IDEALS AND CONGRUENCE RELATIONS IN LATTICES

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Introduction

One of the important tools of the lattice-theoretical researches is the examination of lattice congruences. In connection with lattice congruences arises the necessity of the examination of lattice ideals, for I_Θ — the kernel of the homomorphism induced by the congruence relation Θ — is an ideal (if it is any), and this ideal implicates a lot of properties of Θ.

In this paper our aim is to examine the properties of lattice congruences and the correspondence $\Theta \to I_\Theta$. Our main tools in the discussion are two special types of congruence relations: the minimal congruence relations, induced by a subset of the lattice L, and the separable congruence relations, respectively.

In this paper we deal also with three problems of G. BIRKHOFF [2].[1] We prove a result of J. HASHIMOTO [14] (solving problem 73) in a new and more simple way,[2] and get a new answer to the question raised in problem 72; this has more applications to special cases than the original solution of this problem given by T. TANAKA [18]. We obtained a more general solution of problem 67 than J. JAKUBIK in [15].

The paper consists of four parts. In Part I we deal with congruence relations in distributive lattices. First we prove a theorem that describes the minimal congruence relations in distributive lattices. By the help of this theorem we get a good look at numerous properties of congruence relations in distributive lattices, all of which are able to characterize the distributivity of a lattice. We prove, finally, a theorem which is a far-reaching generalization

[1] Numbers in brackets refer to the Bibliography given at the end of the paper.

[2] In his cited paper J. HASHIMOTO deals with the representations (representation is a homomorphism of a lattice onto a ring of sets) of a lattice, and with topologies which are defined by special representations and inverse representations. Among the applications of these general discussions one can find the solution of problem 73. This explains that if we consider a single theorem independently of the others, then the proof seems to be rather difficult. It has some interest that while J. HASHIMOTO uses the Axiom of Choice during the proof, we succeeded in omitting it.

of many known theorems (which are due to J. HASHIMOTO and M. KOLIBIAR), and contains the solution of G. BIRKHOFF's problem 73 too.

In Part II we discuss with the help of the notion "weak projectivity" (introduced by R. P. DILWORTH [3]) the questions related to congruences in general lattices. After three preliminary lemmas we get an answer to the following question: In which lattices is every congruence relation Θ completely determined by the ideal I_Θ consisting of all x with $x \equiv 0$? Further on we consider the least congruence relation $\Theta[I]$ under which a given ideal is a congruence class. We point out that the correspondence $I \to \Theta[I]$ is a complete join-homomorphism and in case of distributive lattices it is moreover an isomorphism. Finally, we turn our attention to some questions related to weak projectivity.

In Part III we deal with the notion of separable congruence relations. After the definition and typical examples we prove some lemmas of preliminary character, some of which are interesting in their own right. Next we turn to the problem of giving an answer to G. BIRKHOFF's problem 72, by applying the results concerning separable congruence relations. Then we use these results in order to characterize the distributive lattices on which the congruence relations satisfy the dual infinite distributive law.

In Parts II and III we get the results of J. JAKUBIK [15] — concerning problems 72 and 73 of G. BIRKHOFF, in case of discrete lattices — as trivial special cases.

We close Part III by analysing a question raised in problem 67 of G. BIRKHOFF.

After some preliminary theorems we deal in Part IV with the problem: in which distributive lattices may a Boolean ring operation be defined? We describe also in Part IV the types of these lattices and operations.

The kernel of this paper has been published in Hungarian in the papers [6] and [7]. We supplemented the results by several new ones. For instance, the results concerning G. BIRKHOFF's problem 67 are all new.

Preliminaries

Let L be a lattice. The elements of L are denoted by the letters $a, b, c, \ldots, x, y, z$. If the lattice L has a greatest or a least element, then it will be indicated by 1 and 0, respectively. Proper inclusion will be denoted by $a > b$, while the fact that a covers b will be indicated by $a \succ b$. The lattice operations are denoted, as usual, by $\cup$ and $\cap$, while $\bigvee_\alpha a_\alpha$ and $\bigwedge_\alpha a_\alpha$ will mean the complete join and meet of the elements a_α, respectively, if they exist. If a has a complement, it will be denoted by a'.

$\{x, \alpha(x)\}$ designates the set of all elements x in L for which the proposition $\alpha(x)$, defined on the elements of L, is true.

The principal ideal generated by a is $(a] = \{x; x \leqq a\}$, the principal dual ideal generated by a is $[a) = \{x; x \geqq a\}$ and the closed interval $[a, b]$ is $\{x; a \leqq x \leqq b\}$.

The congruence relations on the lattice L are denoted by Θ, Φ, ξ, η. The set of all congruence relations on the lattice L is indicated by $\Theta(L)$. The universal and the identical congruence relations are designated by ι and ω, i. e. $x \equiv y$ (ι) for all $x, y \in L$; $x \equiv y$ (ω) if and only if $x = y$.

Ideals of the lattice L are denoted by the symbols I, J, K. The set of all ideals of the lattice L is indicated by $\mathfrak{L}$.

The sets $\Theta(L)$ and $\mathfrak{L}$ under suitably defined partial orderings form a lattice. This is assured by the following two assertions:

Under the natural partial ordering, $\Theta \leqq \Phi$ $(\Theta, \Phi \in \Theta(L))$ if and only if $x \equiv y$ (Θ) implies $x \equiv y$ (Φ), $\Theta(L)$ form a complete lattice. Moreover, let A be any set of congruence relations Θ on L. We define two new relations ξ and η by

(i) $x \equiv y (\xi)$ means $x \equiv y$ (Θ) for all $\Theta \in A$;

(ii) $x \equiv y (\eta)$ means that for some finite sequence $x = z_0, z_1, \ldots, z_m = y$ we have $z_{i-1} \equiv z_i$ (Θ_i) for some $\Theta_i \in A$.

Then ξ, η are congruence relations; moreover η is the join and ξ is the meet of all $\Theta \in A$.

LEMMA 1. *Let L be a lattice and $\mathfrak{L}$ the set of all ideals of L. Under the set inclusion, $\mathfrak{L}$ is a lattice with complete union. If A is a subset of $\mathfrak{L}$, then we define K as the set of all x for which*

$$x \leqq y_1 \cup y_2 \cup \cdots \cup y_n, \quad y_i \in I_i$$

for some $I_i \in A$. Then K is the complete union of the $I \in A$.

The first assertion is due to G. BIRKHOFF [2]; we were unable to find a proof of the second one in the literature, but the proof is clear from the definitions, so we omit it.

Now we define some special congruence relations. Let S be a subset of L and A the set of all congruence relations under which S lies wholly in one congruence class. By BIRKHOFF's theorem (cited above) the meet of the set of congruences A is again a congruence relation, and under this, too, all the elements of S are in the same congruence class. Hence there exists a least congruence relation under which the elements of S are in the same class. We shall say that this is the congruence relation generated by S, and we shall denote it by $\Theta[S]$. A special case of great importance is when S contains only two elements a and b; in this case $\Theta[S]$ will be designated by $\Theta_{a,b}$. A trivial connection between the notions $\Theta[S]$ and $\Theta_{a,b}$ is the following

LEMMA 2. *Let S be a subset of the lattice L. Then*

$$\Theta[S] = \bigvee_{a,b \in S} \Theta_{a,b}. \tag{1}$$

PROOF. Obviously, $\Theta_{a,b} \leqq \Theta[S]$ for all $a, b \in S$, hence $\bigvee_{a,b\in S} \Theta_{a,b} \leqq \Theta[S]$. On the other hand, S is in one congruence class under $\bigvee_{a,b\in S} \Theta_{a,b}$, for $x, y \in S$ and $x \not\equiv y (\bigvee_{a,b\in S} \Theta_{a,b})$ contradict $\Theta_{x,y} \leqq \bigvee_{a,b\in S} \Theta_{a,b}$. Thus $\Theta[S] = \bigvee_{a,b\in S} \Theta_{a,b}$ by the minimal property of $\Theta[S]$, as asserted.

If L is a lattice, then $\bar{L}$ denotes a homomorphic image of L, under the homomorphism $a \to \bar{a}$, i. e. $\bar{a}$ denotes an element of $\bar{L}$ as well as the class of those elements x of L for which $x \to \bar{a}$. If a congruence relation Θ is given, then the homomorphic image of L induced by Θ (i. e. the lattice of all congruence classes) will be indicated by $L(\Theta)$. If there exists an ideal which is a congruence class under the congruence relation Θ, then we denote it by I_Θ. Clearly, I_Θ is the kernel of the homomorphism induced by Θ.

If in L all bounded chains are finite, then following J. JAKUBIK and M. KOLIBIAR we speak of a discrete lattice. Further, if in L between all comparable pairs of elements there exists a finite maximal chain, then we call the lattice semi-discrete. (These notions coincide in modular, moreover in semi-modular lattices, see e. g. [10].)

At last, we shall denote by S and T the five element lattices generated by the elements x, y, z such that the following identities hold:

(S) $x > y$, $x \cup z = y \cup z = 1$, $x \cap z = y \cap z = 0$;

(T) $x \cup y = x \cup z = y \cup z = 1$, $x \cap y = x \cap z = y \cap z = 0$.

I. CONGRUENCE RELATIONS IN DISTRIBUTIVE LATTICES

§ 1. Description of minimal congruence relations in distributive lattices

It is well known that if Θ is a congruence relation, then $a \equiv b$ (Θ) if and only if $a \cup b \equiv a \cap b$ (Θ) (see [2]). From this trivial fact it follows that we need consider only the problem of determining the comparable pairs of elements congruent under the minimal congruence relation, which collapses a comparable pair of elements. (We say that Θ collapses a and b if they are in one congruence class under Θ)

THEOREM 1. *Given two elements a, b of the distributive lattice L with $a \geqq b$, the elements $c, d \in L$ with $c \geqq d$ satisfy $c \equiv d$ $(\Theta_{a,b})$ if and only if*

$$(a \cup d) \cap c = c \tag{2}$$

and

$$(b \cup d) \cap c = d. \tag{3}$$

PROOF. We define the relation Θ on L by putting

$$x \equiv y \ (\Theta) \tag{4}$$

if and only if $c = x \cup y$ and $d = x \cap y$ satisfy (2) and (3).

From the identities $(a \cup x) \cap x = x$, $(b \cup x) \cap x = x$ it follows that Θ is reflexive, and from the symmetry of x, y in the definition of Θ it follows the symmetric property of Θ. To prove the substitution law for Θ, let us suppose $x \equiv y$ (Θ), and let t be arbitrary, then from the distributivity of L and from (2), (3), (4) we obtain

$$\{a \cup [(x \cup t) \cap (y \cup t)]\} \cap [(x \cup t) \cup (y \cup t)] = \{[a \cup (x \cap y)] \cup t\} \cap$$
$$\cap [(x \cup y) \cup t] = \{[a \cup (x \cap y)] \cap (x \cup y)\} \cup t = (x \cup y) \cup t = (x \cup t) \cup (y \cup t);$$

and in a similar way

$$\{b \cup [(x \cup t) \cap (y \cup t)]\} \cap [(x \cup t) \cup (y \cup t)] = (x \cup t) \cap (y \cup t);$$

furthermore

$$\{a \cup [(x \cap t) \cap (y \cap t)]\} \cap [(x \cap t) \cup (y \cap t)] = \{[a \cup (x \cap y)] \cap (a \cup t)\} \cap$$
$$\cap [(x \cup y) \cap t] = [a \cup (x \cap y)] \cap (x \cup y) \cap [(a \cup t) \cap t] = \{[a \cup (x \cap y)] \cap$$
$$\cap (x \cup y)\} \cap t = (x \cup y) \cap t = (x \cap t) \cup (y \cap t),$$

and likewise

$$\{b \cup [(x \cap t) \cap (y \cup t)]\} \cap [(x \cap t) \cup (y \cap t)] = (x \cap t) \cap (y \cap t).$$

Thus these equations show us that $x \equiv y$ (Θ) implies $x \cup t \equiv y \cup t$ (Θ) and $x \cap t \equiv y \cap t$ (Θ).

We show the transitivity of Θ at first in case $u \geqq v \geqq w$, $u \equiv v$ (Θ), $v \equiv w$ (Θ). By (4) we get

$$(a \cup v) \cap u = u, \tag{5}$$

$$(b \cup v) \cap u = v \tag{6}$$

and

$$(a \cup w) \cap v = v, \tag{7}$$

$$(b \cup w) \cap v = w. \tag{8}$$

We prove

$$(a \cup w) \cap u = u, \tag{9}$$

$$(b \cup w) \cap u = w \tag{10}$$

which are by (4) equivalent to $u \equiv w$ (Θ). Clearly, from (7) we have $a \cup w \geqq v$, applying this fact, $u \geqq v$, $v \geqq w$ and the distributivity of the lattice we get

$$(a \cup w) \cap u = [(a \cup w) \cap u] \cup v = (a \cup w \cup v) \cap (u \cup v) = (a \cup v) \cap u,$$

but by (5) $u = (a \cup v) \cap u$, thus $(a \cup w) \cap u = u$, completing the proof of (9).

From $v \geqq w$ we get $b \cup v \geqq b \cup w$, hence, using (6) and (8),

$$(b \cup w) \cap u = (b \cup w) \cap (b \cup v) \cap u = (b \cup w) \cap v = w,$$

as asserted.

