ON THE ASYMPTOTIC DISTRIBUTION OF THE SUM OF A RANDOM NUMBER OF INDEPENDENT RANDOM VARIABLES

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Introduction

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ denote a sequence of independent random variables and put

$$\zeta_n = \xi_1 + \xi_2 + \cdots + \xi_n \qquad (n = 1, 2, \ldots). \tag{1}$$

Several authors (see e. g. [1] and [2]) investigated the asymptotic distribution of $\zeta_{\nu(t)}$ for $t \to +\infty$ where $\nu(t)$ is a positive integer-valued random variable, for $t > 0$, which converges in probability to $+\infty$ for $t \to +\infty$. The most general results in this direction have been obtained by DOBRUŠIN [3]. In all these investigations it has been supposed that $\nu(t)$ is for any $t > 0$ independent of the random variables ζ_n $(n = 1, 2, \ldots)$. A general and very useful theorem without this supposition has been proved by F. J. ANSCOMBE [4]. In a recent paper [5] TAKÁCS has proved a theorem, which can be considered also as a result on the asymptotic distribution of the sum of a random number of independent random variables, i. e. using the above notations on the asymptotic distribution of $\zeta_{\nu(t)}$ where ζ_n is defined by (1). In this case $\nu(t)$ depends essentially on the variables ζ_n $(n = 1, 2, \ldots)$. The aim of the present paper is to show that the mentioned result of TAKÁCS can be easily deduced from a special case of the theorem of ANSCOMBE mentioned above. To make the paper self-contained, we give in § 1 a short proof of the special case of ANSCOMBE's theorem which is needed for our purpose (Theorem 1). Using this theorem, in § 2 a new and simple proof of the result of TAKÁCS mentioned above is given.

§ 1. A theorem of Anscombe

THEOREM 1 (ANSCOMBE). *Let us suppose that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are independent and identically distributed random variables with mean value* 0 *and variance* 1. *Let us put $\zeta_n = \xi_1 + \xi_2 + \cdots + \xi_n$. Let further $\nu(t)$ denote a positive integer-valued random variable for any $t > 0$ such that $\frac{\nu(t)}{t}$ converges for $t \to +\infty$*

in probability to a constant $c>0$. *Then we have*[1]

$$\lim_{t\to\infty}\mathbf{P}\left(\frac{\zeta_{\nu(t)}}{\sqrt{\nu(t)}}<x\right)=\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{u^2}{2}}du. \tag{1.1}$$

PROOF OF THEOREM 1. Let $0<\varepsilon<\frac{1}{5}$ be arbitrary. First we choose a value $t_1>0$ such that for $t\geqq t_1$ we have

$$\mathbf{P}(|\nu(t)-ct|\geqq c\varepsilon t)\leqq\varepsilon. \tag{1.2}$$

Clearly

$$\mathbf{P}\left(\frac{\zeta_{\nu(t)}}{\sqrt{\nu(t)}}<x\right)=\sum_{n=1}^{\infty}\mathbf{P}\left(\frac{\zeta_n}{\sqrt{n}}<x,\ \nu(t)=n\right). \tag{1.3}$$

It follows from (1.2) and (1.3) that for $t\geqq t_1$

$$\left|\mathbf{P}\left(\frac{\zeta_{\nu(t)}}{\sqrt{\nu(t)}}<x\right)-\sum_{|n-ct|<\varepsilon ct}\mathbf{P}\left(\frac{\zeta_n}{\sqrt{n}}<x,\ \nu(t)=n\right)\right|\leqq\varepsilon. \tag{1.4}$$

Now let us put[2] $N_1=[c(1-\varepsilon)t]$ and $N_2=[c(1+\varepsilon)t]$. Then we have for $|n-ct|\leqq\varepsilon ct$

$$\mathbf{P}\left(\frac{\zeta_n}{\sqrt{n}}<x,\ \nu(t)=n\right)\leqq\mathbf{P}(\zeta_{N_1}<x\sqrt{N_2}+\varrho,\ \nu(t)=n) \tag{1.5}$$

where

$$\varrho=\max_{N_1<n\leqq N_2}\left|\sum_{N_1<k\leqq n}\xi_k\right|.$$

Similarly we obtain

$$\mathbf{P}\left(\frac{\zeta_n}{\sqrt{n}}<x,\ \nu(t)=n\right)\geqq\mathbf{P}(\zeta_{N_1}<x\sqrt{N_1}-\varrho,\ \nu(t)=n). \tag{1.6}$$

According to a well-known inequality due to A. N. KOLMOGOROV [6], we have

$$\mathbf{P}(\varrho\geqq\sqrt[3]{\varepsilon}\sqrt{N_1})\leqq\frac{(N_2-N_1)}{N_1\varepsilon^{2/3}}\leqq5\sqrt[3]{\varepsilon}\quad\text{if}\quad t\geqq\frac{1}{c\varepsilon}. \tag{1.7}$$

Let us denote by R the event $\varrho<\sqrt[3]{\varepsilon}\sqrt{N_1}$ and by E the event $|n-ct|<c\varepsilon t$.

