frac{\sin k r}{k r} dr = -2 \nu E k^2. \qquad \text{(V-20)}$$

4. Invariants

From (V-18) and (V-19) follows an invariant given first by LOITSIANSKII [93]. Namely, multiplying the equations by r^4 and r^2 respectively and integrating over all space yields

$$\frac{\partial}{\partial t} \int r^4 f(r, t) \, dr = 0, \qquad \text{(V-21)}$$

$$\frac{\partial}{\partial t} \int r^2 R(r, t) \, dr = 0 \qquad \text{(V-22)}$$

provided that f, r, k, T vanish sufficiently fast for large values of r. Furthermore, since $R = (r^3 f)'/r^2$

$$\int r^2 R(r, t) \, dr = 0.$$

The integral in (V-21)

$$\int f(r) \, r^4 \, dr = \Lambda \qquad \text{(V-23)}$$

is called LOITSIANSKII's invariant. The existence of such an invariant is a simple consequence of the equation of motion. The continuity, momentum and energy equations in fluid dynamics each have the form of a continuity equation relating the time derivative of a function to a divergence term. The divergence terms can be written in the form

$$\frac{\partial}{\partial r_k} [Q(r) \, r_k] = \frac{1}{r^2} (Q r^3)', \quad \frac{\partial}{\partial r_k} [Q(r) \, r_i \, r_k] = \frac{r_i}{r^3} (Q r^4)', \text{ etc.}$$

and it becomes evident that for homogeneous fields invariants will exist. For the velocity field this is LOITSIANSKII's invariant, for the temperature field in isotropic turbulence CORRSIN [79] has given the appropriate invariant

$$\frac{\partial}{\partial t} \int r^2 \, \overline{\vartheta \, \vartheta'} \, dr = 0, \qquad \text{(V-24)}$$

where ϑ is the temperature fluctuation. For a compressible fluid a similar result holds for the density fluctuation s:

$$\frac{\partial}{\partial t} \int r^2 \overline{s\, s'}\, dr = 0 \qquad \text{(V-25)}$$

as given by CHANDRASEKHAR [77].

The consequence of the existence of the invariant Λ on the spectrum $E(k)$ follows directly from (V-16) and (V-20). Λ, being a fourth moment of f, is essentially determined by the largest elements of turbulence and hence it must determine $E(k)$ in the neighborhood of $k = 0$. Developing (V-16) and (V-20) into powers of k yields

$$E(k) = \frac{2\, k^4\, \Lambda}{3!} + \cdots \qquad \text{(V-26)}$$

since

$$R(r) = \frac{1}{r^2}\,(r^3 f)'.$$

Similarily, remembering that $T(r) = -(r^4 K)'/r^2$ (V-20) yields

$$\frac{\partial E}{\partial t} = \frac{k^6}{16} \int r^5\, T(r, t)\, dr - 2\, \nu\, k^2\, E \quad \text{(for small } k\text{)}$$

consistent with the statement $\partial \Lambda / \partial t = 0$.

Consequently, the spectrum function E is proportional to k^4 near the origin and the coefficient of k^4 is time independent. This result was found by LIN [92] and BATCHELOR [36][1]).

We again draw an analogy to the Maxwell distribution of velocities or the Plank distribution of radiation. In both cases we can also obviously express the distribution functions in such a form that their behavior near the origin remains invariant as the gas or the cavity cools down with time. However, if this is done the whole distribution remains similar, which is typical for a linear system and quite different from the turbulent case in general.

5. The General Problems in Isotropic Turbulence

The discussion of invariants and of the behavior of $E(k)$ for small k, the k^4 law, essentially exhausts the results which can be obtained from the Kármán-Howarth and related equations without the introduction of additional principles or assumptions. The general problem posed to a theory of isotropic turbulence is the determination of the complete functions $E(k, t)$, $f(r, t)$ or $\Phi_{ij}(\vec{k}, t)$, etc. It is immediately questionable whether such a broad problem makes sense. The necessity of a stirring mechanism has to be considered somehow and it is by no means evident that turbulence created in some manner,

[1]) BATCHELOR shows that the k^4 law requires homogeneity only. The coefficient of k^4 is in this case still invariant in time but is a tensor term and not a simple scalar like in the isotropic case.

for example, in a box, will tend toward a unique function $E(k, t)$. Indeed, the existence of the Loitsianskii invariant does establish a connection with the stirring mechanism for all times. For example, the anisotropy introduced by the stirring mechanism can be recovered in the latest stages of decay, as shown by BATCHELOR und STEWART [70]. Hence, while it may be futile to ask for the complete spectrum or correlation function without a complete description of the stirring mechanism, it makes sense to ask for any possible general form the spectrum may have in wave lengths far removed from the ones excited by the stirring mechanism and for times long with respect to a relaxation time. It may be sufficient in these cases to describe the stirring mechanism by a characteristic length only, for example, the mesh size in the case of a grid.