Now let us suppose $u \equiv v$ (Θ), $v \equiv w$ (Θ) for arbitrary $u, v, w \in L$. Applying the substitution law, it follows $u \cup v = (u \cup v) \cup (v \cap w) \equiv u \cup v \cup w$ (Θ), $u \cap v = (u \cap v) \cap (v \cup w) \equiv (u \cap v) \cap (v \cap w) = u \cap v \cap w$ (Θ), i. e.

$$u \cup v \cup w \equiv u \cup v \ (\Theta),$$
$$u \cup v \equiv u \cap v \ (\Theta),$$
$$u \cap v \equiv u \cap v \cap w \ (\Theta).$$

But

$$u \cup v \cup w \geqq u \cup v \geqq u \cap v \geqq u \cap v \cap w,$$

thus from the previous paragraph it follows that $u \cup v \cup w \equiv u \cap v \cap w$ (Θ). From the substitution law by direct computation we obtain $u \cup w = (u \cup w) \cup (u \cap w) = [(u \cup v \cup w) \cap (u \cup w)] \cup (u \cap w) \equiv [(u \cap v \cap w) \cap (u \cup w)] \cup (u \cap w) = (u \cap v \cap w) \cup (u \cap w) = u \cap w$ (Θ). This completes the proof of the transitivity of Θ.

We conclude that Θ is a congruence relation. Furthermore, $a \equiv b$ (Θ), so $\Theta \geqq \Theta_{a,b}$, $\Theta_{a,b}$ being the least congruence relation with $a \equiv b$. Again, $x \equiv y$ (Θ) implies in view of Theorem 1

$$x \cup y = [a \cup (x \cap y)] \cap (x \cup y) \quad \text{and} \quad x \cap y = [b \cup (x \cap y)] \cap (x \cup y).$$

$a \equiv b$ $(\Theta_{a,b})$ and so applying the substitution law twice to the elements $t = x \cup y$ and $t = x \cap y$, we get $[a \cup (x \cap y)] \cap (x \cup y) \equiv [b \cup (x \cap y)] \cap (x \cup y)$ $(\Theta_{a,b})$ which is equivalent to $x \cup y \equiv x \cap y$ $(\Theta_{a,b})$ if we take into consideration the above equations. Hence $x \equiv y$ (Θ) implies $x \equiv y$ $(\Theta_{a,b})$, that is, $\Theta \leqq \Theta_{a,b}$, which compared with the inequality proved above gives the desired result, $\Theta = \Theta_{a,b}$, completing the proof of Theorem 1.

Some of the most important applications of Theorem 1 will be proved in the following section.

§ 2. Characterizations of distributive lattices

Some properties of congruence relations of a lattice are suitable to characterize the distributivity of a lattice. We shall deduce such characterizations from Theorem 1.

THEOREM 2. *Each one of the following conditions is equivalent to the distributivity of the lattice L:*

(a) *if $c \equiv d$ $(\Theta_{a,b})$, then $a \leqq d$ (or $c \leqq b$) is impossible whenever $b \leqq a$, $d < c$ $(a, b, c, d \in L)$;*

(b) *$[b, a]$ is a congruence class under $\Theta_{a,b}$ for all $b \leqq a$ $(a, b \in L)$;*

(c) *$\Theta_{a,b} \cap \Theta_{c,d} = \omega$ for all $a \geqq b \geqq c \geqq d$ $(a, b, c, d \in L)$;*

(d) *$\Theta_{a,b}$ has a complement in $\Theta(L)$ (for all $a \geqq b$) such that $c \geqq a$ implies $c \equiv a$ $(\Theta'_{a,b})$ $(a, b, c \in L)$;*

(e) *if C is a chain of the lattice L, then every congruence relation of C may be extended to L such that the congruence classes on C remain the same;*

(f) *for any ideal I and for any $x \geqq y$, $x \equiv y$ $(\Theta[I])$ if and only if $x = y \cup v$ for some $v \in I$;*

(g) *the condttion* (f) *is valid for all principal ideals I;*

(h) *every ideal is a congruence class under some homomorphism;*

(j) *every principal ideal is a congruence class under some homomorphism.*

PROOF. First we prove that conditions (a)—(j) hold in a distributive lattice L.

Let us suppose that $c \equiv d$ $(\Theta_{a,b})$ $(a \geqq b, c > d)$ and yet $b \geqq c$, then from (3) $d = (b \cup d) \cap c = c$ contrary to $c > d$. We get a contradiction in a similar way from $a \leqq d$ and (2). Thus the validity of (a) is a simple consequence of Theorem 1.

From (a) we can easily deduce (b). Indeed, if $c \equiv a$ $(\Theta_{a,b})$ and $c \notin [b, a]$, then either $c \cup a > a$ or $c \cap b < b$ holds (for, in case $c \cup a = a$ and $c \cap b = b$, we should have $b \leqq c \leqq a$, i. e. $c \in [b, a]$). But from $a < a \cup c$ and $a \equiv c$ $(\Theta_{a,b})$ we get $a \equiv a \cup c$ $(\Theta_{a,b})$, while $b > c \cap b$, $c \equiv b$ $(\Theta_{a,b})$ imply $b \equiv c \cap b$ $(\Theta_{a,b})$, both are in a contradiction to (a).

Now we prove that (c) holds in L. Let $a > b \geqq c > d$, $x > y$ and $x \equiv y$ $(\Theta_{a,b} \cap \Theta_{c,d})$. Then $x \equiv y$ $(\Theta_{c,d})$, hence by Theorem 1

$$(d \cup y) \cap x = y. \tag{11}$$

We assert that $c \cap (d \cup x) > c \cap (d \cup y)$. Indeed, if $c \cap (d \cup x) = c \cap (d \cup y)$, then from the equality $c \cup x = c \cup y$ (it follows from $c \cup x \equiv c \cup y$ $(\Theta_{c,d})$ and from (a)) we get $c \cup (d \cup x) = c \cup (d \cup y)$. Thus $d \cup x$ and $d \cup y$ are both relative complements of c in the interval $[c \cap (d \cup x), c \cup x]$, hence $d \cup x = d \cup y$. From (11) we infer $x = (d \cup x) \cap x = (d \cup y) \cap x = y$, a contradiction. Obviously, $c \cap (d \cup x) \equiv c \cap (d \cup y)$ $(\Theta_{x,y})$ and $\Theta_{x,y} \leqq \Theta_{a,b}$, so we get $c \cap (d \cup x) \equiv \equiv c \cap (d \cup y)$ $(\Theta_{a,b})$ and $a > b \geqq c \geqq c \cap (d \cup x) > c \cap (d \cup y)$, in contradiction to (a). If $a = b$ or $c = d$, then there is nothing to prove.

Next let us consider condition (d). We define Φ as the join of the congruence relations $\Theta[[a)]$ and $\Theta[(b]]$. Then $\Theta_{a,b} \cup \Phi = \iota$, because for all $x \leqq y \in L$, $[x, y] \subseteq [x \cap b, y \cup a]$, thus from $x \cap b \equiv y \cup a$ $(\Theta_{a,b} \cup \Phi)$ we get $x \equiv y$ $(\Theta_{a,b} \cup \Phi)$. Let us suppose that for some $x, y \in L$ $(x \neq y)$ we have $x \equiv y$ $(\Theta_{a,b} \cap \Phi)$. This is equivalent to $\omega < \Theta_{x,y}$ and $\Theta_{x,y} \leqq \Theta_{a,b} \cap \Phi$. From the latter $\Theta_{x,y} = \Theta_{a,b} \cap \Theta_{x,y} \cap \Phi$ and $\Phi = \bigvee_{\substack{u > v \geqq a \\ u \leqq v \leqq b}} \Theta_{u,v}$. Thus (using the infinite distributive law in $\Theta(L)$, see in [2], [4], or in § 1 of Part II) $\Theta_{x,y} = \Theta_{x,y} \cap \cap \Theta_{a,b} \cap \bigvee_{\substack{u > v \geqq a \\ u < v \leqq b}} \Theta_{u,v} = \Theta_{x,y} \cap \bigvee(\Theta_{a,b} \cap \Theta_{u,v})$ and from (c) $\Theta_{a,b} \cap \Theta_{u,v} = \omega$ which is a contradiction. So, $\Phi = \Theta'_{a,b}$. Obviously, $c \geqq a$ implies $c \equiv a(\Phi)$.

To prove the validity of condition (e), let a chain C be given in L, and a congruence relation Φ on C. We define the following congruence relation of L: $\Theta = \bigvee_{\substack{a \geqq b \in C \\ a \equiv b(\Phi)}} \Theta_{a,b}$. We prove that Θ has the desired property. Assume $x \equiv y$ (Θ), $x, y \in C$. Then by BIRKHOFF's theorem cited in the Preliminaries, there exists a finite number of pairs of elements a_i, b_i such that $a_i < b_i$ and $a_i \equiv b_i$ (Φ), furthermore $x \equiv y(\bigvee_{i=1}^{n} \Theta_{a_i, b_i})$. If $[x, y] \subset \bigcup [a_i, b_i]$, then $x \equiv y$ (Φ) is valid too and there is nothing to prove. If $x \not\equiv y(\Phi)$, then there exists a part $[x_1, y_1]$ of $[x, y]$ with the property that for each i either $a_i < b_i \leqq \leqq x_1 < y_1$ or $x_1 < y_1 \leqq a_i < b_i$. Then from (c) $\Theta_{x,y} \cap \bigvee_{i=1}^{n} \Theta_{a_i, b_i} = \omega$ which contradicts $\Theta_{x,y} \leqq \bigvee \Theta_{a_i, b_i}$.

To verify condition (f), let the ideal I of L be given and let $x \geqq y$, $x \equiv y$ $(\Theta[I])$. We prove that $x \equiv y$ $(\Theta_{a,b})$ for some $a, b \in I$. Indeed, from Lemma 2 $\Theta[I] = \bigvee_{\substack{a, b \in I \\ a \geqq b}} \Theta_{a,b}$, and by BIRKHOFF's theorem there exists a finite number of pairs of elements $a_i > b_i$, $a_i, b_i \in I$ $(i = 1, 2, \dots, n)$ such that $x \equiv y(\bigvee_{i=1}^{n} \Theta_{a_i, b_i})$. Let $a = \bigvee a_i$ and $b = \bigwedge b_i$, then $a, b \in I$ and obviously $x \equiv y$ $(\Theta_{a,b})$. Then by Theorem 1 $x = x \cap (a \cup y)$, thus from the distributivity of L we get $x = (x \cap a) \cup y$, hence $v = x \cap a$ has the desired property. On the other hand, it is clear that if $x = y \cup v$ and $v \in I$, then $x \equiv y$ $(\Theta[I])$, so we have proved the validity of (f).

The conditions (g), (h), (j) are special cases of (f).

Now we prove that each one of the conditions (a)—(j) implies the distributivity of L.

If L is not distributive, then it contains as a sublattice a lattice, isomorphic to the lattice S or T, defined formerly. Since a lattice has one of the properties (a)—(j) only if every sublattice of it has this property, so we must prove only that the lattices S and T fail to have this property. Among the conditions (f), (g), (h), (j) the last is the weakest one, hence in this step of the proof we may omit the others. (b) is a consequence of (a), so we may omit condition (a) too.

First we verify that the interval $[0, y]$ is a congruence class under no homomorphism in S and T. Indeed, if $y \equiv 0$ (Θ) for some Θ, then $x = x \cap (y \cup z) \equiv x \cap (0 \cup z) = 0$ and $x \notin [0, y]$, a contradiction. Hence it results that in a non-distributive lattice conditions (a), (b), (f), (g), (h), (j) do not hold. A similar trivial computation shows that conditions (c) (consider in S the chain $0, y, x$ and in T the chain $0, x, 1$), (d) (in S the interval $[y, x]$, in T the interval $[0, x]$ play the role of the interval $[b, a]$), (e) (see the chains described at the condition (c)), do not hold in the lattices S and T. Thus the proof of Theorem 2 is completed.

We mention that the conditions of Theorem 2 play a fundamental role in our researches related to all properties of distributive lattices, not only in this paper, but in the papers [9], [10], [11], [12] too.

Conditions (h) and (j) are the same as those of Theorem 2. 2 of J. Hashimoto [14] (conditions (3) and (4)).

§ 3. A generalized form of G. Birkhoff's problem 73

In his textbook [2] G. Birkhoff proposed the following problem:

Find necessary and sufficient conditions in order that the correspondence between the congruence relations and ideals of a lattice be one-to-one.

More precisely:

Find necessary and sufficient conditions in order that the correspondence $\Theta \rightarrow I_\Theta$ be an isomorphism between $\Theta(L)$ and $\mathfrak{L}$.

Applying Theorems 1 and 2, we get an answer to this question.

Lemma 3 (J. Hashimoto's theorem). *In the lattice L there is a one-to-one correspondence (in the natural way) between the ideals and congruence relations if and only if L is a distributive, relatively complemented lattice with zero element.*

PROOF.

The necessity of the conditions. Obviously, I_ω is the zero ideal of L. Every ideal of L is a congruence class under some homomorphism, so the distributivity of L is assured by condition (h) of Theorem 2.

Now let us suppose that L is a distributive lattice with zero element. We prove that L is relatively complemented. By a theorem of J. VON NEUMANN (see [2], p. 114), it is sufficient to prove that if $b < a$, then b has a complement in the interval $[0, a]$. Let $V_{a,b}$ be the ideal which consists of all u with $u \equiv 0$ $(\Theta_{a,b})$. $V_{a,b}$ is a congruence class under precisely one congruence relation, hence $a \equiv b$ $(\Theta[V_{a,b}])$. From condition (f) it follows that for some $v \in V_{a,b}$ we have

$$b \cup v = a. \tag{12}$$

It is clear that $v \equiv 0$ $(\Theta_{a,b})$, hence from Theorem 1 (v and 0 play the roles of c and d)

$$b \cap v = 0. \tag{13}$$

(12) and (13) show that v is the complement of b in $[0, a]$.

The sufficiency of the conditions. From condition (h) of Theorem 2 it follows that every ideal of L is a congruence class under some homomorphism. Furthermore, every ideal is a congruence class under at most one congruence relation, as it follows from the complementedness of the intervals of type $[0, a]$ (see [2], p. 23, or the Corollary of Theorem 4 in this paper).

Now we are ready to prove the general theorem.

