OSCILLATION AND MONOTONITY THEOREMS CONCERNING NON-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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(*Presented by* P. TURÁN)

Introduction

There are known theorems stating the existence of an oscillating solution of a linear or non-linear differential equation of the second order and also theorems stating the monotonity of the amplitudes (SONIN's and PÓLYA's theorem). Only one will be quoted here: that of W. E. MILNE[1] concerning the non-linear equation

$$y'' + \varphi(x) f(y) = 0.$$

Let $\varphi(x)$ be a positive continuous increasing and bounded function for $x \geqq a$; the function $f(y)$ an increasing odd one and $f(y) \in \mathrm{Lip}\,(1)$ for $|y| \leqq b$. Taking a real value η $(0 < |\eta| \leqq b)$ the theorem states for $x \geqq a$ the existence and uniqueness of a solution subjected to the initial conditions $y(a) = \eta$, $y'(a) = 0$ and this solution oscillates infinitely often for $x \geqq a$, the amplitudes decrease but do not approach zero.

The present paper discusses the generalization of this theorem and certain comparison theorems of Sturmian type concerning the "half-waves", "quarter-waves", "amplitudes" (see below the explication of these expressions) and the distances of the zeros.

§ 1

One can raise the question what a condition imposed on the function $f(x, y, y')$ involves the existence of an oscillatory solution of the equation

$$y'' = f(x, y, y'). \tag{I}$$

It will be shown here that the separability of the variables of $f(x, y, y')$ leads to such conditions. — We shall prove the following generalization of the above-mentioned theorem of W. E. MILNE:

THEOREM 1. *Suppose that in the non-linear differential equation*

$$y'' + \varphi(x) f(y) h(y') = 0 \tag{1}$$

[1] W. E. MILNE, A theorem of oscillation, *Bull. Amer. Math. Soc.*, **28** (1922), pp. 102—104.

1. *$\varphi(x) > 0$ is continuous, increasing and bounded for $x \geqq a$;*

2. *$f(y)$ is an odd*[2] *non-decreasing function and $f(y) \in \mathrm{Lip}\,(1)$ for $|y| \leqq b$;*

3. *$h(u) > 0$ is non-decreasing for $u \leqq 0$ and non-increasing for $u \geqq 0$, further $h(u) \in \mathrm{Lip}\,(1)$ for all u;*[3] *then equation (1) has a unique solution subjected to the initial conditions $y(a) = \eta$, $y'(a) = 0$ $(0 < |\eta| \leqq b)$; this solution exists for all $x \geqq a$, oscillates an infinite number of times, the amplitudes decrease but do not approach zero and y' remains bounded too.*

PROOF. Being $f(x, y, u) = \varphi(x) f(y) h(u)$ continuous in the domain $x \geqq a, |y| \leqq b$ and u arbitrary, the existence of the desired solution in a certain right-hand neighbourhood of a is already clear. In order to prove the uniqueness of the solution, we shall show that the function $f(x, y, u) = \varphi(x) f(y) h(u)$ satisfies in y and u a Lipschitz condition for $|y| \leqq b$ (u arbitrary and $x \geqq a$).

On account of 2 and 3,

$$\begin{aligned} |f(x, y_2, u_2) - f(x, y_1, u_1)| &= \varphi(x)\,|f(y_2)h(u_2) - f(y_1)h(u_1)| \leqq \\ &\leqq \varphi(x)\,|f(y_2) - f(y_1)|\,h(u_1) + |f(y_2)|\,|h(u_2) - h(u_1)| \leqq \\ &\leqq \varphi(x)(K_1|y_2 - y_1|\,h(0) + K_2|u_2 - u_1|\,f(b)) \leqq M(|y_2 - y_1| + |u_2 - u_1|), \end{aligned}$$

where $M = L \max(K_1 h(0), K_2 f(b))$ and L is the least upper bound of $\varphi(x)$, i. e. $L = \lim\limits_{x \to +\infty} \varphi(x)$, further $|y_1| \leqq b$, $|y_2| \leqq b$, u_1 and u_2 are arbitrary, K_1 and K_2 are the Lipschitz constants of $f(y)$ and $h(u)$, respectively. We must show the existence of the solution for all $x \geqq a$. Equation (1) may be written in the form

$$\frac{y' y''}{h(y')} + \varphi(x) f(y) y' = 0. \tag{1'}$$

By means of the notations

$$\int_0^y f(t)\,dt = F(y), \qquad \int_0^u \frac{t}{h(t)}\,dt = H(u) \quad (H(\pm\infty) = +\infty)$$

this can be written as follows:

$$\frac{dH(y')}{dx} + \varphi(x)\frac{dF(y)}{dx} = 0 \quad \text{or} \quad \frac{1}{\varphi(x)}\frac{dH(y')}{dx} + \frac{dF(y)}{dx} = 0.$$

[2] In order to prove the oscillating character of the solution, it is sufficient to assume instead of "$f(y)$ is odd" that $\operatorname{sg} f(y) = \operatorname{sg} y$.

[3] E. g. $h(u)$ may be an even function decreasing for $u \geqq 0$.

Hence, making use of Stieltjes integral, we obtain the following two equations:

$$H(y')+\varphi(x)F(y)=\int_a^x F(y)\,d\varphi(x)+\varphi(a)\,F(\eta), \tag{2}$$

$$\frac{H(y')}{\varphi(x)}+F(y)=-\int_a^x \frac{H(y')}{\varphi(x)^2}\,d\varphi(x)+F(\eta). \tag{3}$$

The function $F(y)$ is positive for $y\neq 0$, $F(0)=0$ and $F(y)$ is an even function and increasing for $y>0$. The function $H(u)$ is positive, except at $u=0$ where $H(0)=0$, and $H(u)$ is increasing for $u\geqq 0$.

With regard to (2) and (3) the first of the functions

$$P(x)=H(y')+\varphi(x)F(y),\quad Q(x)=\frac{H(y')}{\varphi(x)}+F(y)=\frac{P(x)}{\varphi(x)}$$

is obviously increasing, the second one is decreasing.

The solution $y(x)$ in question cannot attain for $x>a$ the boundary $y=\pm b$ of the domain D ($x\geqq a$, $|y|\leqq b$, u arbitrary). For, if a_1' is the next point where $y(a_1')=\pm\eta$ (Fig. 1), then, by the monotonity of $Q(x)$,

$$Q(a)=F(\eta)>Q(a_1')=\frac{H(y'(a_1'))}{\varphi(a_1')}+F(\eta)$$

whence

$$0>\frac{H(y'(a_1'))}{\varphi(a_1')},$$

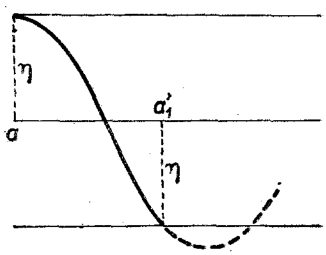

Fig. 1

and this is in contradiction to $H(u)\geqq 0$, $\varphi(x)>0$ also in the case when $y'(a_1')=0$. Therefore $y(x)$ cannot attain the value $\pm\eta$, still less the value $\pm b$.

Equation (1) assures that $y''(x)$ remains finite in every finite interval $[a,c]$ and even as $y'(x)$, being $y'=\int_a^x y''(t)\,dt$. Therefore the solution $y(x)$ may be continued for all $x\geqq a$, i. e. this exists for $x\geqq a$.