The approaches taken to a solution of the problem can be classified into groups.

(*a*) Similarity considerations and dimensional analysis.

(*b*) Attempts to express the triple correlation or the corresponding spectral transfer integrals in terms of the double correlation or the spectrum $E(k)$.

(*c*) Discussion of limiting cases.

6. The Final Stage of Decay

The difficulty in discussing the Kármán-Howarth or related equations arises naturally from the transport term. One may thus first consider a case where this term is negligible, as has already been done by VON KÁRMÁN and HOWARTH [85] and MILLIONSHCHIKOV [94] and more completely by BATCHELOR and TOWNSEND [72]. If the turbulence has decayed far enough for the transport term to be negligible (V-18) or (V-20) may be integrated. The appropriate solutions are

$$E(k, t) = \text{const}\, k^4\, e^{-2\nu k^2 t} \qquad \text{(V-27)}$$

$$f(r, t) = (8\pi\nu t)^{-5/2} \int f\{\vec{r} - \vec{s}, t_0\}\, e^{-s^2/8\nu t}\, s^4\, d\vec{s} \qquad \text{(V-28)}$$

and thus

$$\overline{u^2} = \frac{\Lambda}{48\sqrt{2\pi}} (\nu t)^{-5/2}. \qquad (\text{as } t \to \infty) \qquad \text{(V-29)}$$

(V-27) and (V-29) are in good agreement with experiments [72], [91].

7. Self-Preservation

In the Kolmogoroff region the spectrum and correlation functions depend upon time during decay only indirectly through the dissipation ε. For the larger part of the energy spectrum that contains much of the turbulent energy, time has to enter explicitly. Forms of the spectrum or correlation function which depend upon time but preserve their shape are called self-preserving. VON KÁRMÁN and LIN [87] and STEWART and TOWNSEND [99] have recently discussed problems of self-preservation in detail. HEISENBERG [82] used the concept of a quasi-equilibrium using his form of the exchange term (see below). VON KÁRMÁN and LIN put $E(k, t) = C\, k^4\, \mathfrak{E}(k/k_s)$ where k_s is a function of t. This expression leads to a decay law of the form

$$\overline{u^2} \approx t^{-10/7}\,. \qquad \text{(V-30)}$$

HEISENBERG puts $E(k, t) = E_0(t)\, \mathfrak{E}(k/k_s)$ consistent with an initial decay law of the form

$$\overline{u^2} \approx t^{-1}. \qquad \text{(V-31)}$$

While VON KÁRMÁN and LIN'S expression for E satisfies the proper condition $E \sim k^4$ for $k \to 0$, HEISENBERG'S expression yields $E \sim k$ for small k. The $t^{-10/7}$ law proposed also by KOLMOGOROFF [88] and FRENKIEL [80] has not much experimental support while the t^{-1} law first discussed by BATCHELOR and TOWNSEND appears to fit the experimental results in the initial stage of decay remarkably well.

For a detailed discussion reference is made to the papers by VON KÁRMÁN and LIN, and by STEWARD and TOWNSEND.

8. Form of the Exchange Term

The spectral equation (V-20) integrated from 0 to k reads

$$\frac{\partial}{\partial t} \int_0^k E(k, t)\, dk = \frac{2}{\pi} \int_0^k \int_0^\infty (T\, r^2)'\, k\, \frac{\sin k\, r}{k\, r}\, dr\, dk - 2\, \nu \int_0^k k^2\, E(k, t)\, dk\,. \qquad \text{(V-32)}$$

The time rate of change of the energy contained in the range of wave numbers up to k is due to exchange with elements outside the range, as expressed in the second term, and to viscous dissipation. To make the equation determinate the exchange term has to be expressed in terms of $E(k, t)$ and k. None of the forms of the exchange terms proposed so far have been completely successful. HEISENBERG'S form, or a more general similar expression due to VON KÁRMÁN, seems to have been the most succesful so far. HEISENBERG interprets nonviscous loss of energy from the range of waves numbers 0 to k as due to a turbulent viscosity in the sense of BOUSSINESQ'S exchange coefficient. Using dimensional reasoning he then puts

$$\frac{2}{\pi} \int_0^k \int_0^\infty (T\, r^2)'\, k\, \frac{\sin k\, r}{k\, r}\, dr\, dk = \text{const} \int_k^\infty \left(\frac{E}{k^3}\right)^{1/2} dk \int_0^k k^2\, E\, dk\,. \qquad \text{(V-33)}$$

This form of the exchange term yields the $k^{-5/3}$ law in the nonviscous subrange of KOLMOGOROFF, as any dimensionally correct forms do, but leads to a law of the form k^{-7} in the viscous subrange of KOLMOGOROFF. This latter result is at variance with experiments ([106] and [91]) and also *a priori* unlikely since any power law $E(k) \sim k^{-n}$ for large k leads to infinite moments of E of order $n - 1$. This in turn means that the velocity components have no mean derivative of order $(n - 1)/2$, which is not likely. PROUDMAN has recently discussed the consequences of HEISENBERG's assumptions [96].