THEOREM 3. *Let L be a lattice and "a" a fixed element of L. Every convex sublattice of L containing "a" is a congruence class under precisely one congruence relation if and only if L is distributive, and all the intervals of type $[a, b]$ ($a \geqq b$ or $a < b$) are complemented.*

PROOF.

The necessity of the conditions. First we show the necessity of the distributivity of the lattice L. Let us suppose that L is not distributive; then it contains as a sublattice either the lattice S or the lattice T. (x, y, z will indicate the generators of S or T.)

We prove that $z = a$ is impossible. Indeed, since $x \cap y \neq x \cup y$, that is, $\Theta = \Theta_{x \cap y, x \cup y}$ is not equal to ω, the congruence class which contains a under Θ, is different from the congruence class which contains a under ω. Thus the congruence class under Θ containing a contains a further element x, so we may pick out an element c such that $c \gtrless a$ and

$$c \equiv a \ (\Theta). \tag{14}$$

Now, according as $c > a$ or $c < a$, the interval $[x \cap y \cap a, a]$ or $[a, x \cup y \cup a]$ is a congruence class under no homomorphism. Let us discuss the case $c > a$ (if $c < a$, then the proof goes on the same lines). Then $a \equiv x \cap y \cap a\ (\Phi)$ implies (as in the proof of condition (b) in Theorem 2) $x \equiv y\ (\Phi)$, for any Φ, so $x \cup y \equiv x \cap y\ (\Phi)$, hence $\Phi \geqq \Theta$, $c \equiv a\ (\Phi)$ (see (14)), but $c \notin [x \cap y \cap a, a]$; a contradiction.

Thus we have proved that $x \cup z = y \cup z$ and $x \cap z = y \cap z$ are impossible if $z = a$. So we may suppose by the Duality Principle that $x \cup a \not\equiv y \cup a$. We assert that under these hypotheses the interval $[y \cap z \cap a, y \cup a]$ (which contains a) is no congruence class under any congruence relation. Indeed, if $y \cup a \equiv y \cap z \cap a$, then

$$z = z \cup (y \cap z \cap a) \equiv z \cup (y \cup a) = (z \cup y) \cup a = z \cup x \cup a,$$

furthermore $x \cap z \equiv x \cap (z \cup x \cup a) = x$. But $x \equiv y \cap z, y \cup a \geqq y \cap z \geqq y \cap z \cap a$ and $x \notin [y \cap z \cap a, y \cup a]$, a contradiction.

Summarizing the above proved assertions, we get that the existence of the sublattices S or T contradicts the fact that every convex sublattice of L containing a is a congruence class under precisely one congruence relation.

Our second aim is to prove the complementedness of the intervals of type $[a, b]$ ($a \geqq b$ or $a < b$). Let $b_1 > b_2 > a$. Since $\Theta_{b_1, b_2} \neq \omega$, there exists an element c, comparable with a, such that $c \equiv a\ (\Theta_{b_1, b_2})$. From condition (a) of Theorem 2 we see $c < a$ is impossible. It follows that the congruence class under Θ_{b_1, b_2} which contains a is not empty and it is a part of $[a)$. Hence in $[a)$ the condition of Lemma 3 holds, that is, $[a)$ is relatively complemented. In a similar way we get the relative complementedness of $(a]$ too. The necessity of the conditions is therefore proved.

The sufficiency of the conditions. Let L be a distributive lattice such that, for a fixed a, the lattices $(a]$ and $[a)$ are relatively complemented. First we show that the distributivity of L implies that every convex sublattice is a congruence class under some homomorphism. Let D be a convex sublattice, I and J the ideal resp. dual ideal generated by D. A trivial computation shows that D is a congruence class under $\Theta[I] \cap \Theta[J]$.

Secondly we prove that every convex sublattice containing a is a congruence class under precisely one congruence relation. It is enough to prove in case the convex sublattice consists of a alone, for if D is a congruence class under more than one homomorphism, then let us consider among these the minimal one, $\Theta[D]$, and let $\overline{L} = L(\Theta[D])$ be the corresponding homomorphic image of L. In $\overline{L}$ there are fulfilled all the conditions as in L if the fixed element is $\bar{a}$, furthermore the one element convex sublattice $\bar{a}$ is a congruence class under more than one homomorphism of $\overline{L}$. So we succeeded in reducing the proof to a special case.

Now let us suppose that $x > y$. It is enough to prove the existence of a c with $c \neq a$ and $c \equiv a$ $(\Theta_{x,y})$. From the distributivity of L we obtain $a \cup x > a \cup y$ or $a \cap x > a \cap y$ ($a \cup x = a \cup y$ and $a \cap x = a \cap y$ contradict $x \neq y$). Let c be the relative complement of $a \cup y$ in the interval $[a, a \cup x]$ in the first case, and the relative complement of $a \cap x$ in the interval $[a \cap y, a]$ in the second case. A trivial calculation shows that $x \equiv y$ implies $c \equiv a$ in both cases, that is, the one element sublattice a is a congruence class only under ω. Thus the proof of Theorem 3 is completed.

The proof shows us that Theorem 3 may be sharpened by replacing the condition "every convex sublattice containing $a \ldots$" by the following weaker one: "every interval containing $a \ldots$".

That the relative complementedness of the whole lattice is not a consequence of the condition, it may be illustrated by the following simple counterexample: L is the chain of three elements and a, the fixed element, is the only element different from 0 and 1.

An immediate consequence of our Theorem 3 is the

COROLLARY. *Every convex sublattice of L is a congruence class under precisely one homomorphism if and only if L is a relatively complemented distributive lattice.*

Special cases of Theorem 3 were already known. Lemma 3 (the special case $a = 0$) was first proved by J. HASHIMOTO [14] in 1952; a year later G. J. AREŠKIN [1] has proved Lemma 3, by supposing that the lattice L is distributive and has a zero element. The Corollary was proved independently of us — by supposing the distributivity of the lattice considered — by M. KOLIBIAR [16].

We remark that we may get further theorems, too, as easy consequences of Theorem 3. For instance, in [9] we have pointed out that the following assertion of J. HASHIMOTO [14] is also a simple consequence of Theorem 3:

A relatively complemented lattice L is distributive if and only if L has an element a such that $(a]$ and $[a)$ are prime factorizable.

Using transfinite methods it results [11] that Lemma 3 may be sharpened; in [11] we have published another very simple proof of Lemma 3. Related to these questions we refer to [9] too.

II. CONGRUENCE RELATIONS IN GENERAL LATTICES

§ 1. Some lemmas on congruence relations

In this section we prove three lemmas which will simplify the proofs of several theorems in Parts II and III. A part of the merely technical Lemma 4 was proved already in Theorem 2.

LEMMA 4. *Let* ξ *be a binary relation defined on the lattice* L. ξ *is a congruence relation if and only if*

(a) $x \equiv x$ (ξ) *for all* $x \in L$;

(b) $x \equiv y$ (ξ) *is equivalent to* $x \cup y \equiv x \cap y$ (ξ) *for all* $x, y \in L$;

(c) $x \geqq y \geqq z, x \equiv y$ (ξ) *and* $y \equiv z$ (ξ) *imply* $x \equiv z$ (ξ);

(d) *if* $x \geqq y$ and $x \equiv y$ (ξ), *then* $x \cup t \equiv y \cup t$ (ξ), $x \cap t \equiv y \cap t$ (ξ) *for all* $t \in L$.

PROOF. Obviously, it is sufficient to prove that a relation ξ satisfying conditions (a)—(d) is a congruence relation.

By (a) ξ is *reflexive*, and by (b) it is *symmetric* too.

Let $u \geqq v$, $u \equiv v$ (ξ) and $a, b \in [v, u]$, then we assert $a \equiv b$ (ξ). Indeed, $u \geqq a \cup b \geqq a \cap b \geqq v$ and from (d) $u \cap (a \cup b) \equiv v \cap (a \cup b)$ (ξ) and $u \cap (a \cup b) \geqq$ $\geqq v \cap (a \cup b)$, thus applying again (d), $a \cup b = [u \cap (a \cup b)] \cup (a \cap b) \equiv [v \cap$ $\cap (a \cup b)] \cup (a \cap b) = a \cap b$ (ξ), whence from (b) $a \equiv b$ (ξ), and the assertion is established.

Next let $x \equiv y$ (ξ) and $y \equiv z$ (ξ). On account of (b) $x \cup y \equiv x \cap y$ (ξ), thus from (d) $x \cup y \cup z = (x \cup y) \cup (y \cup z) \equiv (x \cap y) \cup (y \cup z) = y \cup z$ (ξ), similarly, $x \cap y \cap z \equiv y \cap z$ (ξ), that is, $x \cup y \cup z \geqq y \cup z \geqq y \cap z \geqq x \cap y \cap z$ and the consecutive elements are congruent modulo ξ, so applying twice (c) we get $x \cup y \cup z \equiv x \cap y \cap z$ (ξ). Considering that $x, z \in [x \cap y \cap z, x \cup y \cup z]$, we conclude $x \equiv z$ (ξ), i. e. ξ is *transitive*.

The substitution law may easily be proved too, for if we assume $x \equiv y$ (ξ), then from (b) and (d) $x \cup y \equiv x \cap y$ (ξ) and $(x \cup y) \cup t \equiv (x \cap y) \cup t$ (ξ), but $x \cup t, y \cup t \in [(x \cap y) \cup t, x \cup y \cup t]$, hence we obtain $x \cup t \equiv y \cup t$ (ξ) and alike $x \cap t \equiv y \cap t$ (ξ), completing the proof of the Lemma.

We note that the conditions of Lemma 4 are independent and may be weakened, e. g. (a) may be replaced by (a′) $x \equiv y$ (ξ) for some $x, y \in L$, but we need only the above described form of Lemma 4.

Now we prove a lemma which sharpens for lattices a similar result of G. BIRKHOFF for general algebras (see the Preliminaries).

LEMMA 5. *Let A be a subset of $\Theta(L)$. We define the relation $\eta: x \equiv y\ (\eta)$ if and only if there is a finite sequence $x \cup y = u_0 \geqq u_1 \geqq \cdots \geqq u_n = x \cap y$ satisfying $u_i \equiv u_{i-1}\ (\Theta_i)$ for some $\Theta_i \in A$ $(i = 1, \ldots, n)$. Then η is a congruence relation and $\eta = \bigvee_{\Theta_\alpha \in A} \Theta_\alpha$.*

PROOF. It is clear that if η is a congruence relation, then $\eta = \bigvee \Theta_\alpha$. Thus it remains to prove that η is a congruence relation. Obviously, it is reflexive and symmetric. If $x \geqq y \geqq z$, $x \equiv y\ (\eta)$ and $y \equiv z\ (\eta)$, then we have two chains which connect x and y, resp. y and z, having the desired property. Joining these two chains, we get one from x to z with the desired property. At last if $x = z_0 \geqq z_1 \geqq \cdots \geqq z_n = y$, then $t \cup x = t \cup z_0 \geqq t \cup z_1 \geqq \cdots \geqq t \cup z_n = t \cup y$, thus $x \equiv y\ (\eta)$ implies $x \cup t \equiv y \cup t\ (\eta)$, and in a similar way we get that it implies $x \cap t \equiv y \cap t\ (\eta)$ too. We see η satisfies the conditions of Lemma 4, that is, η is a congruence relation.

The importance of Lemma 5 should be revealed by the fact that it decides in the interval $[a, b]$ whether $a \equiv b$ is valid or not. For instance, applying Lemma 5, it may be proved easily[3] the notable theorem of N. FUNAYAMA and T. NAKAYAMA [5], according to which in $\Theta(L)$ unrestrictedly holds the infinite distributive law

(ID) $\Theta \cap \bigvee \Theta_\alpha = \bigvee (\Theta \cap \Theta_\alpha)$.

In proving (ID) it suffices to show that $x \geqq y$ and $x \equiv y\ (\bigvee(\Theta \cap \Theta_\alpha))$ imply $x \equiv y\ (\Theta \cap \bigvee \Theta_\alpha)$. If $x \equiv y\ (\bigvee(\Theta \cap \Theta_\alpha))$, then by Lemma 5 for some finite sequence we have $x = z_0 \geqq z_1 \geqq \cdots \geqq z_n = y$, $z_{i-1} \equiv z_i\ (\Theta \cap \Theta_i)$, hence $z_{i-1} \equiv z_i (\Theta)$, further on $z_{i-1} \equiv z_i\ (\Theta_i)$, so $z_{i-1} \equiv z_i\ (\bigvee \Theta_\alpha)$, consequently $z_{i-1} \equiv z_i\ (\Theta \cap \bigvee \Theta_\alpha)$, that is, $x \equiv y\ (\Theta \cap \bigvee \Theta_\alpha)$, thus $\Theta \cap \bigvee \Theta_\alpha \leqq \bigvee (\Theta \cap \Theta_\alpha)$, q. e. d.

According to Theorem 1, in a distributive lattice under $\Theta_{a,b}$ $(a \geqq b)$ the elements c, d, $(c \geqq d)$ are congruent if and only if $c = (a \cup d) \cap c$, $d = (b \cup d) \cap c$. Now we generalize this theorem to arbitrary lattices.

Obviously, if

$$[\cdots(\{[(a \cup b) \cup x_1] \cap x_2\} \cup x_3) \cap \cdots] \cup x_n = c \cup d, \tag{15}$$

$$[\cdots(\{[(a \cap b) \cup x_1] \cap x_2\} \cup x_3) \cap \cdots] \cup x_n = c \cap d, \tag{16}$$

then $c \equiv d\ (\Theta_{a,b})$, as it follows from the substitution law.

The theory of congruence relations in arbitrary lattices is based upon the notion of weak projectivity due to R. P. DILWORTH [3].

[3] The idea of this proof of the theorem of FUNAYAMA and NAKAYAMA is essentially due to R. P. DILWORTH [3].