Otherwise, the monotonity of $Q(x)$ involves the boundedness of $|y'|$ for all $x\geqq a$, too.

Now we prove the oscillatory character of $y(x)$.

See e. g. the case $\eta>0$. Then considering (1) it is clear that at the place $x=a$ y'' is negative and remains this as long as y is positive. Since $y'=\int_a^x y''\,dx<0$ $(x>a)$, y decreases, its graph is concave downward with negative slope to the right of a (as long as y is positive) and therefore must cut the x axis at a finite point $x_1 > a$ (Fig. 2). In this point the derivative of $y(x)$ is $y_1' = \int_a^{x_1} y'' dx < 0$. To the right of x_1 the value y is negative and therefore (see (1)) $y'' > 0$ (the graph of y is convex downward), *moreover* y'' *increases* as long as y' is negative, because y' (which is < 0) is increasing and thus $h(y')$, too, and again by virtue of (1) y'', increases. Since $y' = y_1' + \int_{x_1}^{x} y'' dx$, y' must vanish for a finite value $x = a_1$ (otherwise y'' would be positive and increasing, $\int_{x_1}^{x} y'' dx$ would surpass $|y_1'|$ and y' would be positive without having been zero). Starting at the place $x = a_1$, $y = y(a_1) < 0$ we can arrive in the same manner at the succeeding zero x_2 and further at a maximum place a_2 (where $y'(a_2) = 0$), etc. Consequently, the oscillatory character for $x \geqq a$ is proved.[4] Zeros and only these are the points of inflexions. If $\eta < 0$, the proof follows exactly the same lines. Let the zeros, the places ($> a$) of the extrema and

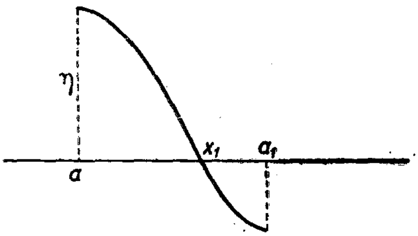

Fig. 2

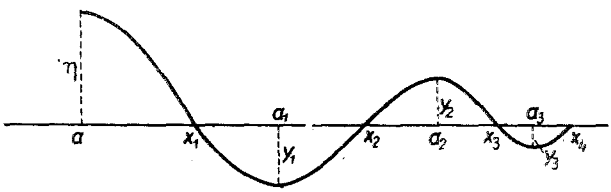

Fig. 3

the corresponding values at these be denoted by x_i, a_i and y_i ($i = 1, 2, 3, \ldots$), respectively. Let $|y_i|$ be called as "amplitudes" of $y(x)$, the graph of $y(x)$ belonging to $[x_i, x_{i+1}]$ as a "half wave", and that belonging to $[x_i, a_i]$ or $[a_i, x_{i+1}]$ as a "quarter-wave" (Fig. 3). Being $Q(x)$ decreasing and $y'(a) = y'(a_i) = 0$ ($i = 1, 2, 3, \ldots$), $H(0) = 0$, we have

$$F(y_i) > F(y_{i+1}) \quad \text{whence} \quad |y_i| > |y_{i+1}| \qquad (i = 1, 2, 3, \ldots),$$

what indicates the decrease of the amplitudes;[5] but these do not approach

[4] Similarly, it may readily be seen that *all* the solutions of (1) have this character, provided that b is large enough compared to η.

[5] If $f(y)$ is like that in footnote [2], then the maxima form a decreasing sequence and similarly the minima form another one.

zero because owing to the increase of $P(x)$

$$\varphi(a)F(\eta)<\varphi(a_1)F(y_1)<\varphi(a_2)F(y_2)<\cdots<\varphi(a_n)F(y_n)<\cdots,$$

$$F(y_n)>\frac{\varphi(a)}{\varphi(a_n)}F(\eta)>\frac{\varphi(a)}{L}F(\eta)\qquad (L=\lim_{x\to+\infty}\varphi(x)).$$

COROLLARIES.

1. At places of equal $|y|$ (not only at zeros) the sequence $\frac{H(y')}{\varphi(x)}$ is decreasing.

2. At places characterized by equal $\frac{H(y')}{\varphi(x)}$ the values $|y|$ decrease.

3. At places of equal y' the values $\varphi(x)F(y)$ increase. If $h(u)$ is an even function, these values corresponding to equal $|y'|$ are increasing too.

4. At zeros and at places of equal $\varphi(x)F(y)$ of the ascending branchs of the curve of y, the values y' increase. A similar statement is valid for the descending branchs. If $h(u)$ is even, $|y'|$ increases at places of equal $\varphi(x)F(y)$.

5. On account of the decrease of $Q(x)$ we obtain

$$F(\eta)>\frac{H(y_1')}{\varphi(x_1)}>F(y_1)>\frac{H(y_2')}{\varphi(x_2)}>F(y_2)>\cdots\qquad (y_i'=y'(x_i)).$$

This is the relation between the amplitudes and the slopes at the neighbouring zeros.

6. Being $P(x)$ increasing we get $\varphi(a)F(\eta)<H(y_1')<\varphi(a_1)F(y_1)<$ $<H(y_2')<\varphi(a_2)F(y_2)<\cdots$.

7. In the case of MILNE's theorem $h(u)\equiv\text{const}$

$$P(x)=\frac{y'^2}{2}+\varphi(x)F(y),\quad Q(x)=\frac{y'^2}{2\varphi(x)}+F(y)$$

and considering the linear equation $y''+\varphi(x)y=0$ $(f(y)\equiv y, h(u)=\text{const})$

$$P(x)=\frac{y'^2}{2}+\varphi(x)\frac{y^2}{2},\quad Q(x)=\frac{P(x)}{\varphi(x)}$$

and some of the above formulae will be simplified.

8. Assumption $\varphi(x)=k=\text{const}>0$ is possible too. In this case the functions $P(x)$ and $Q(x)$ remain constant, therefore the amplitudes, the values $H(y')$ at zeros and at places of equal y remain constant too, consequently all the "half-waves" are congruent, *finally $y(x)$ is periodic* and y' is bounded.[6] If $h(u)$ is even, all the half-waves are symmetrical relative to their centres.

[6] If $f(y)$ is like that in footnote [2], then the maxima are equal and the minima too, but their absolute values may be different.