The proposed forms of OBOUKHOV [95] and KOVASZNAY [89] are less successful than HEISENBERG's.

9. Temperature Fluctuations in Isotropic Turbulence

CORRSIN [79] has studied the case of temperature fluctuations in a field of isotropic, homogeneous turbulence of an incompressible fluid. In this case he obtains an equation similar to the Kármán-Howarth equation obtained for the temperature correlation function

$$\overline{\vartheta(\vec{R})\, \vartheta(\vec{R} + \vec{r})} = \theta(r) \tag{V-34}$$

and the temperature velocity correlations

$$\overline{\vartheta(\vec{R})\, \vartheta(\vec{R} + \vec{r})\, u_i(\vec{R} + r)} = \theta_i(r)\ . \tag{V-35}$$

Similarily equations for the spectrum can be written. The difference due to the scalar character of ϑ as compared to the vector character of u_i becomes apparent in the form of the spectrum for small k. The temperature spectrum begins with k^2 corresponding to the invariance of

$$\int \overline{\vartheta\, \vartheta'}\, r^2\, dr \tag{V-36}$$

during decay. The discussion of the final state of decay, the subranges, etc. is similar to that for the velocity field, as shown in CORRSIN's paper.

10. Pressure Fluctuations in Isotropic Turbulence

The pressure of an incompressible fluid p satisfies the equation

$$\nabla^2 p = -\varrho\, \frac{\partial^2 u_i u_j}{\partial x_i\, \partial x_j} \equiv f\,. \tag{V-37}$$

(V-37) is a very simple form of the general type of (III-6). Thus we immediately obtain from (III-11) the relation between the spectra of the pressure and the second velocity derivatives. Denote by $\Pi(k)$ the pressure spectrum and by $F(k)$

the spectrum of f in (V-37). Then

$$\Pi(k) = \frac{1}{k^4} F(k) \tag{V-38}$$

since $\Gamma(k) = k^{-2}$. (V-38) has been given by BATCHELOR [75]. BATCHELOR introduces the spectrum tensor $\Phi_{ijlm}(\vec{k})$ corresponding to the quadruple correlation[1])

$$\varphi_{ijlm}(r) = \overline{u_i(\vec{R})\, u_j(\vec{R})\, u_l(\vec{R}+\vec{r})\, u_m(\vec{R}+\vec{r})} - \overline{u_i(\vec{R})\, u_j(\vec{R})} \cdot \overline{u_l(\vec{R}+\vec{r})\, u_m(\vec{R}+\vec{r})},$$

(V-38) then becomes

$$k^4\, \Pi(k) = k_i\, k_j\, k_l\, k_m\, \Phi_{ijlm}(\vec{k}) \tag{V-39}$$

and

$$\overline{p^2} = 4\,\pi\,\varrho^2 \int k^2\, \Pi(k)\, dk\,, \tag{V-40a}$$

$$\overline{(\text{grad}\, p)^2} = 4\,\pi\,\varrho^2 \int k^4\, \Pi(k)\, dk\,. \tag{V-40b}$$

However, to evaluate φ_{ijlm} or Φ_{ijlm} it is necessary to introduce further assumptions. BATCHELOR and also earlier HEISENBERG [83] assume the relation between the quadruple and double correlations to be the same as the one resulting from a normal joint probability of $\vec{u}(\vec{R})$ and $\vec{u}(\vec{R}+\vec{r})$. With this assumption $\Pi(k)$ can be related to the energy spectrum function $E(k)$ of isotropic turbulence and approximately evaluated.

To relate the statistical properties of the substantial acceleration in hydrodynamics to the derivatives at a fixed position we must know the properties of the pressure gradient. A study of the pressure field is thus very important for the relation between the Eulerian and Lagrangian correlations and thus for the diffusion problem in turbulence. Besides BATCHELOR'S paper [75] reference is made to recent work of COLLIS [78] and UBEROI and CORRSIN [107].

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Zusammenfassung

Die Arbeit gibt einen Überblick über einige Probleme der neueren Turbulenzforschung. Nach einer allgemeinen Einleitung werden zuerst elementare, für die Turbulenztheorie wichtige Resultate der Theorie homogener, stochastischer Vorgänge diskutiert. Im dritten Teil werden sodann Beispiele linearer Systeme behandelt, bei denen die äusseren Kräfte von turbulenten Schwankungen herrühren. Im vierten Teil werden turbulente Transportphänomene gestreift, und im fünften Teil wird ein kurzer Überblick über Resultate und Fragen der Theorie der isotropen Turbulenz gegeben.

(Received: July 7, 1952.)