DEFINITION 1.[4] Let L be a lattice and $a, b, c, d \in L$. The pair of elements a, b is weakly projective into the pair of elements c, d if for some $x_1, \ldots, x_n \in L$ the equations (15) and (16) hold.

In what follows $\overline{a, b} \to \overline{c, d}$ will denote that a, b is weakly projective into c, d. Obviously, the relation $\to$ is transitive.

With the help of this notion we can easily describe the congruence relation $\Theta_{a,b}$.

LEMMA 6 (DILWORTH [3]). *$c \equiv d$ $(\Theta_{a,b})$ in the lattice L if and only if for some finite sequence*

(17) $c \cup d = y_0 \geqq y_1 \geqq \cdots \geqq y_k = c \cap d$ *one has* $\overline{a, b} \to \overline{y_{i-1}, y_i}$ $(i = 1, 2, \ldots, k)$.

PROOF. It is clear that if c, d satisfy (17), then $c \equiv d$ $(\Theta_{a,b})$. On the other hand, let us define the relation ξ such that $u \equiv v$ (ξ) if and only if some sequence $\{y_i\}$ and $c = u$, $d = v$ satisfy (17). Repeating word for word the trivial calculation of Lemma 5 we get (applying Lemma 4) that ξ is a congruence relation, completing the proof of this lemma.

COROLLARY 1. *Let L be a lattice and S a subset of L. S is a congruence class under some congruence relation if and only if $a, b, c \in S$ and $\overline{a, b} \to \overline{c, d}$ imply $d \in S$.*

PROOF. The assertion "only if" is trivial from the definition, and "if" is obvious from Lemma 2 and Lemma 6.

From Corollary 1 and from Theorem 1 it results the well-known fact that every convex sublattice of a distributive lattice is a congruence class under some congruence relation. This proof gives perhaps more insight into the cause of the validity of the above statement.

Another trivial consequence of this lemma is

COROLLARY 2. *A lattice L is simple if and only if for all $a, b, c, d \in L$ there exists a finite sequence $c \cup d = z_0 \geqq z_1 \geqq \cdots \geqq z_n = c \cap d$ such that $\overline{a, b} \to \overline{z_{i-1}, z_i}$ $(i = 1, 2, \ldots, n)$.*

If L is a modular lattice and a covers b, then $\overline{a, b} \to \overline{c, d}$ implies that $c = d$ or c covers d. Thus we are led to

COROLLARY 3. *If in the simple modular lattice L there exists a pair of elements a, b such that a covers b, then L is discrete.*

[4] Definition 1 is that of [6], but it may be shown easily that it is equivalent to that of R. P. DILWORTH. The notation is the same as in [6].

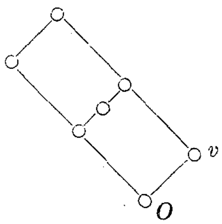
§ 2. Weakly complemented lattices

The notion of weak complementedness was introduced by H. WALLMAN [19] for distributive lattices. Now we define the notion of weak complementedness[5] in general such that for distributive lattices this is equivalent to that of H. WALLMAN.

DEFINITION 2. A lattice L with 0 is weakly complemented if to all pairs of elements a, b $(a \neq b;\ a, b \in L)$ there exists an element $c \neq 0$ such that $\overline{a, b} \to \overline{c, 0}$.

A trivial computation shows[6] that in a distributive lattice $a > b$ and $\overline{a, b} \to \overline{c, 0}$ $(c \neq 0)$ imply $a \cap c > 0$, $b \cap c = 0$. On the other hand, if $a \cap c > 0$ and $b \cap c = 0$, then putting $c' = a \cap c$ we obviously have $\overline{a, b} \to \overline{c', 0}$ and $c' \neq 0$. This coincides with the original definition of weak complementedness in distributive lattices.

Weak complementedness is not a homomorphic invariable property, that is, there exists a lattice which is weakly complemented, but a suitable homomorphic image of it is not relatively complemented. If this lattice is distributive, then it is necessarily infinite (see the example in [11]), but in the non-distributive case there are finite examples too; e. g. let L be the following lattice:



We can easily verify that this lattice is weakly complemented, yet $L(\Theta_{v,0})$ — which is isomorphic to the chain of three elements — is not weakly complemented.

G. J. AREŠKIN [1] proved the following assertion:

Let L be a distributive lattice with zero element. Every ideal of L is the kernel of at most one homomorphism if and only if every homomorphic image of L is weakly complemented.

[5] It seems to be unreasonable to change the definition of weakly complemented lattices, for it is a well-known notion. Our motivation is: the original notion of weak complementedness was successfully used only in distributive lattices in discussing the connection of topological spaces and distributive lattices [19] and in the researches of the congruence relations of distributive lattices [1]. In general lattices only some theorems were known which are based on the original definition. For this reason we propose the notion of "weakly complemented in the stronger sense" for the original one.

[6] It follows trivially from Theorem 1 too.

Now we show that this theorem is valid for arbitrary lattices with the above defined notion of weakly complementedness. This is a solution of the most natural generalization of G. BIRKHOFF's problem 73.[7]

THEOREM 4. *In the lattice L every congruence relation is the minimal one of a suitable ideal if and only if L has a zero and every homomorphic image of L is weakly complemented.*

For the proof a preliminary lemma is needed.

LEMMA 7. *Let L be a lattice with zero element. The zero ideal is a congruence class under precisely one congruence relation if and only if L is weakly complemented.*

PROOF. If the lattice L is weakly complemented, then the zero ideal is a congruence class only under ω, for if $x \neq y$, then there exists a $z \neq 0$, with $\overline{x, y} \rightarrow \overline{z, 0}$, that is, $z \equiv 0\ (\Theta_{x, y})$, i. e. the zero ideal is not a congruence class. On the other hand, let us assume that to the elements x, y there is no element z with $\overline{x, y} \rightarrow \overline{z, 0}$. Then $z \equiv 0\ (\Theta_{x, y})$, $z > 0$, is impossible, for if this held, then from Lemma 6 it would follow the existence of a $z_1 > 0$ with $\overline{a, b} \rightarrow \overline{z_1, 0}$. Thus the zero ideal is a congruence class under ω and $\Theta_{x, y}$ too, a contradiction.

Now we prove Theorem 4. Let I be an ideal which is a congruence class under at least one congruence relation. Obviously, I is a congruence class under more than one congruence relation if and only if the zero ideal of $L(\Theta[I])$ is a congruence class under more than one congruence relation. Thus the proof of Theorem 4 is completed.

If L is distributive, then we get from Theorem 4 the above theorem of of G. J. AREŠKIN.[8] On the other hand, we want to point out that every relatively complemented lattice with zero element is weakly complemented, so as a trivial special case of Theorem 4 we get a result of G. BIRKHOFF (see [2], p. 23):

COROLLARY. *In a lattice with 0, where all closed intervals $[0, a]$ are complemented, every congruence relation is determined by the ideal consisting of all x with $x \equiv 0$.*

We can get another answer to the above-mentioned problem.

[7] J. JAKUBIK [15] and J. HASHIMOTO [14] have also formulated in such a way the more natural generalization.

[8] Comparing Theorem 4 with Lemma 3 we get the following theorem of G. J. AREŠKIN [1]: A distributive lattice with zero is relatively complemented if and only if every homomorphic image of it is weakly complemented. (A far-reaching generalization of this theorem may be found in our paper [11].)

If $a, b \in L$, then $V_{a,b}$ will denote the ideal which is generated by all x with $\overline{a, b} \to \overline{x, 0}$.

THEOREM 5. *In the lattice L every congruence relation has an ideal as a congruence class and every ideal is a congruence class under at most one congruence relation if and only if L is a weakly complemented lattice with zero element and to all $a, b \in L$ there exist a $y \in V_{a,b}$ and a sequence $a \cup b = d_0 \geqq d_1 \geqq \cdots \geqq d_n = a \cap b$ with $\overline{y, 0} \to \overline{d_{i-1}, d_i}$ $(i = 1, \ldots, n)$.*

PROOF. We already know the necessity of the existence of a zero element and of weak complementedness. The third condition is necessary too, because if for a, b it did not hold, then $V_{a,b}$ would be a congruence class under more than one homomorphism. Indeed, if $V_{a,b}$ were the kernel of precisely one homomorphism, then $a \equiv b$ $(\Theta[V_{a,b}])$ would be valid, and this means just by Lemma 6 the validity of the third condition.

The sufficiency of the conditions follows from the fact that under these conditions

$$a \equiv b \ (\Theta) \quad \text{if and only if} \quad V_{a,b} \subseteq I_\Theta,$$

that is, I_Θ determines the congruence relation. Indeed, if $a \equiv b$ (Θ), then $\overline{a, b} \to \overline{c, 0}$ implies $c \equiv 0$ (Θ), that is, $V_{a,b} \subseteq I_\Theta$. On the other hand, if $V_{a,b} \subseteq I_\Theta$, then there exist a $y \in V_{a,b}$ and a finite sequence $a \cup b = y_0 \geqq y_1 \geqq \cdots \geqq y_n = a \cap b$ with $\overline{y, 0} \to \overline{y_{i-1}, y_i}$, but from $y \in V_{a,b} \subseteq I_\Theta$ it follows $y \equiv 0$ (Θ) and so $a \equiv b$ (Θ), q. e. d.

Theorem 5 is a generalization of a theorem of J. JAKUBIK [15]. J. JAKUBIK dealt with discrete lattices and he got the conditions of Theorem 5 with the small difference that the conditions on a, b are supposed only if a covers b. An easy computation shows that these conditions are equivalent in discrete lattices, and what is more it becomes trivial that it is valid not only in discrete lattices but under that weakened condition, too, that L is semi-discrete.

§ 3. Minimal congruence relations generated by ideals

In this section we deal with the correspondence $I \to \Theta[I]$.

THEOREM 6. *The congruence relation generated by the ideal $\bigvee I_\alpha$ is $\bigvee \Theta[I_\alpha]$, that is,*

$$\Theta[\bigvee I_\alpha] = \bigvee \Theta[I_\alpha]. \tag{18}$$

PROOF. First we verify that if $a \leqq b$ and $a \leqq c$, then

$$\Theta_{a,b} \cup \Theta_{a,c} = \Theta_{a,b \cup c}. \tag{19}$$

Since $a \equiv b\ (\Theta_{a, b \cup c})$ and $a \equiv c\ (\Theta_{a, b \cup c})$, thus $\Theta_{a, b} \cup \Theta_{a, c} \leqq \Theta_{a, b \cup c}$; on the other hand, $a \equiv b\ (\Theta_{a, b})$ and $a \equiv c\ (\Theta_{a, c})$, hence $a \equiv b \cup c\ (\Theta_{a, b} \cup \Theta_{a, c})$, that is, $\Theta_{a, b \cup c} \leqq \Theta_{a, c} \cup \Theta_{a, c}$. These inequalities prove (19).

By Lemma 2, (18) is equivalent to

$$\bigvee_{x, y \in \bigvee_{\alpha \in A} I_\alpha} \Theta_{x, y} = \bigvee_{\alpha \in A} \bigvee_{a, b \in I_\alpha} \Theta_{a, b}. \tag{20}$$

Let us suppose that $\Theta_{a, b}$ occurs in the right side of (20), then $a, b \in I_\alpha$ for some $\alpha \in A$, hence $a, b \in \bigvee_{\alpha \in A} I_\alpha$, thus we obtain that $\Theta_{a, b}$ occurs in the left side of (20), i. e. (20) holds with $\geqq$ instead of $=$.

Conversely, let $\Theta_{x, y}$ be a congruence relation which occurs in the left side of (20). By Lemma 1 this means the existence of such i_{α_r} ($\in I_{\alpha_r}$, $\alpha_r \in A$; $r = 1, 2, \ldots, n$) that $x, y \leqq i_{\alpha_1} \cup \cdots \cup i_{\alpha_r}$. Let $u = \bigwedge_{r=1}^{n} i_{\alpha_r} \cap (x \cap y)$. Obviously, $u \in I_{\alpha_r}$ ($r = 1, \ldots, n$), hence $\Theta_{u, i_{\alpha_r}}$ occurs in the right side of (20). By (19) $\bigvee_{r=1}^{n} \Theta_{u, i_{\alpha_r}} = \Theta_{u, \bigvee_{r=1}^{n} i_{\alpha_r}} \geqq \Theta_{x, y}$, and so

$$\bigvee_{x, y \in \bigvee_{\alpha \in A} I_\alpha} \Theta_{x, y} \leqq \bigvee_{\alpha \in A} \bigvee_{a, b \in I_\alpha} \Theta_{a, b},$$

that is, (20) is valid.

Let us denote by $\Theta_0[I]$ the least congruence relation under which I is a congruence class. Obviously, $\Theta[I] = \Theta_0[I]$ if $\Theta_0[I]$ exists.[9]

COROLLARY. *If $\Theta_0[I_\alpha]$ exists for all $\alpha \in A$, and also $\Theta_0[\bigvee_{\alpha \in A} I_\alpha]$ exists, then*

$$\bigvee \Theta_0[I_\alpha] = \Theta_0[\bigvee I_\alpha].$$

In Theorem 6 we have proved that the join of minimal congruence relations of ideals is a minimal congruence relation of an ideal. The analogous assertion for the meet is not true as it may be shown by the example of the lattice S. It would have some interest to give conditions under which the meet of minimal congruence relations remains minimal. We assert only

LEMMA 8. *Let L be a lattice with 0 on which every ideal is a congruence class under at most one congruence relation. Let $\bar{\mathfrak{L}}$ denote the lattice of all ideals in L which are congruence classes under some congruence relation. Then $\bar{\mathfrak{L}}$ is isomorphic to $\Theta(L)$, i. e.*

$$\Theta[\bigvee_{\alpha \in A} I_\alpha] = \bigvee_{\alpha \in A} \Theta[I_\alpha] \qquad (I_\alpha \in \bar{\mathfrak{L}}), \tag{21}$$

$$\Theta[\bigwedge_{\alpha \in A} I_\alpha] = \bigwedge_{\alpha \in A} \Theta[I_\alpha] \qquad (I_\alpha \in \bar{\mathfrak{L}}); \tag{22}$$

[9] It would be of great interest to examine the lattice of all ideals for which $\Theta_0[I]$ exists. One can easily prove that they form a distributive lattice.

let ($\bar{I}, \bar{K}, \bar{X}_i, \bar{Y}_j \in \bar{\mathfrak{L}}$)

$$\bar{I} = \bar{X}_0 \supset \bar{X}_1 \supset \cdots \supset \bar{X}_n = \bar{K}$$

and

$$\bar{I} = \bar{Y}_0 \supset \bar{Y}_1 \supset \cdots \supset \bar{Y}_s = \bar{K},$$

then there are refinements of these chains of common length.