Provided that $\varphi(x)=k=\text{const}>0$, equation (1) will be of the form

$$y''+kf(y)h(y')=0$$

and from this

$$H(y')+kF(y)=c=kF(\eta),$$

hence denoting the inverse function of $H(u)$ by $H^{-1}(u)$ (it is a bivalent function)

$$y'=H^{-1}(k(F(\eta)-F(y))),$$

therefore

$$I(y)=\int_{\eta}^{y}\frac{du}{H^{-1}(k[F(\eta)-F(u)])}=x-a.$$

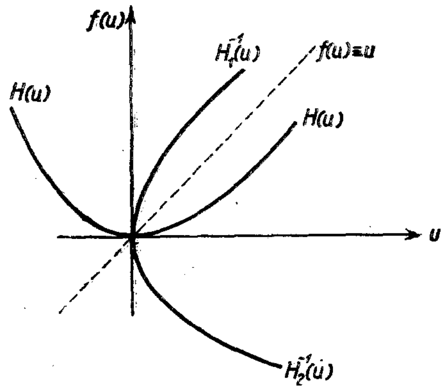

Fig. 4

According to the above theorem, *the solution of this equation* (the inverse function) *exists for all $x\geqq a$ and it is periodic. If $h(u)$ is even, all the half-waves are symmetrical to their centres.* Let the inverse of the function $H(u)$ be denoted by $H_1^{-1}(u)$ for $u\geqq 0$ and by $H_2^{-1}(u)$ for $u\leqq 0$ (Fig. 4). Then

$$x_1-a=\int_{\eta}^{0}\frac{dy}{H_2^{-1}(k[F(\eta)-F(y)])},\quad a_1-x_1=\int_{0}^{-\eta}\frac{dy}{H_2^{-1}(k[F(\eta)-F(y)])}.$$

Similarly

$$x_2-a_1=\int_{-\eta}^{0}\frac{dy}{H_1^{-1}(k[F(\eta)-F(y)])},\quad a_2-x_2=\int_{0}^{\eta}\frac{dy}{H_1^{-1}(k[F(\eta)-F(y)])}$$

etc. The length of a period is

$$p=a_2-a=\int_{-\eta}^{\eta}\frac{dy}{H_1^{-1}(\cdots)}+\int_{\eta}^{-\eta}\frac{dy}{H_2^{-1}(\cdots)}.$$

Let $\lim\limits_{x\to+\infty}\varphi(x)$ be denoted by k, then we see that *the solution of* (1) *in question tends to the above periodic function and the distance of two consecutive zeros tends to $\frac{p}{2}$ (moreover decreasing as we shall show later) as $x\to+\infty$.* See, for example, the equation

$$y''+ky\frac{1}{1+y'^2}=0\qquad(k>0).$$

We get herefrom

$$I(y) = -\int_{\eta}^{y} \frac{du}{\sqrt{-1+\sqrt{1+2k(\eta^2-u^2)}}} = x-a \qquad (a \leqq x \leqq a_1).$$

Introducing the notation $\sqrt{1+2k\eta^2} = K$ and carrying out the transformation $1-\frac{2k}{K^2}u^2 = v^2$

$$I(y) = \frac{K}{\sqrt{2k}} \int_{1/K}^{v_y} \frac{v\,dv}{\sqrt{1-v^2)(Kv-1)}} = x-a \quad (a \leqq x \leqq x_1), \quad v_y = \frac{\sqrt{1+2k(\eta^2-y^2)}}{K}.$$

This integral is an elliptic one and its inverse is an elliptic doubly periodic function and the theorem states that one of the periodes is real, etc., and we recognize all these without the explicit form of the solution. As another example we take the equation

$$y'' + kye^{-y'^2} = 0 \qquad (k > 0).$$

Hence

$$I(y) = -\int_{\eta}^{y} \frac{du}{\sqrt{\log(1+k\eta^2-ku^2)}} = x-a \qquad (a \leqq x \leqq x_1).$$

The above statements are valid here too and this is already more interesting because otherwise there is very little known about this function.

§ 2

Take now the general form of the explicit differential equation of the second order

(1) $$y'' = f(x, y, y').$$

It will be formulated here an *oscillation theorem* concerning this:

THEOREM 2. *Let $f(x,y,u)$ be defined for $x \geqq a$ and for arbitrary y and u with the following properties:*

1. *$f(x,y,u)$ is continuous and* $\operatorname{sg} f(x,y,u) = -\operatorname{sg} y$;
2. *$f_x(x,y,u)$, $f_y(x,y,u)$, $f_u(x,y,u)$ exist and are continuous;*
3. $\operatorname{sg} f_x = -\operatorname{sg} y$;
4. $f_y(x,y,u) < 0$ *and* $f_u(x,y,u) \begin{cases} > 0 & \textit{if} \quad \operatorname{sg} u = \operatorname{sg} y \neq 0, \\ = 0 & \textit{if} \quad y = 0, \\ < 0 & \textit{if} \quad \operatorname{sg} u = -\operatorname{sg} y \neq 0 \end{cases} (x \geqq a);$

5. *let $f(x, y, u)$ satisfy the Nagumo condition, i. e. let a continuous positive function $\Phi(u)$ $(u \geqq 0)$ exist satisfying the conditions*

$$|f(x, y, u)| \leqq \Phi(|u|) \quad \text{and} \quad \int_0^\infty \frac{u}{\Phi(u)} du = +\infty;$$

then there exists for $x \geqq a$ a unique solution of (1) satisfying the initial conditions $y(a) = \eta \neq 0$, $y'(a) = 0$ and oscillating an infinite number of times.

The proof follows exactly the same lines as in § 1. The theorem of NAGUMO[7] assures the existence of the solution for $x \geqq a$ stating the boundedness of y'. (The question arises whether there exist at all functions $f(x, y, u)$ satisfying 1—5. The function $f(x, y, u) = -\varphi(x)f(y)h(u)$ of § 1 is an example for a function of this kind. Another example is $f(x, y, u) = -\varphi(x)\frac{y^3}{y^2+y'^2}$ $(|y| \leqq b)$. The corresponding equation[8] has an oscillatory solution etc.) The increase and decrease of y'' in the right-hand neighbourhood of the zeros x_1 and x_2, respectively (see Figures 2—3), are to read off from the formula $y''' = f_x + f_y y' + f_{y'} y''$ in which all the terms of the right member are positive or negative, respectively. It is easy to see, even as in the linear case, that the zeros cannot have a finite limit point. If we restrict y by $|y| \leqq b$ $(b > 0)$, then the graph of y can leave the domain $x \geqq a$, $|y| \leqq b$ attaining the boundary $y = -b$ (supposed $\eta > 0$) and we cannot assert its further existence.

§ 3

Theorem 1 may be extended to the equation

$$y'' + \sum_{i=1}^{n} \varphi_i(x) f_i(y) h_i(y') = 0 \tag{1}$$

where the functions φ_i, f_i, h_i have the same properties as in § 1. Theorem 2 cannot be applied here immediately without any hypothesis on the derivability of the functions φ_i, f_i, h_i. In general, it seems to be impossible to find the analogues of the functions $P(x)$ and $Q(x)$. Let the restriction

$$h_1 \equiv h_2 \equiv h_3 \equiv \cdots \equiv h_n \equiv h(y')$$

be assumed. Then

$$\frac{y'y''}{h(y')} + \sum_{i=1}^{n} \varphi_i(x) f_i(y) y' = 0$$

[7] M. NAGUMO, Über die Differentialgleichung $y'' = f(x, y, y')$, *Proc. of the Phys.-math. Soc. of Japan* (3), **19** (1937), pp. 861—865.