Taking (1.4), (1.5), (1.6), (1.7) into account, it follows

$$\mathbf{P}\left(\frac{\zeta_{\nu(t)}}{\sqrt{\nu(t)}}<x\right)\leqq\mathbf{P}\left(\frac{\zeta_{N_1}}{\sqrt{N_1}}<x\sqrt{\frac{N_2}{N_1}}+\sqrt[3]{\varepsilon},\ RE\right)+6\sqrt[3]{\varepsilon} \tag{1.8}$$

[1] We denote by $\mathbf{P}(\ldots)$ the probability of the event in the brackets.

[2] We denote by $[\ldots]$ the integral part of the number in the square brackets.

and

$$\mathbf{P}\left(\frac{\zeta_{\nu(t)}}{\sqrt{\nu(t)}} < x\right) \geqq \mathbf{P}\left(\frac{\zeta_{N_1}}{\sqrt{N_1}} < x - \sqrt[3]{\varepsilon}, RE\right) - \varepsilon. \tag{1.9}$$

It follows that

$$\mathbf{P}\left(\frac{\zeta_{N_1}}{\sqrt{N_1}} < x - \sqrt[3]{\varepsilon}\right) - 7\sqrt[3]{\varepsilon} \leqq \mathbf{P}\left(\frac{\zeta_{\nu(t)}}{\sqrt{\nu(t)}} < x\right) \leqq \mathbf{P}\left(\frac{\zeta_{N_1}}{\sqrt{N_1}} < x\sqrt{\frac{1+2\varepsilon}{1-2\varepsilon}} + \sqrt[3]{\varepsilon}\right) + 6\sqrt[3]{\varepsilon}.$$

By the central limit theorem we have (see e. g. [7], p. 215)

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\zeta_n}{\sqrt{n}} < x\right) = \Phi(x)$$

where $\Phi(x)$ is defined by (1. 1). Thus we obtain, as $\Phi(x)$ is continuous, (1. 1).

§ 2. New proof of a theorem of L. Takács

In his paper [5] TAKÁCS has considered stochastic processes of the following type: $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ are random points on the real axis,

$$\tau_0 = 0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$$

such that putting

$$\tau_{2n+1} - \tau_{2n} = \xi_n \quad \text{and} \quad \tau_{2n+2} - \tau_{2n+1} = \eta_n \qquad (n = 0, 1, \ldots) \tag{2.1}$$

the positive random variables ξ_n, η_n are all independent, the variables ξ_n are all identically distributed with the cumulative distribution function $\mathbf{P}(\xi_n < x) = A(x)$ and the variables η_n are also identically distributed with the cumulative distribution function $\mathbf{P}(\eta_n < x) = B(x)$.

For any positive number $t > 0$ let us put

$$\alpha(t) = \begin{cases} \xi_1 + \xi_2 + \cdots + \xi_n & \text{if} \quad \tau_{2n-1} \leqq t < \tau_{2n} \qquad (n = 1, 2, \ldots), \\ \xi_1 + \xi_2 + \cdots + \xi_n + t - \tau_{2n} & \text{if} \quad \tau_{2n} \leqq t < \tau_{2n+1} \quad (n = 0, 1, \ldots) \end{cases} \tag{2.2}$$

and $\beta(t) = t - \alpha(t)$. By other words, if we interpret t as time, and consider a system which is at time t in state $\mathcal{A}$ if $\tau_{2n} \leqq t < \tau_{2n+1}$ $(n = 0, 1, \ldots)$ and in state $\mathcal{B}$ if $\tau_{2n-1} \leqq t < \tau_{2n}$ $(n = 1, 2, \ldots)$, then $\alpha(t)$ and $\beta(t)$ denotes the total time which the system has spent in state $\mathcal{A}$ and $\mathcal{B}$, respectively, during the time interval $(0, t)$. TAKÁCS investigated the limiting distribution of the random variables $\alpha(t)$ and $\beta(t)$, respectively, for $t \to \infty$, and proved that if the first two moments of the random variables ξ_n and η_n exist, and if we put

$$\alpha = \int_0^\infty x\,dA(x) \quad \text{and} \quad \beta = \int_0^\infty x\,dB(x), \tag{2.3}$$

further

(2. 4) $$\sigma_\alpha^2=\int_0^\infty (x-\alpha)^2 dA(x) \quad \text{and} \quad \sigma_\beta^2=\int_0^\infty (x-\beta)^2 dB(x),$$

and finally

(2. 5) $$a=\frac{\alpha}{\alpha+\beta}, \qquad b=\frac{\beta}{\alpha+\beta} \quad \text{and} \quad D=\sqrt{\frac{\beta^2\sigma_\alpha^2+\alpha^2\sigma_\beta^2}{(\alpha+\beta)^3}},$$

then $\frac{\alpha(t)-at}{D\sqrt{t}}$ and $\frac{\beta(t)-bt}{D\sqrt{t}}$ are asymptotically normal for $t\to+\infty$ with mean value 0 and variance 1.