PROOF. (21) was proved in Theorem 6. (22) follows from the fact that the existence of $0 \in L$ implies the existence of $\bigwedge I_\alpha$; furthermore $\bigwedge I_\alpha$ is a congruence class under $\bigwedge \Theta[I_\alpha]$. But $\bigwedge I_\alpha$ is a congruence class under at most one congruence relation, hence $\bigwedge \Theta[I_\alpha] = \Theta[\bigwedge I_\alpha]$. Thus we have proved that the correspondence $I \to \Theta[I]$ ($I \in \bar{\mathfrak{L}}$) is an isomorphism, between $\bar{\mathfrak{L}}$ and $\Theta(L)$, and so $\bar{\mathfrak{L}}$ is distributive. Hence the JORDAN—DEDEKIND theorem is applicable to $\bar{\mathfrak{L}}$, and this assures the validity of the last statement.

Now we give a simple answer to the problem formulated above.

THEOREM 7. *The congruence relations of the form $\Theta[I]$ form a sublattice of $\Theta(L)$ if*

$$\Theta_{a,b} \cap \Theta_{a,c} = \Theta_{a,b \cap c} \quad \textit{for all} \quad a \leqq b,\ a \leqq c. \tag{23}$$

PROOF. We must prove only

$$\Theta[I_1] \cap \Theta[I_2] = \Theta[I_1 \cap I_2], \tag{24}$$

for the same statement for joins was proved in Theorem 6. Applying Lemma 2, (24) get the following form:

$$\bigvee_{a,\, b \in I_1} \Theta_{a,b} \cap \bigvee_{c,\, d \in I_2} \Theta_{c,d} = \bigvee_{x,\, y \in I_1 \cap I_2} \Theta_{x,y},$$

thus from the infinite distributive law we conclude

$$\bigvee_{a,\, b \in I_1;\, c,\, d \in I_2} (\Theta_{a,b} \cap \Theta_{c,d}) = \bigvee_{x,\, y \in I_1 \cap I_2} \Theta_{x,y}. \tag{25}$$

If $\Theta_{x,y}$ occurs in the right side of (25), i. e., $x, y \in I_1 \cap I_2$, then $\Theta_{x,y}$ occurs in the left side too, i. e.

$$\bigvee_{a,\, b \in I_1;\, c,\, d \in I_2} (\Theta_{a,b} \cap \Theta_{c,d}) \geqq \bigvee_{x,\, y \in I_1 \cap I_2} \Theta_{x,y}.$$

On the other hand, we see that if $t \leqq a \cap b \cap c \cap d$ ($a, b \in I_1$, $c, d \in I_2$), then by (23)

$$\Theta_{a,b} \cap \Theta_{c,d} \leqq \Theta_{a \cup b,\, t} \cap \Theta_{c \cup d,\, t} = \Theta_{(a \cup b) \cap (c \cup d),\, t},$$

where $(a \cup b) \cap (c \cup d) \in I_1 \cap I_2$ and $t \in I_1 \cap I_2$, i. e. every member of the left side is less than or equal to a suitable member of the right side, and so the inequality holds in the reversed sense too, completing the proof of (25).

The condition (23) is not necessary, not even in modular lattices.

As an easy consequence of Theorems 1 and 7 we get:

COROLLARY. *In a distributive lattice the congruence relations of type* $\Theta[I]$ *form a sublattice of* $\Theta(L)$.

PROOF. Let $a \leqq b$ and $a \leqq c$, then $u \equiv v\ (\Theta_{a,b})$ and $u \equiv v\ (\Theta_{a,c})$ under the condition $u \geqq v$ are equivalent to (Theorem 1)

$$(a \cup v) \cap u = v, \tag{26}$$

$$(b \cup v) \cap u = u, \tag{27}$$

$$(c \cup v) \cap u = u. \tag{28}$$

From (27) and (28) by the distributive law

$$u = u \cap u = (b \cup v) \cap u \cap (c \cup v) \cap u = [(b \cap c) \cup v] \cap u. \tag{29}$$

(26) and (29) together mean by Theorem 1 that $u \equiv v\ (\Theta_{a,b \cap c})$. Thus $\Theta_{a,b} \cap \Theta_{a,c} \leqq \Theta_{a,b \cap c}$; the converse inequality is an immediate consequence of $\Theta_{a,b}, \Theta_{a,c} \geqq \Theta_{a,b \cap c}$; the proof is completed.

We remark that this Corollary is an immediate consequence of condition (f) of Theorem 2 too.

The validity of (21) and (22) is assured under a lot of restrictions by

THEOREM 8. *Let* L *be a dual infinite distributive lattice with zero element. Then the congruence relations* $\Theta[I]$ *form a complete sublattice of* $\Theta(L)$, *that is,* (21) *and* (22) *are valid.*

PROOF. It is enough to prove (22). This may be treated in a similar manner as the Corollary of Theorem 7. It suffices to note that the zero element assures the existence of $\bigwedge I_\alpha$, and the dual infinite distributive law is used in the proof of

$$\bigwedge \Theta_{a,b_\alpha} = \Theta_{a,\wedge b_\alpha}$$

which is analogous to (23). We omit here the detailed proof.

§ 4. Remarks on weak projectivity

Let four elements a, b, c, d be given in L such that $\overline{a,b} \to \overline{c,d}$ and $\overline{c,d} \to \overline{a,b}$. Then $a \equiv b$ is equivalent to $c \equiv d$ under every congruence relation. The situation is the same if a, b and c, d are projective[10] (which shall be denoted by $[a,b]\,\Pi\,[c,d]$ if $a \leqq b$ and $c \leqq d$). The following problem arises: give a necessary and sufficient condition under which a, b is

[10] In the literature one speaks about the projectivity of intervals. We say that a, b and c, d are projective if the intervals $[a \cap b,\ a \cup b]$ and $[c \cap d,\ c \cup d]$ are projective in the usual sense. Our definition is more convenient in the sequel.

projective into c, d if and only if

$$\overline{a, b} \to \overline{c, d} \quad \text{and} \quad \overline{c, d} \to \overline{a, b}. \tag{30}$$

Now we consider two classes of lattices in which this condition holds.

THEOREM 9. *Let L be a*

A) *distributive, or*

B) *discrete, modular*

lattice, then a, b and c, d are projective if and only if (30) *holds.*

PROOF. Evidently, in both cases it is enough to verify that (30) implies the projectivity of a, b and c, d.

A) By Theorem 1, (30) is equivalent to (we suppose that $a \leqq b, c \leqq d$)

$$(a \cup c) \cap d = c; \tag{31}$$

$$(b \cup c) \cap d = d; \tag{32}$$

$$(c \cup a) \cap b = a; \tag{33}$$

$$(d \cup a) \cap b = b. \tag{34}$$

Let us prove the equation $b \cup (a \cup c) = d \cup (a \cup c)$, i. e.

$$b \cup c = d \cup a. \tag{35}$$

From (32) $b \cup c \geqq d$ and from $b \geqq a$ we get $b \cup c \geqq d \cup a$ and, on the other hand, from (34) $a \cup d \geqq b$ and from $d \geqq c$ we get $a \cup d \geqq b \cup c$; these inequalities prove (35). The equations (31), (33), (35) show that the consecutive members of the sequence of intervals $[a, b]$, $[a \cup c, b \cup c]$, $[c, d]$ are transposed, that is, $[a, b] \Pi [c, d]$, q. e. d.

We see that we have proved more than it was required by Theorem 9. In addition we get

THEOREM 9′. *$\Theta_{a,b} = \Theta_{c,d}$ in a distributive lattice L if and only if a, b and c, d are projective.*

B) The proof may be decomposed into two assertions. These may be proved by trivial induction, hence the detailed proofs may be omitted.

LEMMA 9. *If L is a modular, discrete lattice and $a, b, c \in L$, $a \leqq b$, then under condition* B *we have*[11]

$$l[a, b] \geqq l[a \cup c, b \cup c] \quad \textit{and} \quad l[a, b] \geqq l[a \cap c, b \cap c],$$

and if a sign of equality holds, then the corresponding intervals are transposed.

PROOF by induction on $l[a, b]$.

[11] $l[a, b]$ denotes the length of a maximal chain from a to b.

LEMMA 10. *If L is a modular, discrete lattice and $a, b, c, d \in L$, $a \leqq b$, $c \leqq d$, $\overline{a, b} \to \overline{c, d}$, then*

$$l[a, b] \geqq l[c, d].$$

PROOF by Lemma 9 and by an induction on the number of steps in the definition of weak projectivity (the number n in (15) and (16)).

The proof of case B) may be completed as follows: if $a \leqq b$ and $c \leqq d$ and condition (30) holds, then $l[a, b] = l[c, d]$ from Lemma 10, hence $[a, b]\, \Pi\, [c, d]$ from Lemma 9, q. e. d.

By repeated use of distributivity it is clear that if in a distributive lattice $\overline{a, b} \to \overline{c, d}$ $(a \geqq b,\ c \geqq d)$, then for suitable p and $q (\in L)$ the following two equations hold (see Theorem 1 too):

$$(a \cup p) \cap q = c, \tag{36}$$

$$(b \cup p) \cap q = d. \tag{37}$$

Now we prove that this property characterizes the distributivity of the lattice L.

THEOREM 10. *The condition $\overline{a, b} \to \overline{c, d}$ $(a \geqq b,\ c \geqq d)$ is equivalent to* (36) *and* (37) *if and only if L is distributive.*

PROOF. The sufficiency of the distributivity is obvious. Therefore we may restrict ourselves to the necessity.

Let us suppose that the stated condition holds. We prove that $c > d \geqq a$ is impossible. Indeed, if $d \geqq a$, then $b \cup p \geqq a$, and so $a \cup p \leqq b \cup p$, that is, $a \cup p = b \cup p$, consequently $c = (a \cup p) \cap q = (b \cup p) \cap q = d$, a contradiction.

It follows from Lemma 6, that $c > d \geqq a$, $c \equiv d$ $(\Theta_{a, b})$ is impossible, hence from condition (a) of Theorem 2 we get the distributivity of L.

III. SEPARABLE CONGRUENCE RELATIONS

§ 1. The definition of separable congruence relations; examples

In this section we introduce the notion of separable congruence relation. This notion will enable us to solve many problems.

DEFINITION 3. Let L be a lattice and Θ a congruence relation on L. Θ is separable if to all $a \leqq b$ in L there exists a chain $a = z_0 \leqq z_1 \leqq \cdots \leqq z_n = b$ such that for each i either $z_{i-1} \equiv z_i$ (Θ) or ($z_{i-1} \not\equiv z_i (\Theta)$ and) $x, y \in [z_{i-1}, z_i]$, $x \equiv y$ (Θ) imply $x = y$.

We also say that this chain $\{z_i\}$ is separated modulo Θ, or Θ separates the chain $\{z_i\}$, or a and b are separated modulo Θ by the chain $\{z_i\}$.

We get immediately from the definition:

LEMMA 11. *If the lattice L is semi-discrete, then all congruence relations on L are separable.*

Now let us consider an example[12] of a non-separable congruence relation. Let L be the chain of all positive integers, together with $+\infty$. We define $x \equiv y$ (Θ) if and only if $x = 2i$, $y = 2i+1$ for some $i = 1, 2, \ldots$. Obviously, Θ is non-separable, e. g. no chain separates 1 and $+\infty$.

From the definition it is also clear that if Θ is separable, then between all a, b $(a \geqq b)$ there exists a maximal chain such that on this chain there is but a finite number of congruence classes with more than one element under Θ. Indeed, every maximal chain which refines a separating chain has the required property.

The converse statement is in general not true. It is neither true that if Θ is a congruence relation such that between all $a > b$ there is a maximal chain with the property described above, then Θ is necessarily separable. The statement: if Θ is separable, then all maximal chains between any $a \geqq b$ have the property described above, is also false. Counterexamples may be found in § 4 of this Part, see examples (A) and (B).

Some typical examples on separable and non-separable congruence relation will be shown by the following lemmas.

LEMMA 12. *Let L be a lattice with the greatest element* 1, *and I a neutral ideal*[13] *of L. $\Theta[I]$ is separable if and only if I is a principal ideal.*

PROOF. If 1 and $y (\in I)$ are separated under $\Theta[I]$ $(I \neq L)$ by the chain $\{z_i\}$ $(i = 1, \ldots, n)$, then it may be supposed that $z_1 = y$, $z_3 = 1$ $(n = 3)$. There is no subinterval of $[z_2, 1]$ which is congruent under $\Theta[I]$, thus z_2 is the generating element of I. On the other hand, if $I = (a]$, then $x \geqq y$ may be separated under $\Theta[I]$ by the chain $y \leqq x \cap (y \cup a) \leqq x$.[14]

The following is a significant example of a non-separable congruence relation of distributive lattices.

LEMMA 13. *Let an infinite sequence of elements $a = a_1 < b_1 < \cdots < a_i < < b_i < \cdots < b$ be given in the distributive lattice L. Then $\bigvee_{i=1}^{\infty} \Theta_{a_i, b_i}$ is not separable.*

[12] This example is generalized by Lemma 13.