[8] See the discussion of this equation in a forthcoming paper of the author.

and with the notations

$$F_i(y)=\int_0^y f_i(t)dt, \quad H(u)=\int_0^u \frac{t}{h(t)}dt$$

we have

$$\frac{dH(y')}{dx}+\sum_{i=1}^{n}\varphi_i(x)\frac{dF_i(y)}{dx}=0 \tag{2}$$

whence, making use of Stieltjes integral,

$$H(y')+\sum\varphi_i(x)F_i(y)=\sum\int_a^x F_i(y)d\varphi_i(x)+\sum\varphi_i(a)F_i(\eta).$$

Thus the function

$$P(x)=H(y')+\sum_{i=1}^{n}\varphi_i(x)F_i(y)$$

is increasing, and similar conclusions may be made as in § 1. Without further restriction one cannot find the analogue of the function $Q(x)$. If one of the functions $\varphi_i(x)$, say $\varphi_1(x)$, is more quickly increasing, then the other ones, i. e. $\frac{\varphi_i(x)}{\varphi_1(x)}$, are decreasing for $i=2, 3, \ldots, n$, then from (2)

$$\frac{1}{\varphi_1(x)}\frac{dH(y')}{dx}+\sum_{i=1}^{n}\frac{\varphi_i(x)}{\varphi_1(x)}\frac{dF_i(y)}{dx}=0,$$

and so we obtain by integration

$$\frac{H(y')}{\varphi_1(x)}+\sum_{i=1}^{n}\frac{\varphi_i(x)}{\varphi_1(x)}F_i(y)=-\int_a^x H(y')\frac{1}{\varphi_1(x)^2}d\varphi_1(x)+\sum\int_a^x F_i(y)d\left(\frac{\varphi_i}{\varphi_1}\right)+$$
$$+\sum\frac{\varphi_i(a)}{\varphi_1(a)}F_i(\eta).$$

Clearly, the function

$$Q(x)=\frac{H(y')}{\varphi_1(x)}+\sum_{i=1}^{n}\frac{\varphi_i(x)}{\varphi_1(x)}F_i(y)$$

is decreasing. In this case we cannot state the decrease of the amplitudes (however, see this problem later), only the decrease of the sequence $\sum\frac{\varphi_i(x)}{\varphi_1(x)}F_i(y)$ formed at the amplitudes, etc.

The equation

$$\frac{d}{dx}(p(x)y')+q(x)f(y)h(p(x)y')=0 \tag{3}$$

may be written by the transformation $\xi=\int\limits_a^x \frac{dx}{p(x)}$ in the form

$$\frac{d^2\bar{y}}{d\xi^2}+\bar{p}(\xi)\bar{q}(\xi)f(\bar{y})h\left(\frac{d\bar{y}}{d\xi}\right)=0 \tag{3'}$$

where $\bar{y}$, $\bar{p}$, $\bar{q}$ mean y, p, q as functions of ξ. Since $p(x)$ must have a constant sign (see the above transformation), $p(x)$ must be positive (in the opposite case ξ would be negative for $x>a$). Being

$$\frac{d(\bar{p}\bar{q})}{d\xi}=\frac{d(pq)}{dx}\frac{dx}{d\xi}=\frac{d(pq)}{dx}p,$$

Theorem 1 may be applied:

THEOREM 1'. *Let the functions $p(x)$ and $q(x)$ be positive continuous, pq increasing and bounded for $x \geqq a$, further let $f(y)$ and $h(u)$ be the same as in § 1, then the same is true as under conditions of Theorem 1. The inflexions are not necessarily on the x axis.*

The function $P(x)=H(py')+pqF(y)$ is increasing and $Q(x)= =\frac{H(py')}{pq}+F(y)$ is decreasing and the results of § 1 hold concerning these functions.

In the linear case

$$P(x)=\frac{(py')^2}{2}+pq\frac{y^2}{2}, \quad Q(x)=\frac{(py')^2}{2pq}+\frac{y^2}{2}.$$

$Q(x)$ is the function used in proving the SONIN—PÓLYA theorem. This is thus included in the above theorem.

The equation

$$\frac{d}{dx}(py')+\sum_{i=1}^{n} q_i(x)f_i(y)h_i(py')=0$$

may be discussed similarly.

§ 4

Let us consider again the equation

$$y''+\varphi(x)f(y)h(y')=0 \tag{1}$$

with the premises of § 1 and the graph of the solution obtained there.

THEOREM 3. *If $h(u)$ is an even function, then regarding equal values of y (equal levels) on a half-wave, the slope $|y'|$ is greater to the right of the maximum (extremum) point than to the left of it. Consequently, the symmet-*

rical of the left-hand quarter-wave to the ordinate $x = x_m$ *of the extremum will lie entirely above the right-hand quarter-wave* (Fig. 5), *hence its area is greater too, finally*

$$x_m - x_1 > x_2 - x_m .$$

PROOF. Equation (1) may be written in the form

$$\frac{dH(y')}{dx} = -\varphi(x)f(y)y'. \tag{2}$$

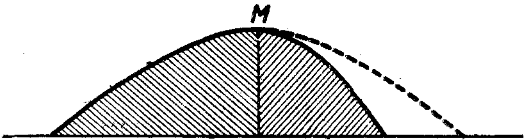

Fig. 5

The coordinates and the derivative of the left-hand branch will be denoted by x, y and y', and those of the right-hand branch by ξ, η and η', respectively. Then (2) relates to the left-hand branch and

$$\frac{dH(\eta')}{d\xi} = -\varphi(\xi)f(\eta)\eta' \tag{3}$$

to the right-hand one.

Integrating (2) as a function of x from x_0 to x_m and (3) as a function of ξ from ξ_0 to x_m, we have (being $H(0)=0$) (Fig. 6)

$$H(y_0') = \int_{x_0}^{x_m} \varphi(x)f(y)y'\,dx = \int_{y_0}^{y_m} \varphi(x)f(y)\,dy \qquad (y_0' = y'(x_0))$$

and

$$H(\eta_0') = \int_{\xi_0}^{x_m} \varphi(\xi)f(\eta)\eta'\,d\xi = \int_{\eta_0}^{y_m} \varphi(\xi)f(\eta)\,d\eta \qquad (\eta_0' = \eta'(\xi_0)),$$

respectively. The variables of the integration of the right sides (i. e. y and η) pass the same interval (y_0, y_m), but ξ passes the interval (x_m, ξ_0) and x the interval (x_0, x_m). On account of the monotonity of $\varphi(x)$, there will be on the same level $0 \leqq y_0 = \eta_0 \leqq y_m$

$$H(\eta_0') > H(y_0')$$

and, being $H(u)$ an even function,

$$|\eta_0'| > |y_0'| .$$

The further conclusion is obvious.

The case of MILNE's theorem and that of the linearity are included too.

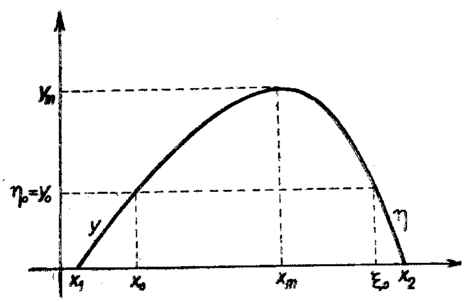

Fig. 6

§ 5

THEOREM 4. *Holding the premises of the previous § but omitting the restriction $h(-u)=h(u)$ it will be proved that a quarter-wave lying before a zero may be brought by a rotation of* 180° *or* —180° *around the common endpoint quite over the succeeding half-wave, i. e.* (Fig. 7) *the arc $\widehat{AM'}$ is outside of the area bounded by the arc $\widehat{ANB}$ and the part $\overline{AB}$ of the x axis:*