Using Theorem 1 of § 1 we give a new proof of this fact which is somewhat simpler than that given by TAKÁCS.

Thus we prove the following

THEOREM 2 (TAKÁCS). *If α, β, σ_α^2 and σ_β^2 exist, we have*

(2. 6a) $$\lim_{t\to+\infty} \mathbf{P}\left(\frac{\alpha(t)-at}{D\sqrt{t}}<x\right)=\Phi(x)$$

and

(2. 6b) $$\lim_{t\to+\infty} \mathbf{P}\left(\frac{\beta(t)-bt}{D\sqrt{t}}<x\right)=\Phi(x)$$

for $-\infty<x<+\infty$ *where*

(2. 7) $$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}du.$$

The proof is based besides Theorem 1 on three simple lemmas of which Lemma 1 and Lemma 2 are well known.

LEMMA 1. *If $\chi(t)$, $\varepsilon(t)$ and $\delta(t)$ are random functions* $(0<t<+\infty)$ *and are such that the asymptotic distribution of $\chi(t)$ exists, $\varepsilon(t)$ converges in probability to* 1 *and $\delta(t)$ converges in probability to* 0 *for $t\to+\infty$, then the asymptotic distribution of $\chi(t)\varepsilon(t)+\delta(t)$ exists also for $t\to+\infty$ and coincides with that of $\chi(t)$.*

Lemma 1 is contained in a theorem of H. CRAMÉR ([7], p. 255), and therefore may be omitted.

LEMMA 2. *If $\chi_n^{(1)}$, χ_n and $\chi_n^{(2)}$ are sequences of random variables such that $\chi_n^{(1)}\leqq\chi_n\leqq\chi_n^{(2)}$ and the sequences $\chi_n^{(1)}$ and $\chi_n^{(2)}$ have the same asymptotic distribution for $n\to+\infty$, then χ_n has also the same asymptotic distribution.*

The proof of Lemma 2 is evident and may be left to the reader. (As a matter of fact, Lemma 2 can be deduced also from Lemma 1.)

LEMMA 3. *Let* $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ *denote a sequence of identically distributed random variables, having the distribution function* $F(x)$, *and let us suppose that the second moment* $\int_{-\infty}^{+\infty} x^2 dF(x)$ *of the variables* ξ_n *exists. Let* $\nu(t)$ *denote a positive integer-valued random variable for* $t>0$, *for which* $\frac{\nu(t)}{t}$ *converges in probability to* $c>0$ *for* $t \to +\infty$. *Then* $\frac{\xi_{\nu(t)}}{\sqrt{\nu(t)}}$ *converges in probability to* 0.

PROOF OF LEMMA 3. Let us choose an $\varepsilon>0$. Then we can find to any $\delta>0$ a $t_1>0$ such that for $t \geqq t_1$

$$\mathbf{P}(|\nu(t)-ct| \geqq \varepsilon c) \leqq \delta. \tag{2.8}$$

Put further

$$N_1=[c(1-\varepsilon)t] \quad \text{and} \quad N_2=[c(1+\varepsilon)t]. \tag{2.9}$$

We have evidently

$$\mathbf{P}\left(\frac{|\xi_{\nu(t)}|}{\sqrt{\nu(t)}}>\varepsilon\right) \leqq \delta+\mathbf{P}\left(\frac{\underset{N_1<n\leqq N_2}{\operatorname{Max}}|\xi_n|}{\sqrt{N_1}}>\varepsilon\right) \tag{2.10}$$

and thus

$$\mathbf{P}\left(\frac{|\xi_{\nu(t)}|}{\sqrt{\nu(t)}}>\varepsilon\right) \leqq (N_2-N_1)(1-F(\varepsilon\sqrt{N_1})+F(-\varepsilon\sqrt{N_1}))+\delta. \tag{2.11}$$

As the existence of $\int_{-\infty}^{+\infty} x^2 dF(x)$ implies that

$$\lim_{x\to+\infty} x^2(1-F(x))=\lim_{x\to+\infty} x^2 F(-x)=0$$

and $\frac{N_2-N_1}{N_1}$ is bounded, further δ may be chosen as small as we like, it follows that

$$\lim_{t\to\infty} \mathbf{P}\left(\frac{|\xi_{\nu(t)}|}{\sqrt{\nu(t)}}>\varepsilon\right)=0 \quad \text{for any} \quad \varepsilon>0.$$

Thus Lemma 3 is proved.