[13] The ideal I of the lattice L is called neutral if for any ideals J, K of L, the sublattice of the lattice of all ideals of L generated by I, J, K is distributive. If I is neutral, then $x \equiv y$ under $\Theta[I]$ if and only if $(x \cap y) \cup i \geqq x \cup y$ for some $i \in I$. For this fact we refer to [2], pp. 28, 79, 119 and 124, or to [14], p. 167.

[14] If in Lemma 14 we omit the condition that L has a unit element, then the assertion does not remain valid.

PROOF. Suppose that $\Theta = \bigvee \Theta_{a_i, b_i}$ is separable, and let $\{z_i\}$ be a chain which separates a and b $(a = z_0 < z_1 < \cdots < z_n = b)$. If $z_{i-1} \equiv z_i$ (Θ), then $z_{i-1} \equiv z_i (\bigvee_{l=1}^{n} \Theta_{a_{j_l}, b_{j_l}})$, that is, already a *finite number* of the $[a_i, b_i]$ generates all congruences on the chain $\{z_i\}$. Let $[a_t, b_t]$ be an interval different from the above ones. Let Θ'_{a_t, b_t} be the complement of Θ_{a_t, b_t} (see condition (d) of Theorem 2), then $a \equiv b$ $(\Theta_{a_t, b_t} \cup \Theta'_{a_t, b_t})$, $a \not\equiv b$ (Θ'_{a_t, b_t}) (for $a_t \not\equiv b_t$ (Θ'_{a_t, b_t})). Hence for a suitable index j we have $z_{j-1} \not\equiv z_j$ (Θ'_{a_t, b_t}). According to Lemma 5 applied to $z_{j-1} \equiv z_j$ $(\Theta'_{a_t, b_t} \cup \Theta_{a_t, b_t})$, there is a pair of elements u, v such that $z_{j-1} \leqq u < v \leqq z_j$ and $u \equiv v$ (Θ_{a_t, b_t}). On the other hand, $z_{j-1} \equiv z_j$ (Θ), that is, $u \equiv v$ (Θ) whence $u \equiv v$ $(\bigvee_{l=1}^{n} \Theta_{a_{j_l}, b_{j_l}})$. Comparing this with the above congruence we get $u \equiv v$ $(\Theta_{a_t, b_t} \cap \bigvee_{l=1}^{n} \Theta_{a_{j_l}, b_{j_l}})$, that is, $u \equiv v$ $(\bigvee_{l=1}^{n} (\Theta_{a_{j_l}, b_{j_l}} \cap \Theta_{a_t, b_t}))$. From the conditions of the Lemma and from $[a_{j_l}, b_{j_l}] \neq [a_t, b_t]$, we get for each l either $a_{j_l} < b_{j_l} < a_t < b_t$ or $a_t < b_t < a_{j_l} < b_{j_l}$. Thus by condition (c) of Theorem 2 we get $\Theta_{a_{j_l}, b_{j_l}} \cap \Theta_{a_t, b_t} = \omega$. Hence the above congruence becomes $u \equiv v$ (ω), i. e. $u = v$, in contradiction to $u < v$. The proof is completed.

Now we prove

LEMMA 14. *The separable congruence relations on L form a sublattice $\Theta_s(L)$ of $\Theta(L)$. $\Theta_s(L)$ contains* ι and ω.

PROOF. It is clear that ι and ω are separable, so $\Theta_s(L)$ is not the void set. Furthermore, let $\Theta, \Phi \in \Theta_s(L)$, and let $a \geqq b$ $(a, b \in L)$. The chain $\{z_i\}_i$ separates a and b modulo Θ, and let $\{u_{ij}\}_j$ be a chain which separates z_i and z_{i-1} modulo Φ. A rather simple computation shows that the chain $\{u_{i,j}\}_{i,j}$ separates a and b modulo $\Theta \cup \Phi$ as well as modulo $\Theta \cap \Phi$, completing the proof.

$\Theta(L)$ is distributive, hence its center $\Theta_z(L)$ is the set of all congruence relations having a complement. It is well known that $\Theta_z(L)$ is a sublattice of $\Theta(L)$. (It is trivial from the identities $(\Theta \cup \Phi)' = \Theta' \cap \Phi'$ and $(\Theta \cap \Phi)' = \Theta' \cup \Phi'$.)

LEMMA 15. *If the congruence relation Θ has a complement, then it is separable, that is, $\Theta_z(L) \subseteq \Theta_s(L)$.*

PROOF. By Lemma 5, to all $a > b$ there exists a chain $a = z_0 \geqq \cdots \geqq z_n = b$ such that either $z_i \equiv z_{i-1}$ (Θ) or $z_i \equiv z_{i-1}$ (Θ') for every i. We assert that the chain $\{z_i\}$ separates a and b modulo Θ. Indeed, if $x, y \in [z_i, z_{i-1}]$ and $z_i \not\equiv z_{i-1}$ (Θ), furthermore $x \equiv y$ (Θ), then from $x \equiv y$ (Θ') we get $x \equiv y$ $(\Theta \cap \Theta')$, that is, $x = y$, q. e. d.

COROLLARY. *In a distributive lattice all congruence relations of the form $\Theta_{a,b}$ are separable.*

This is an immediate consequence of condition (d) of Theorem 2 and of Lemma 15.

§ 2. Weakly modular lattices and G. Birkhoff's problem 72

First of all we introduce the notion of weakly modular lattices. It plays an important role in the discussion of problem 72 as well as in our researches concerning the so-called standard ideals (see [8]).

DEFINITION 4. The lattice L is weakly modular if $\overline{a, b} \to \overline{c, d}$ ($a < b, c \neq d, a, b, c, d \in L$) implies the existence of elements a_1, b_1 ($a \leqq a_1 < b_1 \leqq b$) such that $\overline{c, d} \to \overline{a_1, b_1}$.

The weakly modular lattices are a common generalization of the modular and relatively complemented lattices[15] as it is assured by

LEMMA 16. *If L is a*

(a) *modular, or*

(b) *relatively complemented*

lattice, then it is weakly modular.

PROOF. The case (a) is an immediate consequence of the isomorphism theorem for modular lattices (we refer to [2], p. 73). Now consider case (b). Let $a > b$ and $a \cap x > y \geqq b \cap x$. Then denoting by z the relative complement of y in the interval $[b \cap x, a \cap x]$, we have $\overline{a \cap x, y} \to \overline{b, b \cup z}$ and $b < b \cup z \leqq a$, for $[(a \cap x) \cap z] \cup b = b \cup z$ and $(y \cap z) \cup b = b$. In the same lines it may be proved that in case $a > b, a \cup x \geqq y > b \cup x$, we have $\overline{b \cup x, y} \to \overline{a_1, a}$ for

[15] The necessity of a common generalization of modular and relatively complemented lattices has arisen in many cases. Let us consider an illustrative example. DILWORTH and HALL [4] proved — generalizing a theorem of G. BIRKHOFF — that every weakly atomic (a lattice is called weakly atomic, if any $a > b$ implies the existence of c, d with $a \geqq c \succ d \geqq b$) modular lattice is the subdirect product of projective lattices (a projective lattice is a lattice in which all prime intervals are projective). J. HASHIMOTO [13] proved a similar result for relatively complemented lattices. Thus the necessity of a theorem arises which is a common generalization of the above mentioned ones. Let us call the lattice L weakly projective if for any pair of prime intervals p, q the relations $p \to q$ and $q \to p$ hold (the notations are that of § 3). Obviously, any weakly projective modular or relatively complemented lattice is projective. We assert: Any weakly atomic, weakly modular lattice is a subdirect product of weakly projective lattices. The proof goes on the same lines as the proof of the assertion of DILWORTH and HALL, or, what is the same, the proof of J. HASHIMOTO. This is also a consequence of Lemma 18.

suitable $a > a_1 \geqq b$. The proof may be completed by an easy induction on n of Definition 1.

From Lemma 16 it is also clear that the weakly modular lattices generalize the modularity in another way than the semi modularity. We remark that by the Corollary 2 of Lemma 6 all simple lattices are also weakly modular.

An important property of weakly modular lattices is proved in

LEMMA 17. *Let L be a weakly modular lattice and Θ a congruence relation on L. Define $x \equiv y$ (Θ^*) if and only if in the interval $[x \cap y, x \cup y]$ every congruence class under Θ consists of a single element. Then Θ^* is a congruence relation, furthermore Θ^* is the pseudo-complement*[16] *of Θ in $\Theta(L)$.*

PROOF. Owing to the definition of Θ^*, it is reflexive and satisfies the condition (b) of Lemma 4. Let $u > v > w, u \equiv v$ (Θ^*) and $v \equiv w$ (Θ^*) and let us suppose that for some $u \geqq x > y \geqq v$ we have $x \equiv y$ (Θ). Since $x \equiv y$ $(\Theta_{u,v} \cup \Theta_{v,w})$, from Lemma 5 it follows the existence of x_1, y_1 such that $x \geqq x_1 > y_1 \geqq y$ and either $\overline{u, v} \rightarrow \overline{x_1, y_1}$ or $\overline{v, w} \rightarrow \overline{x_1, y_1}$. From the weak modularity it results that $\overline{x_1, y_1} \rightarrow \overline{v_1, w_1}$ for some $v \geqq v_1 > w_1 \geqq w$ or $\overline{x, y} \rightarrow \overline{u_1, v_1}$ for some $u \geqq u_1 > v_1 \geqq v$. But $x \equiv y$ (Θ) implies $v_1 \equiv w_1$ (Θ) or $u_1 \equiv v_1$ (Θ), in contradiction to $v \equiv w$ (Θ^*) or to $u \equiv v$ (Θ^*). The cases $u = v$ and $v = w$ are trivial. Finally, we prove that $x \geqq y$ and $x \equiv y$ (Θ^*) imply $x \cup t \equiv y \cup t$ (Θ^*). Indeed, if $x \cup t \equiv y \cup t$ (Θ^*) is not true, then $u \equiv v$ (Θ) is valid for some $x \cup t \geqq u > v \geqq y \cup t$. From the weak modularity it follows that $\overline{u, v} \rightarrow \overline{x_1, y_1}$ for some $x \geqq x_1 > y_1 \geqq y$, that is, $x \equiv y$ (Θ^*) is false. Thus we have proved the validity of the conditions (a)—(d) of Lemma 4, and so Θ^* is a congruence relation. The last assertion of the lemma is clear.

COROLLARY. *Any separable congruence relation of a weakly modular lattice has a complement, that is, $\Theta_s(L) = \Theta_z(L)$.*

PROOF. Let Θ be separable; we assert that the congruence relation Θ^* of Lemma 17 is the complement of Θ. Indeed, if $a \geqq b$ $(a, b \in L)$, then let $a = z_0 \geqq z_1 \geqq \cdots \geqq z_n = b$ be a chain which separates a and b modulo Θ. If $z_i \not\equiv z_{i-1}$ (Θ), then by the definition of Θ^* it follows $z_i \equiv z_{i+1}$ (Θ^*), whence $a \equiv b$ $(\Theta \cup \Theta^*)$, completing the proof of $\Theta \cup \Theta^* = \iota$.

Now we proceed to problem 72 of G. BIRKHOFF (see [2], p. 153):

Find necessary and sufficient conditions on a lattice L that its congruence relations should form a Boolean algebra.

[16] Let L be a lattice with 0. The element a^* is called the pseudo-complement of a if $x \cap a = 0$ is equivalent to $x \leqq a^*$.

First T. TANAKA [18] gave an answer to this question.[17] He got the following interesting theorem which is a generalization of a theorem of R. P. DILWORTH [3]:

The congruence relations of the lattice L form a Boolean algebra if and only if L is a discrete subdirect product of simple lattices. (Discrete subdirect product is a subdirect product in which any two elements differ only in a finite number of components.)

The theorem of T. TANAKA may be considered as the structure theorem of lattices L for which $\Theta(L)$ is a Boolean algebra. However, in some respects the following theorem is more applicable to interesting special cases:

THEOREM 11. *The congruence relations of the lattice L form a Boolean algebra if and only if*

(W) *L is weakly modular*

and

(S) *all congruence relations on L are separable.*

PROOF.

The necessity of the conditions. Let us suppose that $\Theta(L)$ is a Boolean algebra for the lattice L. Then by Lemma 15 all congruence relations are separable, hence (S) is necessary.

Let us suppose that $\overline{a, b} \rightarrow \overline{c, d}$ $(a > b, c \neq d)$. $\Theta_{c,d}$ has a complement, let us denote it by Φ. Now, $a \equiv b$ $(\Theta_{c,d} \cup \Phi)$, but in case $a \equiv b$ (Φ), it follows that $c \equiv d$ (Φ) which is impossible, so $a \not\equiv b$ (Φ). Thus Lemma 5 implies that for some $a \geqq a_1 > b_1 \geqq b$ the relation $a_1 \equiv b_1$ $(\Theta_{c,d})$ holds. By Lemma 6 this implies that for some $a_1 \geqq a_2 > b_2 \geqq b_1$, $\overline{c, d} \rightarrow \overline{a_2, b_2}$ is valid, thus in case $\Theta(L)$ is complemented, (W) holds.

The sufficiency of the conditions. By (W), $\Theta_z(L) = \Theta_s(L)$, as it was proved in the Corollary of Lemma 17. Condition (S) is equivalent to $\Theta(L) = \Theta_s(L)$, thus $\Theta(L) = \Theta_z(L)$, as we wished to prove.

We get from Theorem 11 a lot of Corollaries.

COROLLARY 1. *The lattice of all congruence relations of a*

(a) *modular, or*

[17] The result of T. TANAKA remains valid in abstract algebras, too, this explains that for lattices one can get sharper results. We note that while the result of T. TANAKA depends on the Axiom of Choice, our result does not use it.