$$|y(x_2-u)|>|y(x_2+u)| \qquad (0<u\leqq \min(x_2-x_M, x_3-x_2)).$$

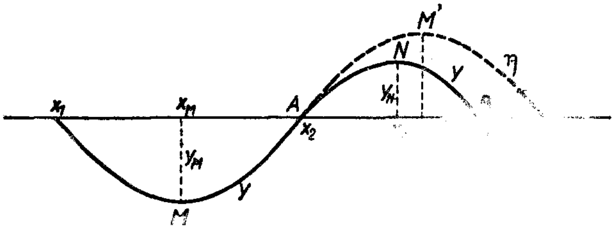

Fig. 7

Here x_M is the abscissa of the point M, x_N is that of the point N. Placing the origin in x_2 (carrying out a linear transformation) the form of the equation will not be changed. Let e. g. $y(x)$ be the left-hand half-wave under the x axis and let the ordinate of the rotated half-wave be denoted by $\eta(x)$ as a function of x. Then

$$\eta(x)=-y(-x), \quad \eta'(x)=y'(-x), \quad \eta''(x)=-y''(-x)$$

whence

$$y(x)=-\eta(-x), \quad y'(x)=\eta'(-x), \quad y''(x)=-\eta''(-x).$$

Writing these in the equation

(1) $$y''+\varphi(x)f(y)h(y')=0$$

we have

$$-\eta''(-x)+\varphi(x)f(-\eta(-x))h(\eta'(-x))=0.$$

Putting here $-x$ instead of x and taking into account that $f(-\eta)=-f(\eta)$ we obtain as an equation satisfied by $\eta(x)$

(2) $$\eta''(x)+\varphi(-x)f(\eta)h(\eta')=0.$$

Now it will be proved that on the same level ($y=\eta$) $\eta'>y'$ from the level $y=\eta=0$ up to the level $\eta=y=y_N$ where $y_N=y(x_N)$ (Fig. 8). Equations (1) and (2) may be

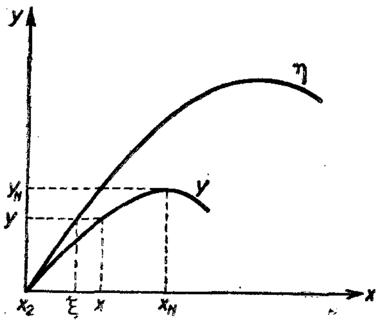

Fig. 8

written, denoting the abscissae by x and ξ, in the form

$$\frac{dH(y')}{dx} = -\varphi(x)f(y)y' \tag{1'}$$

and

$$\frac{dH(\eta')}{d\xi} = -\varphi(-\xi)f(\eta)\eta', \tag{2'}$$

respectively. Integrating from 0 to the common level $y=\eta \leqq y_N$ and taking into consideration that $y'(0)=\eta'(0)=y_0$, we obtain

$$H(y') = H(y_0') - \int_0^x \varphi(x)f(y)y'\,dx = H(y_0') - \int_0^y \varphi(x)f(y)\,dy,$$

$$H(\eta') = H(y_0') - \int_0^\xi \varphi(-\xi)f(\eta)\eta'\,d\xi = H(y_0') - \int_0^y \varphi(-\xi)f(\eta)\,d\eta,$$

hence

$$H(\eta') - H(y') = \int_0^y [\varphi(x) - \varphi(-\xi)]f(y)\,dy,$$

but

$$\varphi(x) > \varphi(-\xi) \quad (x > x_2) \quad \text{and} \quad f(y) \geqq 0,$$

therefore $H(\eta') > H(y)$, consequently $\eta' > y'$ (being $\eta' > 0$, $y' > 0$) up to the level $y = y_N$, i. e.

$$|y(x_2 - u)| > |y(x_2 + u)| \qquad (0 < u \leqq \min(x_2 - x_M, x_3 - x_2)).$$

On this level $y'=0$, $\eta' > 0$ and η is still further increasing up to its maximum; thus its arc will be over that of y.

We shall prove, restricting $f(y)$, in § 7 that an extremum lying to the left of a zero is farther hereform than the extremum lying to the right of this zero, what means: the left-hand quarter-wave may be brought *quite* over the right-hand one; its area and "length" are greater. Similar facts will be stated concerning the half-waves too.

§ 6

Now we extend one of STURM's comparison theorems to the equations

$$y'' + \varphi(x)f(y)h(y') = 0, \tag{1}$$

$$\eta'' + \psi(x)f(\eta)h(\eta') = 0. \tag{2}$$

THEOREM 5. *We assume the following conditions:*

1. *$\varphi(x)$ and $\psi(x)$ are positive continuous increasing bounded functions in $[a, b]$ and $\varphi(x) \geqq \psi(x)$ but $\varphi(x) \not\equiv \psi(x)$ in any subinterval of $[a, b]$, further $\varphi(x) - \psi(x) \geqq O(x-a)$, $x - a \to a + 0$;*

2. *$f(u) \in \mathrm{Lip}\,(1)$, $f(u)$ is increasing and $\mathrm{sg}\, f(u) = \mathrm{sg}\, u$, $\frac{f(u)}{u} = O(1)$ $(u \to 0)$, further $\frac{f(u)}{u}$ is non-increasing for $u > 0$ and non-decreasing for $u < 0$ (e. g. $f(u) \equiv \mathrm{Arctg}\, u$);*

3. *$h(u) > 0$ is an even function non-increasing for $u > 0$, non-decreasing for $u < 0$ and $h(u) \in \mathrm{Lip}\,(1)$;*

4. *let $y(x)$ and $\eta(x)$ be oscillatory solutions of (1) and (2), respectively, and*

$$y(a) \geqq 0, \quad \eta(a) \geqq 0, \quad y'(a) > 0, \quad \eta'(a) > 0, \quad y(b) = 0,$$
$$\eta'(a) y(a) - y'(a) \eta(a) \geqq 0;$$

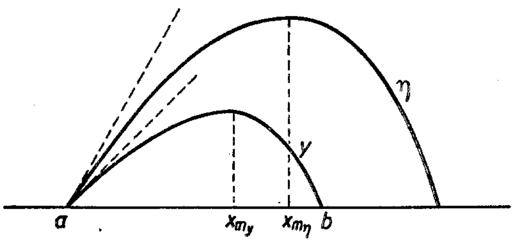

Fig. 9

then (Fig. 9) $\frac{\eta(x)}{y(x)}$ *is increasing in $a < x \leqq b$ and assuming $\eta(a) \geqq y(a)$ we have*

$$\eta'(x) > y'(x) \qquad (a < x < x_{m_\eta} + \delta)$$

and

$$x_{m_\eta} > x_{m_y},$$

where δ is a certain positive number, x_{m_η} and x_{m_y} mean the abscissae of the first maximum points of $\eta(x)$ and $y(x)$, respectively, succeeding a, finally

$$\eta(x) > y(x) \qquad (a < x \leqq b).$$

PROOF. It will be dealt with here only the case when

$$y(a) = \eta(a) = 0, \quad \eta'(a) \geqq y'(a) > 0.$$

The general case may be treated in the same way.