Now we are in the position to prove Theorem 2.

Let us put

$$\zeta_n=\xi_n+\eta_n=\tau_{2n+1}-\tau_{2n-1} \qquad (n=1,2,\ldots) \tag{2.12}$$

and let the positive integer-valued random variable $\nu(t)$ be defined for $t>0$ by the inequality

$$\tau_{2\nu(t)-1} \leqq t < \tau_{2\nu(t)+1}. \tag{2.13}$$

Clearly

$$\mathbf{P}(\nu(t) \leqq N) = \mathbf{P}(\zeta_1 + \zeta_2 + \cdots + \zeta_N > t) \qquad (N = 1, 2, \ldots) \tag{2. 14}$$

which implies that

$$\mathbf{P}\left(\frac{\nu(t)}{t} < x\right) = \mathbf{P}\left(\frac{\zeta_1 + \zeta_2 + \cdots + \zeta_{[tx]}}{[tx]} > \frac{t}{[tx]}\right). \tag{2. 15}$$

As the law of large numbers (see e. g. [7], p. 253) clearly applies to the random variables ζ_n, which are independent, identically distributed, and their mean value is $\alpha + \beta$, we have

$$\lim_{n \to \infty} \mathbf{P}\left(\frac{\zeta_1 + \zeta_2 + \cdots + \zeta_n}{n} > y\right) = \begin{cases} 0 & \text{for } y > \alpha + \beta, \\ 1 & \text{for } y < \alpha + \beta. \end{cases} \tag{2. 16}$$

(2. 15) and (2. 16) imply that $\frac{\nu(t)}{t}$ *converges in probability to* $\frac{1}{\alpha + \beta}$ *for* $t \to +\infty$. (This fact is well known (see e. g. [8]); we proved it only for the sake of completeness.) Now let us put

$$\vartheta_k = \frac{\beta \xi_k - \alpha \eta_k}{(\alpha + \beta)^{3/2}} \qquad (k = 1, 2, \ldots). \tag{2. 17}$$

Then we have

$$\sum_{k=1}^{\nu(t)} \vartheta_k - \frac{\xi_{\nu(t)+1}}{\sqrt{\alpha + \beta}} \leqq \frac{\alpha(t) - at}{\sqrt{\alpha + \beta}} \leqq \sum_{k=1}^{\nu(t)} \vartheta_k + \frac{\xi_{\nu(t)+1} + \eta_{\nu(t)}}{\sqrt{\alpha + \beta}}. \tag{2. 18}$$

As the random variables ϑ_k are independent, identically distributed, and have the mean value 0 and the variance D^2, it follows by Theorem 1 that

$$\lim_{t \to \infty} \mathbf{P}\left(\frac{\sum_{k=1}^{\nu(t)} \vartheta_k}{D\sqrt{\nu(t)}} < x\right) = \Phi(x). \tag{2. 19}$$

On the other hand, it follows from Lemma 3 (which can be applied as the random variables ξ_n and η_n have finite variances) that

$$\frac{\xi_{\nu(t)+1}}{\sqrt{\nu(t)}} \quad \text{and} \quad \frac{\xi_{\nu(t)+1} + \eta_{\nu(t)}}{\sqrt{\nu(t)}}$$

are converging in probability to 0. Thus, by Lemma 1, the random variables

$$\frac{\sum_{k=1}^{\nu(t)} \vartheta_k + \frac{\xi_{\nu(t)+1} + \eta_{\nu(t)}}{\sqrt{\alpha + \beta}}}{D\sqrt{\nu(t)}} \quad \text{and} \quad \frac{\sum_{k=1}^{\nu(t)} \vartheta_k - \frac{\xi_{\nu(t)+1}}{\sqrt{\alpha + \beta}}}{D\sqrt{\nu(t)}}$$

are both asymptotically normal for $t \to +\infty$. By virtue of (2. 18) and Lemma 2 we obtain

$$\lim_{t\to\infty} \mathbf{P}\left(\frac{\alpha(t)-at}{D\sqrt{(\alpha+\beta)\nu(t)}}<x\right)=\Phi(x). \tag{2.20}$$

Taking into account that $\frac{\nu(t)(\alpha+\beta)}{t}$ converges in probability to 1 for $t\to+\infty$, and using again Lemma 1, we may replace $(\alpha+\beta)\nu(t)$ by t in (2. 20), what proves (2. 6a). Clearly, (2. 6b) follows from (2. 6a), in view of

$$\beta(t)-bt=at-\alpha(t). \tag{2.21}$$

This completes the proof.

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