(b) *relatively complemented*[18]

lattice is a Boolean algebra if and only if the condition (S) *holds.*

It follows from Theorem 11 and from Lemma 16.

In case the distributivity of L is assumed, we can get further improvements.

COROLLARY 2 (Theorem of J. HASHIMOTO [14]). *The lattice of all congruence relations of a distributive lattice L is a Boolean algebra if and only if L is discrete.*

PROOF. By Corollary 1 it is enough to prove that in distributive lattices (S) is equivalent to the discreteness of L. Indeed, if L is discrete, then by Lemma 11 (S) holds. On the other hand, if L is not discrete, then by the usual method of bisection of intervals we get a sequence of elements required in Lemma 13, so that, by this lemma it follows the existence of a nonseparable congruence relation, that is, condition (S) is false.[19]

In case of modular complemented lattices, SHIH-CHIANG WANG got a condition for the complementedness of $\Theta(L)$.

COROLLARY 3 (Theorem of SHIH-CHIANG WANG [20]). *The lattice of all congruence relations of a complemented modular lattice is a Boolean algebra if and only if all neutral ideals are principal.*

PROOF. By a theorem of G. BIRKHOFF, every congruence relation of a complemented modular lattice is a minimal congruence relation of a neutral ideal. By Lemma 12, the minimal congruence relation of a neutral ideal is separable if and only if the ideal is principal, and so Corollary 3 follows.

It is surprising that Corollary 3 which seems to be true only in complemented modular lattices remains true after omitting the condition of modularity, provided that we replace neutral ideals by standard[20] ones. In [8] we proved that every congruence relation of a relatively complemented lattice with 0 and 1 is a minimal congruence relation of a standard ideal, thus the proof of Corollary 3 may be applied to establish

[18] The results of this section were published in [7] in 1957. At the same time, J. HASHIMOTO published in [13] the following result: If in L the restricted chain condition holds (that is, in every (closed) interval of L the maximum *or* the minimum condition holds) and L is relatively complemented, then $\Theta(L)$ is a Boolean algebra. Indeed, the restricted chain condition is a special case of semi-discreteness, further, on any semi-discrete lattice all congruence relations are separable (Lemma 11), thus the assertion follows from the part (b) of Corollary 1.

[19] For a direct proof of Corollary 2 see our paper [12].

[20] We have introduced the notion of standard ideals in [8]. Among the more than seven equivalent definitions now we formulate only the following two: (a) the ideal I is called standard if for any ideals J, K of L the relation $J \cap (I \cup K) = (J \cap I) \cup (J \cap K)$ holds;

COROLLARY 4. *The lattice of all congruence relations of a relatively complemented lattice with 0 and 1 is a Boolean algebra if and only if all standard ideals are principal.*

We get other types of Corollaries if we restrict our consideration to discrete or semi-discrete lattices. In semi-discrete lattices (S) is valid (Lemma 11). We prove that in semi-discrete lattices (W) is equivalent to

(J) weak projectivity between prime intervals is symmetric.

(The interval $[a, b]$ is called prime if b covers a.) Indeed, if (W) holds and b covers a, d covers c, $\overline{a, b} \rightarrow \overline{c, d}$, then for some $b \geqq b_1 > a_1 \geqq a$, $\overline{c, d} \rightarrow \overline{a_1, b_1}$ is valid, but from the covering relations we infer $b = b_1$ and $a = a_1$, hence (J) is valid. On the other hand, assume the validity of (J), and let $\overline{a, b} \rightarrow \overline{c, d}$, $a < b$. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a finite maximal chain between a and b. Then $c \equiv d \ (\bigcup_{i=1}^{n} \Theta_{x_i, x_{i-1}})$, so that by Lemmas 4 and 5, for some $c \cap d \leqq c_1 \prec d_1 \leqq c \cup d$ and for some i, $\overline{x_{i-1}, x_i} \rightarrow \overline{c_1, d_1}$ is valid. But then by (J) $\overline{c_1, d_1} \rightarrow \overline{x_i, x_{i-1}}$ and the assertion follows. Thus we have

COROLLARY 5. *The lattice of all congruence relations of a semi-discrete lattice is a Boolean algebra if and only if the relation of weak projectivity between prime intervals is symmetric.*

Corollary 5 in case of discrete lattices was firstly proved by J. JAKUBIK [15]. We shall weaken the conditions of Corollary 5 in the following section.

§ 3. Special properties of $\Theta(L)$

If we can construct from the lattice L a new lattice, then it is always interesting to characterize those lattices for which the new lattice has some special properties. So, for instance, the characterization of those lattices for which $\Theta(L)$ is a Boolean algebra was the content of § 2. Now we consider further problems of this kind.

(b) the ideal I is said to be standard if $x \equiv y$ under $\Theta[I]$ if and only if $(x \cap y) \cup t = x \cup y$ for some $t \in I$. From (a) it is clear that the notion of standard ideals is a generalization of the neutral ideals; from (b) we see that the proof of Lemma 12 for standard ideals remains valid.

In [8] we have proved Corollary 4 in another way. We can sharpen Corollary 4, for in [8] we have proved that in a weakly modular lattice all standard elements are neutral and in a relatively complemented lattice all ideals which are congruence classes under some congruence relation are standard, thus we get: the lattice of all congruence relations of a relatively complemented lattice with 0 and 1 is a Boolean algebra if and only if every ideal which is a congruence class under some congruence relation is a principal ideal.

We know that in $\Theta(L)$ the infinite distributive law

(ID) $\Theta \cap \bigvee \Theta_\alpha = \bigvee(\Theta \cap \Theta_\alpha)$

holds, but, as it was pointed out by N. FUNAYAMA and T. NAKAYAMA [5], the dual law

(DID) $\Theta \cup \bigwedge \Theta_\alpha = \bigwedge(\Theta \cup \Theta_\alpha)$

does not hold in general. Let us consider the lattices in which (DID) does hold. First we prove

LEMMA 18. *Let Θ be a separable congruence relation of L. Then for any subset A of $\Theta(L)$*

$$\Theta \cup \bigwedge_{\Theta_\alpha \in A} \Theta_\alpha = \bigwedge_{\Theta_\alpha \in A} (\Theta \cup \Theta_\alpha)$$

is valid.

PROOF. Since $\Theta \cup \bigwedge \Theta_\alpha \leqq \bigwedge(\Theta \cup \Theta_\alpha)$ is true in any complete lattice, it is enough to verify that $\Theta \cup \bigwedge \Theta_\alpha \geqq \bigwedge(\Theta \cup \Theta_\alpha)$. Let $x \equiv y$ $(\bigwedge(\Theta \cup \Theta_\alpha))$; since Θ is separable, there exists a chain $x \cup y = z_0 \geqq z_1 \geqq \cdots \geqq z_n = x \cap y$ separating $x \cup y$ and $x \cap y$ modulo Θ. If $z_i \not\equiv z_{i-1}$ (Θ) for some i, then from $z_{i-1} \equiv z_i$ $(\bigwedge(\Theta \cup \Theta_\alpha))$ we get $z_i \equiv z_{i-1}$ $(\bigwedge \Theta_\alpha)$. Thus for every i either $z_i \equiv z_{i-1}$ (Θ) or $z_i \equiv z_{i-1}$ $(\bigwedge \Theta_\alpha)$, that is, $x \equiv y$ $(\Theta \cup \bigwedge \Theta_\alpha)$, which we intended to prove.

COROLLARY. *If all congruence relations on L are separable, then* (DID) *holds unrestrictedly.*

Lemma 18 or its Corollary may not be conversed, as it will be shown in § 4 by a counterexample (example (C)).

Now we characterize the distributive lattices L such that in $\Theta(L)$ (DID) holds.

THEOREM 12. *In the lattice of all congruence relations of a distributive lattice* (DID) *holds unrestrictedly if and only if L is discrete.*[21]

PROOF. If L is discrete, then all congruence relations on L are separable by Lemma 11, thus, by the Corollary of Lemma 18, (DID) holds in $\Theta(L)$.

On the other hand, assume that (DID) holds in $\Theta(L)$. Let $\Theta \in \Theta(L)$. then $\Theta = \bigvee_{a \equiv b(\Theta)} \Theta_{a,b}$. By condition (d) of Theorem 2 any $\Theta_{a,b}$ has a complement $\Theta'_{a,b}$. Put $\Phi = \bigwedge_{a \equiv b\,(\Theta)} \Theta'_{a,b}$, then by (ID)

$$\Theta \cap \Phi = \bigvee \Theta_{a,b} \cap (\bigwedge \Theta'_{a,b}) = \bigvee(\Theta_{a,b} \cap \bigwedge \Theta'_{a,b}) \leqq \bigvee(\Theta_{a,b} \cap \Theta'_{a,b}) = \bigvee \omega = \omega,$$

hence $\Theta \cap \Phi = \omega$, and from (DID)

$$\Theta \cup \Phi = \bigvee \Theta_{a,b} \cup (\bigwedge \Theta'_{a,b}) = \bigwedge(\Theta'_{a,b} \cup \bigvee \Theta_{a,b}) \geqq \bigwedge(\Theta'_{a,b} \cup \Theta_{a,b}) = \iota,$$

[21] A simple proof of Theorem 12 was published in our paper [12].

(the $\bigwedge$ and $\bigvee$ are extended to all a, b with $a \equiv b$ (Θ) in all above formulae), thus $\Theta \cup \Phi = \iota$, that is, Φ is the complement of Θ. Thus $\Theta(L)$ is a Boolean algebra, hence from Corollary 2 of Theorem 11 L is discrete and the theorem follows.

*

Now we consider a question related to problem 67 of G. BIRKHOFF [2].

Let P be the set of all prime intervals of the lattice L; the elements of P are denoted by p, q. If $p = [a, b]$ and $q = [c, d]$, furthermore $\overline{a, b} \to \overline{c, d}$, then we write $p \to q$. The elements of P under the relation $\to$ are quasi-ordered, thus if we identify those p, q for which $p \to q$ and $q \to p$ simultaneously, then we get a partially ordered set which will be denoted also by P. Now we are seeking for a condition under which $\Theta(L) \simeq \mathbf{2}^P$. ($\mathbf{2}$ denotes the lattice of two elements. The definition of $\mathbf{2}^P$ may be found in [2], p. 8.)

LEMMA 19. *For any lattice L, $\mathbf{2}^P$ is a complete homomorphic image of $\Theta(L)$.*

PROOF. We say that the congruence relation Θ collapses the prime interval p, if $p = [a, b]$ and $a \equiv b$ (Θ). We call a subset A of P s-ideal, if $p \in A$ and $p \to q$ imply $q \in A$. We assert that every s-ideal may be regarded as the set of all prime intervals collapsed by some congruence relation. Indeed, if Θ is a congruence relation, then the set of all collapsed prime intervals A form an s-ideal, for if $p \in A$ and $p \to q$, then q is also collapsed by Θ. On the other hand, let A be an s-ideal of P, and let us define $\Theta = \bigvee_{p=[a, b] \in A} \Theta_{a, b}$. Under Θ the prime intervals of A are collapsed, furthermore if q is collapsed by Θ, then $q = [a, b]$, $a \equiv b$ (Θ), thus, by Lemmas 4 and 5, $p \to q$ for some $p \in A$, hence $q \in A$.

In a similar way we get that if under Φ and Θ the collapsed prime intervals are A_Φ and A_Θ, respectively, then under $\Theta \cup \Phi$ and $\Theta \cap \Phi$ the collapsed prime intervals are $A_\Phi \cup A_\Theta$ and $A_\Phi \cap A_\Theta$, where $\cup$ and $\cap$ denote the set theoretical meet and join. Thus the set B of all s-ideals of P, partially ordered under set-inclusion is a homomorphic, moreover, a complete homomorphic image of $\Theta(L)$ (naturally the void set is also regarded as an s-ideal). It is evident that B is isomorphic to $\mathbf{2}^P$, completing the proof of Lemma 19.

A trivial condition concerning the problem under discussion follows from Lemma 19.

THEOREM 13. *The isomorphism $\Theta(L) \simeq \mathbf{2}^P$ holds if and only if to any pair $\Theta > \Phi$ $(\Theta, \Phi \in \Theta(L))$ there exists some $p \in P$ collapsed by Θ but not by Φ.*

PROOF. Since $\mathbf{2}^P$ is a homomorphic image of $\Theta(L)$, the condition of Theorem 13 is necessary and sufficient in order that this homomorphism may be an isomorphism. Q. e. d.

As a trivial consequence of Theorem 13 we get immediately a sharpened form of a theorem of J. JAKUBIK [15] (he restricts himself to discrete lattices; in § 4 we prove by examples that the following Corollary is more sharpen than JAKUBIK's theorem):

COROLLARY 1. *If L is a semi-discrete lattice, then* $\Theta(L) \simeq \mathbf{2}^P$.

Instead of proving it we shall verify a more general assertion.

COROLLARY 2. *Let L be a weakly atomic lattice with separable congruence relations. Then* $\Theta(L) \simeq \mathbf{2}^P$.

PROOF. It is enough to prove that if $\Theta > \Phi$, then there exists a prime interval p which is collapsed by Θ but not by Φ. As a matter of fact, there exists a pair of elements a, b with $a > b$, $a \equiv b$ (Θ) and $a \not\equiv b$ (Φ), and there exists a chain which separates a and b modulo Φ; let $a = z_0 \geqq \cdots \geqq z_n = b$ be this chain. Choose an index i for which $z_i \not\equiv z_{i-1}$ (Φ). Then no subinterval of $[z_i, z_{i-1}]$ is congruent under Φ. By weak atomicity there is a prime interval p in $[z_i, z_{i-1}]$; thus p is not collapsed by Φ but is collapsed by Θ, completing the proof.

Theorem 13 and Corollary 2 may be regarded as a general solution of G. BIRKHOFF's problem 67.