In the first place it will be shown that in a certain right-hand neighbourhood of a $\eta' > y'$ and so $\eta > y$. Applying the finite Taylor formula we have

$$\left.\begin{aligned} y(x) &= y'(a)(x-a) + \frac{1}{2} y''(a + \theta(x-a))(x-a)^2, \\ \eta(x) &= \eta'(a)(x-a) + \frac{1}{2} \eta''(a + \theta'(x-a))(x-a)^2 \end{aligned}\right\} \qquad (0 < \theta, \ \theta' < 1).$$

Being $\eta''(x)$, $y''(x)$ continuous (see (1) and (2))

$$\left.\begin{aligned} y(x) &= y'(a)(x-a) + o((x-a)^2), \\ \eta(x) &= \eta'(a)(x-a) + o((x-a)^2) \end{aligned}\right\} \qquad (x \to a+0) \quad (\text{being } \eta''(a) = y''(a) = 0)$$

and

$$\eta(x) - y(x) = (\eta'(a) - y'(a))(x-a) + o((x-a)^2) \qquad (x \to a+0).$$

Now we have two cases:

1. $\eta'(a) > y'(a)$, thus $\eta'(x) > y'(x)$ and $\eta(x) > y(x)$ for sufficiently small $x-a>0$;

2. $\eta'(a) = y'(a)$, consequently

$$\eta(x)-y(x) = o((x-a)^2) \qquad (x \to a+0)$$

and we get similarly

$$\eta'(x)-y'(x) = o((x-a)) \qquad (x \to a+0).$$

By virtue of these and 2—3 we obtain

$$\left.\begin{aligned} |f(\eta(x))-f(y(x))| &\leqq K_1|\eta(x)-y(x)| = o((x-a)^2), \\ |h(\eta'(x))-h(y'(x))| &\leqq K_2|\eta'(x)-y'(x)| = o(x-a), \\ f(\eta(x)) &= f(y(x))+o((x-a)^2), \\ h(\eta'(x)) &= h(y'(x))+o(x-a), \\ f(\eta(x))h(\eta'(x)) &= f(y(x))h(y'(x))+o((x-a)^2) \end{aligned}\right\} \qquad (x \to a+0),$$

being $f(y(x)) = O(x-a)$. Therefore

$$\begin{aligned} \eta''(x)-y''(x) &= \varphi(x)f(y)h(y')-\psi(x)f(\eta)h(\eta') = \\ &= [\varphi(x)-\psi(x)]f(y)h(y')+o((x-a)^2) \qquad (x \to a+0) \end{aligned}$$

but

$$f(y) = O(x-a), \quad \varphi(x)-\psi(x) \geqq O(x-a) \qquad (x \to a+0),$$

consequently

hence

finally

$$\left.\begin{aligned} \eta'' &> y'', \\ \eta' &> y', \\ \eta &> y \end{aligned}\right\} \quad \text{for sufficiently small } x-a>0.$$

It will easily be found from equations (1) and (2) that

$$\eta''y-y''\eta = \varphi(x)f(y)h(y')\eta-\psi(x)f(\eta)h(\eta')y$$

or

$$(\eta'y-y'\eta)' = \eta y\left(\varphi(x)\frac{f(y)}{y}h(y')-\psi(x)\frac{f(\eta)}{\eta}h(\eta')\right)$$

whence

$$\Delta(x) \equiv \eta'y-y'\eta = \int_a^x \eta y\left(\varphi(x)\frac{f(y)}{y}h(y')-\psi(x)\frac{f(\eta)}{\eta}h(\eta')\right)dx. \tag{3}$$

Taking 1—3 into consideration we have for sufficiently small $x-a>0$ (at least as long as $\eta' \geqq y' \geqq 0$)

$$\Delta(x) \equiv \eta'y-\eta y' > 0.$$

We state that $\eta' > y'$ holds up to $x = \max(x_{m_\eta}, x_{m_y})+\delta$ at least with some

$\delta > 0$. In the opposite case there would exist a first place c $(a < c < b)$ where

$$\eta'(c) = y'(c), \quad \eta(c) > y(c) > 0$$

and

$$\Delta(c) = \eta'(c)y(c) - y'(c)\eta(c) > 0,$$

what is a contradiction, provided that $\eta'(c) = y'(c) \geqq 0$. Therefore $x_{m_\eta} > x_{m_y}$ and $\eta' > y'$ for $a < x < x_{m_\eta} + \delta$ and $c > x_{m_\eta}$ when c exists at all.

See now the sign of $\Delta(x)$. This could be negative only for $x > x_{m_\eta}$. We state that $\Delta(x)$ remains non-negative at least up to b. In the opposite case there would be a first place d $(x_{m_\eta} < c < d < b)$ where

$$\frac{\eta'(d)}{\eta(d)} = \frac{y'(d)}{y(d)}, \quad \eta'(d) < 0, \quad y'(d) < 0$$

hold but $\Delta(x) = y\eta\left(\frac{\eta'}{\eta} - \frac{y'}{y}\right) < 0$ in a certain right-hand neighbourhood of d. $\frac{\eta}{y}$ increases up to d, thus $\eta(d) > y(d) > 0$, $|\eta'(d)| > |y'(d)|$, consequently the integrand of (3) is positive at d and in some right-hand neighbourhood of it. Therefore

$$\Delta(x) = \int_a^d \cdots + \int_d^x \cdots = \int_d^x \cdots > 0 \qquad (x > d),$$

what is in contradiction to the definition of d. Moreover $\Delta(d) > 0$. For, if $\Delta(d) = 0$, the function $\Delta(x)$ has a minimum at d where $\Delta'(x)$ (the integrand of (3) at $x = d$) must vanish in contradiction to the above statement. Thus the theorem is proved.

We cannot decide whether or not η' remains greater than y' for $a < x \leqq b$. Assuming $\eta'(a) > y'(a)$ the functions $\varphi(x)$ and $\psi(x)$ may be identical, i. e. *a solution of* (1) *having a greater initial slope remains greater up to b, provided that at least one of* $\frac{f(y)}{y}$ *and* $h(u)$ *is strictly monotone.*

§ 7

Now we can solve the problem of the quarter and half-waves. We shall prove the following

THEOREM 6.[9] *All the half-waves of the solution, obtained in § 1, of the equation*

(1) $$y'' + \varphi(x)f(y)h(y') = 0$$

[9] This is a generalization of a theorem of E. MAKAI concerning the linear equation $y'' + \varphi(x)y = 0$: On a monotonity property of certain Sturm—Liouville functions, *Acta Math. Acad. Sci. Hung.*, 3 (1952), pp. 165—172.

may be rotated (in the sense of § 5) over the succeeding one (see Fig. 7), *provided that the premises of § 6 hold, further one of* $D_+\varphi(x)$ *and* $D_-\varphi(x)$ *is positive at every place* $x \geqq a$ *(*$D_\pm\varphi(x)$ *mean the one-sided derivatives). A similar statement is valid concerning the quarter-waves too (without supposing that* $h(u)$ *is even).*

PROOF. As we have seen, placing the common endpoint in the origin, the ordinates of the rotated curve satisfy the equation

$$\eta'' + \varphi(-x)f(\eta)h(\eta') = 0. \tag{2}$$

Let e. g. the left-hand half-wave be under the x axis. Now we have

$$\eta'(a) = y'(a) > 0, \quad \eta(a) = y(a) = 0, \quad \lim_{x\to 0}\frac{\eta(x)}{y(x)} = 1, \quad \varphi(x) > \varphi(-x) \quad (x > a)$$

and

$$\varphi(x) - \varphi(-x) = \left(\frac{\varphi(x)-\varphi(0)}{x} + \frac{\varphi(0)-\varphi(-x)}{x}\right)x$$

but

$$\lim_{x\to+0}\frac{\varphi(x)-\varphi(0)}{x} = D_+\varphi(0) \quad \text{and} \quad \lim_{x\to+0}\frac{\varphi(0)-\varphi(-x)}{x} = D_-\varphi(0),$$

thus $\varphi(x)-\varphi(-x) \geqq O(x)$ $(x\to+0)$ and Theorem 5 can be applied whereby Theorem 6 is proved.

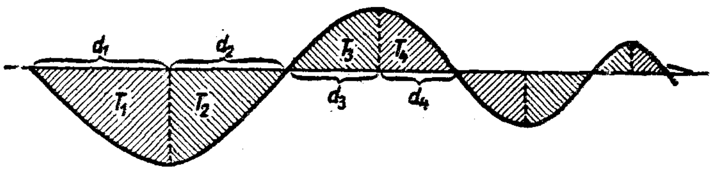

Fig. 10

Denoting the areas and "lengths" of the successive quarter-waves by T_i and d_i $(i = 1, 2, 3, \ldots)$, respectively, we have (Fig. 10)

$$T_1 > T_2 > T_3 > \cdots,$$

$$d_1 > d_2 > d_3 > \cdots \quad \text{(convexity of the zeros).}$$

Of course, all these are valid in the MILNE's case and in the linear case, too. Simultaneously, the present proof is a new one concerning the decrease of the amplitudes too (although making use of more restrictive conditions).

§ 8

Consider again the equation of § 3

$$(py')' + qf(y)h(py') = 0 \tag{1}$$

with the premises assumed there and further let us assume $\frac{f(u)}{u}$ to be as in § 6 and $p(x)$ a *decreasing* function. Then the decrease of the area of the *half*-waves holds.

Namely, as we have seen in § 3, by the transformation $\frac{dx}{p} = d\xi$ equation (1) will be of the form

$$\frac{d^2\bar{y}}{d\xi^2} + \bar{p}\bar{q}f(\bar{y})h\left(\frac{d\bar{y}}{d\xi}\right) = 0. \tag{2}$$

The results of § 7 concerning $\bar{y}(\xi)$ hold, i. e. denoting the zeros of $\bar{y}(\xi)$ by $\xi_1, \xi_2, \xi_3, \ldots$ and those of $y(x)$ by $x_1, x_2, x_3, \ldots$, the sequence

$$\left|\int_{\xi_i}^{\xi_{i+1}} \bar{y}(\xi)d\xi\right| = \left|\int_{x_i}^{x_{i+1}} y(x)\frac{1}{p(x)}\,dx\right| \qquad (i = 1, 2, 3, \ldots)$$

is decreasing. Omitting the increasing factor $\frac{1}{p(x)}$ the sequence $\left|\int_{x_i}^{x_{i+1}} y(x)dx\right|$ $(i = 1, 2, 3, \ldots)$ is a fortiori decreasing.

We cannot decide in this way whether over-rotation of the half-waves is possible or not, because by the above transformation the distances of the zeros increase compared to the corresponding ones of $\bar{y}(\xi)$, hence the distances in question can be increased too (while the amplitudes remain decreasing). Proofs and results of §§ 4—7 may be extended to the equation

$$y'' + \sum \varphi_i(x)f_i(y)h_i(y') = 0$$

inclusive the decrease of the amplitudes *what we could not prove in § 3.* STURM's theorem can also be formulated and proved. The above Sturm theorem may otherwise be extended to the equations

$$y_i'' + \varphi_i(x)f_i(y_i)h_i(y_i') = 0 \qquad (i = 1, 2)$$

with different φ_i, f_i and h_i, but we do not deal with this.

§ 9

The question arises how STURM's comparison theorem will be formed concerning the equation

$$(py')' + q(x)f(y)h(py') = 0. \tag{1}$$

Holding the premises already often used relative to p, q, $f(u)$, $\frac{f(u)}{u}$, $h(u)$ and the initial conditions of § 6 we have for the solutions y_1 and y_2 of the equations

$$(py_i')' + q_i f(y_i) h(py_i') = 0 \qquad (q_2 \geqq q_1)$$
$$(i = 1, 2)$$

(but not identically in any interval) the inequalities

$$y_1' > y_2' \qquad (a < x \leqq x_{m_\eta} + \delta),$$
$$x_{m_\eta} > x_{m_y},$$
$$y_1 > y_2 \qquad (a < x \leqq b).$$

It is more interesting that assuming $q_1 = q_2 = q$ (a unique equation) these inequalities hold, provided that $y_1(a) = y_2(a) = 0$, $y_1'(a) > y_2'(a) > 0$ and that one of the functions $\frac{f(y)}{y}$, $h(u)$ is strictly monotone. The proof follows previous lines.

COROLLARIES. Two particular solutions of (1) can have a zero in common without having only common zeros (differently from the linear case).

In fact, $y_1(a) = y_2(a)$ and $y_1'(a) = y_2'(a)$ imply $y_1 \equiv y_2$ by virtue of the uniqueness of the solution, but $y_1(a) = y_2(a)$ and $y_1'(a) > y_2'(a) > 0$ result in $y_1 > y_2$ up to the next zero of y_2. Hence two consecutive zeros of y_1 cannot be consecutive zeros of y_2. y_2 cannot twice intersect y_1 without vanishing one or other between these points of intersection. If y is a solution, $-y$ is also that. Therefore, comparing the zeros of y_1 and y_2, these functions may be assumed of the same sign in the initial part of the interval of the comparison and so the comparison can be carried out.

In order to compare the solutions of the equations

$$(p_i y_i')' + q_i f(y_i) h(p_i y_i') = 0 \qquad (i = 1, 2)$$

one can deduce in the usual way the analogues of the formulae of STURM and PICONE. These are the following:

$$\frac{d}{dx}(p_1 y_1' y_2 - p_2 y_2' y_1) = \left[q_2 \frac{f(y_2)}{y_2} h(p_2 y_2') - q_1 \frac{f(y_1)}{y_1} h(p_1 y_1')\right] y_1 y_2 + (p_1 - p_2) y_1' y_2',$$

$$\frac{d}{dx}\left[y_1^2\left(\frac{p_1 y_1'}{y_1} - \frac{p_2 y_2'}{y_2}\right)\right] =$$
$$= \left[q_2 \frac{f(y_2)}{y_2} h(p_2 y_2') - q_1 \frac{f(y_1)}{y_1} h(p_1 y_1')\right] y_1^2 + (p_1 - p_2) y_1'^2 + p_2 \left(\frac{y_1' y_2 - y_2' y_1}{y_2}\right)^2.$$

However, applying these either in this form or taking $p_1 = p_2$, $q_1 = q_2$ (considering one equation with two particular solutions) it will not be obtained new results because the first term of the right-hand member of the Picone formula cannot be asserted to be positive when $q_2 > q_1$ and $p_1 > p_2$.