From Corollary 1 one can deduce Corollary 5 of Theorem 11 using only the fact that $\mathbf{2}^P$ is a Boolean algebra if and only if P is unordered.[22] Thus from Corollary 2 of Theorem 13 we get a generalization of Corollary 5 of Theorem 11:

Let L be a weakly atomic lattice with separable congruence relations. $\Theta(L)$ is a Boolean algebra if and only if weak projectivity is a symmetric relation among its prime intervals.

§ 4. Counterexamples

Now we construct some counterexamples to questions raised in Part III.

(A) There is a lattice having a congruence relation Θ and a maximal chain C such that Θ induces on C an infinity of congruence classes of more than one element.

[22] Let us prove that $\mathbf{2}^P$ is a Boolean algebra if and only if P is unordered. Indeed, if P is unordered and $f \in \mathbf{2}^P$, i. e. f is an isotone function from P to $\mathbf{2}$, then define g by $g(a) = 0$ if $f(a) = 1$ and $g(a) = 1$ if $f(a) = 0$. Obviously, g is the complement of f in $\mathbf{2}^P$. On the other hand, if $x, y \in P$ and $x > y$, then consider the function f for which $f(x) = 1$, $f(y) = 0$. If g is the complement of f, then $\max(f(a), g(a)) = 1$ for all $a \in P$, that is, $g(y) = 1$, $\min(f(a), g(a)) = 0$ for all $a \in P$, whence $g(x) = 0$, g is not isotone; a contradiction.

EXAMPLE. Let P be the chain of all non-positive integers together with $-\infty$, with the natural ordering. In the cardinal product of P with itself let us consider the congruence relation $\Theta=\Theta_{(0,0),(-\infty,0)}$ and a maximal chain C which consists of all elements of type (x,x). By the Corollary of Lemma 15 Θ is separable, yet on the chain C it induces an infinity of congruence classes with more than one element.

(B) There exists a lattice L on which there is a non-separable congruence relation Θ with the property that any a, b $(a \geqq b)$ may be connected by a maximal chain on which there is but a finite number of congruence classes of more than one element.

EXAMPLE. Let P be the chain of all non-negative integers and let L be the lattice $P \cdot P$ bounded with I, and $\Theta=\bigvee_{i=1}^{\infty} \Theta_{(2i,0)(2i+1,0)}$. By Lemma 13, Θ is non-separable. Let $a>b$. We may suppose $a=I$ unless $[b,a]$ is finite. If $a=I$, then a chain with the required properties is formed by the elements (b_1, x), where $b=(b_1, b_2)$ and x runs over the numbers $b_2, b_2+1, b_2+2, \ldots$.

It is of some interest that examples (A) and (B) could be constructed among distributive lattices.

(C) There is a lattice L with the property that in $\Theta(L)$, although the dual infinite distributive law unrestrictedly holds, yet there are non-separable congruence relations.

EXAMPLE. Let P be again the set of all non-negative integers and let L consist of P and of three new elements I, x, y. L will be a lattice if the partial ordering of P remains the usual and the following relations hold:

$$x \cup i=y \cup i=x \cup y=x \cup I=y \cup I=I,$$
$$x \cap i=y \cap i=I \cap 0=0 \quad \text{for all} \quad i \in P.$$

Let us have a look over the congruence relations of this lattice L. It is easy to verify that $I \equiv i$ and $i \equiv 0$ $(i \neq 0)$ hold only under ι. This implies that with the exception of ι all congruence relations of L are those of P, in the sense that the congruence relations of P are extended to congruence relations of L such that the congruence classes outside P consist of one element only. In $\Theta(P)$ the law (DID) is satisfied as we proved it in Theorem 12, thus a trivial calculation shows its validity in $\Theta(L)$ too. Yet in $\Theta(L)$ there are non-separable congruence relations, for instance, let $x \equiv y$ (Θ) if and only if $x=2i+1$ and $y=2i$ (i is arbitrary, $i \in P$, $i \neq 0$), then one cannot separate e. g. 1 and I.

Naturally, all counterexamples of type (C) are non-distributive, for if L were distributive, then by Theorem 12 it would follow that L is discrete, hence by Lemma 11 all congruence relations on L are separable.

(D) There is a semi-discrete but not discrete lattice L, with the property that $\Theta(L)$ is a Boolean algebra.[23]

EXAMPLE. Let L be the set of all non-negative integers. We partially order L by putting

$$\left.\begin{aligned} 2i-1 &< 2i < 0, \\ 2i-1 &< 2i+1 \end{aligned}\right\} \qquad (i=1,2,\ldots).$$

Then L is a lattice which is obviously non-discrete but semi-discrete, furthermore L is simple, that is, it has all the required properties.

(E) There is a weakly atomic, not semi-discrete lattice L with separable congruence relations such that $\Theta(L)$ is a Boolean algebra.[24]

EXAMPLE. Let L be the lattice of all partitions of an infinite set. Then L is a simple, weakly atomic lattice (for the proof we refer to O. ORE [17]), thus it satisfies the required properties.

IV. BOOLEAN RING OPERATIONS ON DISTRIBUTIVE LATTICES

§ 1. A characterization of relatively complemented distributive lattices

In this section we prove a theorem which enables us to prove the main theorem of this part without complicated computations.

Let

$$\left.\begin{aligned} &f_i(u_1,\ldots,u_n,x_1,\ldots,x_m) \\ \text{and}\quad &\psi_i(u_1,\ldots,u_n,x_1,\ldots,x_m) \end{aligned}\right\} \qquad (i=1,2,\ldots,k)$$

be lattice-polynomials with the variables x_j.

THEOREM 14. *In a relatively complemented lattice L the system of equations*

$$f_i=\psi_i \qquad (i=1,2,\ldots,k) \tag{38}$$

has a solution for any

$$u_1=a_1,\ldots,u_n=a_n \qquad (a_j\in L;\, j=1,2,\ldots,n)$$

if and only if (38) *has a solution in* **2** *for any*

$$u_1=b_1,\ldots,u_n=b_n \qquad (b_j\in \mathbf{2};\, j=1,2,\ldots,n).$$

We remark that if $m=0$ (that is the set of unknowns is void), then (38) is a system of identities, the validity of which is in question.

[23] Example (D) shows that Corollary 5 of Theorem 11 is applicable to more lattices than the original theorem of J. JAKUBIK.

[24] (E) shows that the assertion formulated at the end of § 3 is actually stronger than Corollary 5 of Theorem 11.

First we prove

LEMMA 20. *Let L be a relatively complemented distributive lattice and $x_1, \ldots, x_n \in L$. L has a sublattice B_n which is a finite Boolean algebra containing $x_1, \ldots, x_n$ (and $O(B_n) \leqq 4^n$).*[25]

PROOF. The assertion for $n=1$ is true. Now we make an induction on n. Let us suppose that we have already constructed B_{n-1} which contains $x_1, \ldots, x_{n-1}$. Let O_{n-1}, I_{n-1} be the least and greatest elements of B_{n-1}, respectively. Let us consider in the interval $[O_{n-1}, u_n \cup I_{n-1}]$ the relative complement I'_{n-1} of I_{n-1} and let A_1 and A_2 be sublattices of L consisting of O_{n-1}, I'_{n-1} and $x_n \cup I_{n-1}, x_n$, respectively. Let $B_n = (B_{n-1} \cdot A_1) \circ A_2$ where $\circ$ denotes the cardinal product, but if B_n is regarded as a sublattice of L, then the embedding B_n in L is effected by $(x, y) \rightarrow x \cap y$. Obviously, B_n is a finite Boolean algebra and $x_1, \ldots, x_n \in B_n$. The calculation on the number of the elements of B_n is very easy by the construction.

Now we prove Theorem 14.

The necessity of the condition. Consider a finite Boolean algebra B_{n+m} containing the elements $x_1, \ldots, x_m$; $a_1, \ldots, a_n$. (38) is solvable in B_{n+m}, thus it is solvable in **2** too. Any choice of $\bar{a}_i$ may be regarded as a homomorphic image of a suitable chosen a_i.

The sufficiency of the condition. Let us suppose that (38) may be solved for some $x_i = b_i$ in **2**. Then (38) is solvable in all finite Boolean algebras, for (38) is solvable componentwise. Let B_n be a finite Boolean algebra containing $a_1, \ldots, a_n$. (38) is solvable in B_n, thus it is solvable in L too.

From Theorem 14 we get easily a theorem which characterizes the relatively complemented distributive lattices.

THEOREM 15. *The solvability of* (38) *in L is equivalent to the solvability in* **2** *if and only if L is relatively complemented and distributive.*

PROOF. The case "if" was proved in Theorem 14. Now prove the "only if". The identity

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$$

holds in **2**, thus it must hold in L, that is, L is distributive. Furthermore, the equation system

$$(a \cup b) \cup x = a \cup b \cup c,$$
$$(a \cup b) \cap x = a$$

is solvable in **2**, thus it must be solvable in L too, hence L is relatively complemented, q. e. d.

[25] The number of the elements of the finite lattice L is indicated by $O(L)$.

§ 2. Boolean ring operations

In Corollary of Theorem 3 we have shown that among distributive lattices just the relatively complemented ones have the property that every congruence relation is determined by any congruence class of it. It is well known that the rings have the same property. We prove that this connection between the rings and relatively complemented distributive lattices is not accidental. We shall see that any relatively complemented distributive lattice may be regarded as a Boolean ring, hence the validity of the above statement becomes very natural.

DEFINITION 5. Let A be a set of equations on the distributive lattice L, containing a finite number of equations, parameters and the unknowns x, y and z. If A has a unique solution with respect to z for any fixed values of x and y in any homomorphic image of L, then we write $z = x + y$. If the operation $+$ satisfies the group axioms, then we speak of a group operation defined on the lattice L. If, furthermore, in a similar way (that is, with an equation system, having unique solution in any homomorphic image of L) there is defined another operation denoted by $\cdot$ such that $+$ and $\cdot$ satisfy the ring axioms, then we speak of a ring operation defined on the lattice L.[26]

THEOREM 16. *On the distributive lattice L one may define a Boolean ring operation if and only if L is relatively complemented.*[27] *All Boolean ring operations may be defined in the following way:*

Let a be a fixed element of L. Let $x \cdot y$ *be equal to* $(x \cup a) \cap (x \cup y) \cap (a \cup y)$ *and let* $x + y$ *be the relative complement of* $x \cdot y$ *in the interval* $[a \cap x \cap y, a \cup x \cup y]$.

PROOF. First we prove that the operations defined in the Theorem are ring operations. Applying Theorem 14 we get that it suffices to prove in case of the lattice **2**.

In the lattice **2** the above operations may be given by the following tables:

$$a = 0 \qquad \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

[26] The conditions of Definition 5 are satisfied if we define $+$ and $\cdot$ only with the operations: join, meet, and taking the relative complement of an element in an interval.

[27] One can easily show that if we restrict ourselves to that special case in Definition 5 in which the equations of A contain x and y only in the form $x \cup y$ and $x \cap y$, then the existence of a ring operation characterizes not only the relative complementedness of a distributive lattice, but even the distributivity of the lattice. It immediately follows from the fact that with the above definition $+$ is ambiguous in the lattices S and T.

Thus it is clear that we get Boolean ring operations. The case $a=1$ is the dual of the above.

Now we prove that in a relatively complemented distributive lattice one cannot define other Boolean ring operations. First we prove this for finite lattices. If the lattice considered is **2**, then the assertion is trivial, there is only two types of Boolean ring operations. Let us consider the Boolean algebra $B_n = \mathbf{2}^n$. A system of equations is uniquely solvable in a lattice which is a cardinal product of lattices if and only if the same is true in all of its cardinal-components. Thus the Boolean ring operations in B_n are in a one-to-one correspondence with the Boolean ring operations of the n components. In all of its components two operations may be defined, thus in B_n the number of different operations is 2^n (these are all different from each other, for the zero elements are unequal). On the other hand, the construction in the Theorem gives also 2^n different operations, for a may be chosen in 2^n different ways and these are also different from each other, for the zeros of the rings (a) are unequal. Thus the definition of Theorem 16 exhausts all the Boolean ring operations in the finite case.

Now, let us turn to the general case. Let x, y and u, v be two pairs of elements of L, and $B_{x,y}$, $B_{u,v}$ will denote the finite Boolean algebras containing x, y resp. u, v and the parameters of the operations. $B_{x,y}$ and $B_{u,v}$ are finite, thus they have elements a and b which characterize the operations in $B_{x,y}$ and in $B_{u,v}$, respectively. We get a contradiction from $a \neq b$, and this will complete the proof of the statement according to which all Boolean ring operations may be defined in the way described in the Theorem. Indeed, if $a \neq b$, then consider an element s common to $B_{x,y}$ and to $B_{u,v}$. Now, $s+s=a$ considered in $B_{x,y}$ and $s+s=b$ in $B_{u,v}$, thus necessarily $a=b$.

We shall use the following note: if L is a distributive lattice and $x_1, \ldots, x_n \in L$, then there exists a finite sublattice L_n of L, containing $x_1, \ldots, x_n$. Indeed, by an obvious induction ($n=1$ is trivial) if L_{n-1} is already constructed such that $x_1, \ldots, x_{n-1} \in L_{n-1}$, then let L_n consist of L_{n-1}, from x_n and from the elements of the form $x_n \cap u$ and $x_n \cup u$ where u runs over the elements of L_{n-1}.

At last we prove that if on the distributive lattice L one may define a Boolean ring operation, then L is relatively complemented. Let $L_{x,y}$ be a finite sublattice of L which contains x, y and all the parameters. $L_{x,y}$ is a subdirect product of replicas of **2** and in **2** the operations are defined as in the Theorem, thus in $L_{x,y}$ the operations are defined by the definition of our Theorem too. From the fact that $L_{x,y}$ is closed under the operations $+$ and $\cdot$, it follows the relative complementedness of $L_{x,y}$ and in the same way the relative complementedness of L, q. e. d.

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