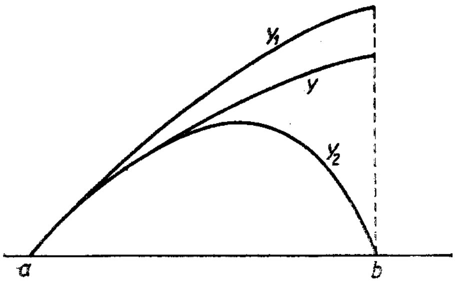

Fig. 11

According to Sturm's theorem we can enclose any solution of

$$y'' + \varphi(x) f(y) h(y') = 0$$

by the solutions of the equations

$$y_1'' + (\min \varphi) f(y_1) h(y_1') = 0,$$
$$y_2'' + (\max \varphi) f(y_2) h(y_2') = 0$$

with the same initial conditions, i. e. we have (Fig. 11)

$$y_2 < y < y_1 \qquad (a < x \leqq b).$$

A separability theorem in the sense of Sturm is not valid here. Rather there are particular solutions of (1) situated to each other as on the Figures 12. The existence of the first and second configurations is obvious. It will be shown the existence of the third too. Corollary 5 of § 1 ensures that the value of a maximum and the slope on the next zero may be made as small as wanted by decreasing the slope at the zero preceding the maximum place.

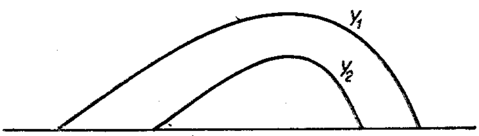

Fig. 12a

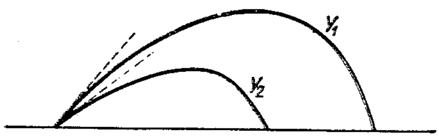

Fig. 12b

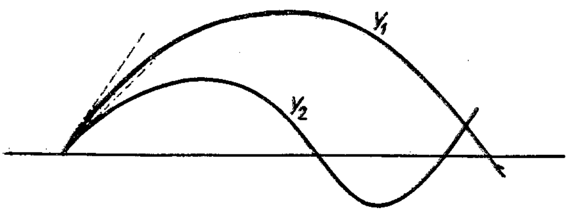

Fig. 12c

Corollary 8 of the same § shows that the distance of two consecutive zeros tends to zero with this slope, provided that $\varphi(x)$ is constant, but also for variable $\varphi(x)$ on account of our statement relative to enclosing of a solution by solutions of the equations with constant $\varphi(x)$. Thus we obtain the interesting result: *given a solution* $y(x)$ *of* (1), *then there are also solutions having an arbitrary number of zeros between two consecutive zeros of* $y(x)$.

Application of § 1. On account of the decrease of[10] $Q(x)=\frac{H(y')}{\varphi(x)}+$ $+F(y)$, we have

$$H(y_n')<F(\eta)\varphi(x_n)<LF(\eta).$$

This implies[11] a bound for $|y_n'|$ and considering that $|y'|$ assumes its maximum at the zeros the mentioned bound of $|y_n'|$ is that of $|y'|$ too. The inequality $F(y_n)>\frac{\varphi(a)}{L}F(\eta)$ involves *a lower bound for* $|y_n|$. *In the linear case* we have

$$|y'|<|\eta|\sqrt{L},\quad |y_n|>\sqrt{\frac{\varphi(a)}{L}}|\eta|.$$

Let $\varDelta$ be the distance between the zero x_n and the next extremum place (Fig. 13). Then

$$|\eta|\sqrt{L}\varDelta>|y_n'|\varDelta>y_n>\sqrt{\frac{\varphi(a)}{L}}|\eta|.$$

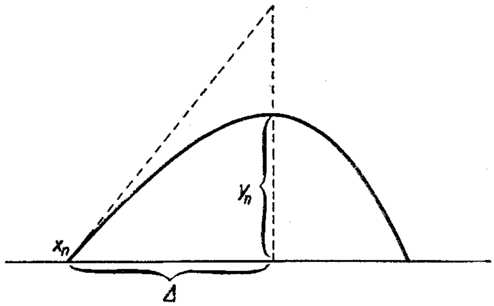

Fig. 13

Thus we obtain a lower bound for $\varDelta$ *and for the distances of the zeros:*

$$\varDelta>\frac{\sqrt{\varphi(a)}}{L}\quad \textit{and}\quad \frac{2\sqrt{\varphi(a)}}{L},$$

respectively. E. g., the function $\sqrt{x}J_\nu(x)$ satisfies

$$y''+\varphi(x)y=0.$$

The function $\varphi(x)=1+\frac{\frac{1}{4}-\nu^2}{x^2}$ is increasing for $|\nu|>\frac{1}{2}$. Denoting a maximum place, where already $\varphi(x)>0$, by a_ν, the distances of the zeros of $\sqrt{x}J_\nu(x)$ and those of $J_\nu(x)$ (which are the same) remain greater than

$$2\sqrt{1+\frac{1-4\nu^2}{4a_\nu^2}}=\frac{\sqrt{1+4(a_\nu^2-\nu^2)}}{a_\nu}\qquad (L=1).$$

I. e. $\varDelta>1$ and the distances of the zeros are greater than 2. The increase of $P(x)$ and decrease of $Q(x)$ imply

$$F(y_{i-1})>F(y_i)>\frac{\varphi(a_{i-1})}{\varphi(a_i)}F(y_{i-1}).$$

In the linear case

$$y_{i-1}>y_i>\sqrt{\frac{\varphi(a_{i-1})}{\varphi(a_i)}}\,y_{i-1}.$$

[10] See the notations in § 1.

[11] Viz. $H(u)$ and $F(u)$ are increasing functions for $u>0$ and have also increasing inverse functions.

Application of § 7. Let us denote *any* solution of the Bessel equation

$$y''+\frac{1}{x}y'+\left(1-\frac{\nu^2}{x^2}\right)y=0$$

by $Z_\nu(x)$. The function $\sqrt{x}Z_\nu(x)$ satisfying the equation

$$y''+\left(1+\frac{1-4\nu^2}{4x^2}\right)y=0$$

is of character of § 7 for $|\nu|>\frac{1}{2}$ and $x\geqq 0$ being $\varphi'(x)>0$. Similarly, the function $\sqrt{x}Z_\nu\left(2\nu x^{\frac{1}{2\nu}}\right)$ for $0<\nu<\frac{1}{2}$, $x\geqq 0$ (this satisfies $y''+x^{\frac{1}{\nu}-2}y=0$) and, in general, $\sqrt{x}Z_\nu(x^\alpha)$ for $\alpha>1$, $x>\left(\frac{1-4\alpha^2\nu^2}{4\alpha^2(\alpha-1)}\right)^{\frac{1}{2\alpha}}$ and *arbitrary* ν $\left(\text{this satisfies } y''+\frac{1-4\alpha^2\nu^2+4\alpha^2x^{2\alpha}}{4x^2}y=0\right)$ are also of this character. Concerning $J_\nu(x)$ the decrease of the area of the *half*-waves holds, provided that $x\geqq 0$ and $\nu>-1$ as R. Cooke[12] showed. E. Makai[13] proved the same for $Z_\nu(x)$, provided that $|\nu|>\frac{1}{2}$, $x\geqq 0$, moreover also the same is shown by G. Szegő[14] for all ν but only for $x>\sqrt[3]{2\left(\frac{1}{9}-\nu^2\right)}$.

Of course, for $\nu<0$ the lower limit of x is positive in all the above cases. The property of the quarter-waves of $\sqrt{x}Z_\nu(x)$ etc. cannot be extended to $Z_\nu(x)$.

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[12] R. Cooke, A monotonity property of Bessel functions, *Journal London Math. Soc.*, **12** (1937), pp. 180—185.

[13] Loc. cit. in § 7.

[14] Still unpublished.