III. SPATIAL NETWORK EQUILIBRIA

SPATIAL COMPETITION FACILITY LOCATION MODELS: DEFINITION, FORMULATION AND SOLUTION APPROACH

R.L. TOBIN*

Environmental Research Division, Argonne National Laboratory, Argonne, Illinois 60439, USA

and

T.L. FRIESZ

Department of Civil Engineering, University of Pennsylvania, Philadelphia, Pennsylvania 19104, USA

Abstract

Models are presented for locating a firm's production facilities and determining production levels at these facilities so as to maximize the firm's profit. These models take into account the changes in price at each of the spatially separated markets that would result from the increase in supply provided by the new facilities and also from the response of competing firms. Two different models of spatial competition are presented to represent the competitive market situation in which the firm's production facilities are being located. These models are formulated as variational inequalities; recent sensitivity analysis results for variational inequalities are used to develop derivatives of the prices at each of the spatially separated markets with respect to the production levels at each of the new facilities. These derivatives are used to develop a linear approximation of the implicit function relating prices to productions. A heuristic solution procedure making use of this approximation is proposed.

Keywords and phrases

Spatial competition, heuristic, location, profit maximization, variational inequalities.

*Currently at GTE, Laboratories, Inc., Waltham, Massachusetts 02254, USA.

1. Introduction and notation

In this paper we are concerned with locating a firm's production facilities and determining production levels at these facilities so as to maximize the firm's profits, taking into account the effect the firm's production will have on market prices. We assume that competition exists among all firms and, in particular, between the locating firm and those already in place. Furthermore, we restrict the possible locations to a subset of nodes of a graph representing the transportation network and require that movements of the commodity produced and resulting prices correspond to a competitive equilibrium.

Most formulations for optimally locating production facilities assume a fixed demand at the markets to be served and that the prices at these markets will not be changed by the introduction of the new production. Exceptions to this are formulations presented by Hansen and Thisse [4] and by Erlenkotter [2]. In both of these formulations, although the market price is related to the locating firm's production, there is no interaction among firms, since these authors consider a spatial monopolist. The models presented here take into account the changes in prices at each of the spatially separated markets that would result from the increase in supply provided by the new facilities and also from the response of the competing firms.

In sect. 2, two different models of spatial competition are presented to represent the competitive market situation in which the firm's production facilities are being located. For both of these models, it is shown how they can be represented as variational inequalities. In sect. 3, a profit maximization location model is formulated assuming the firm is locating facilities in a competitive spatial equilibrium. In sect. 4, sensitivity analysis results for variational inequalities are presented and applied to the models of spatial competition to obtain derivatives of the prices at each of the spatially separated markets with respect to the production levels at each of the new facilities. In sect. 5, these derivatives are used to develop a linear approximation of the implicit function relating prices to productions and a heuristic solution procedure using this approximation is proposed.

The following notation will be used throughout the paper:

a	denotes an arc of the network
p	denotes a path of the network
ℓ, m, i	denotes nodes of the network
$T(\ell)$	is the set of arcs leaving node ℓ
$H(\ell)$	is the set of arcs entering node ℓ
P	is the full set of paths
$P_{\ell m}$	is the set of paths connecting origin-destination pair (ℓ, m)

δ_{ap}	is an element of the arc-path incidence matrix; $\delta_{ap} = 1$ if arc a belongs to path p, 0 otherwise
h_p	is the flow (of a single commodity) on path p
h	$= (\ldots, h_p, \ldots)$
f_a	is the flow (of a single commodity) on arc a; note $f_a = \Sigma_p \delta_{ap} h_p$
f_a^ℓ	is the flow on arc a originating at node ℓ
f	$= (\ldots, f_a, \ldots)$
$c_a(f)$	is the unit cost of transportation on arc a as a function of flow
$c(f)$	$= (\ldots, c_a(f), \ldots)$
$c_p(h)$	is the unit cost of transportation on path p as a function of flow; note $c_p(h) = \Sigma_a \delta_{ap} c_a$
D_ℓ	is the demand (for the single commodity) at node ℓ
D_ℓ^k	is the demand at node ℓ for the commodity originating at node k
D	$= (\ldots, D_\ell, \ldots)$
$\theta_\ell(D)$	is the inverse demand function at node ℓ
$\theta(D)$	$= (\ldots, \theta_\ell(D), \ldots)$
S_ℓ	is the supply (of the single commodity) at node ℓ
S	$= (\ldots, S_\ell, \ldots)$
$\psi_\ell(S)$	is the inverse supply function at node ℓ
$\psi(S)$	$= (\ldots, \psi_\ell(S), \ldots)$
π_ℓ	is the (single commodity) price at ℓ
y_ℓ	is a discrete location decision variable; $y_\ell = 1$ if a production facility is located at node ℓ, 0 otherwise
N	denotes the set of all nodes of the network
$N_0 \subset N$	denotes those nodes at which the firm of interest may locate
$N_D \subset N$	denotes those nodes at which there is a demand
$N_S \subset N$	denotes those nodes at which there is a supply
F_ℓ	is the fixed cost of establishing a production facility at ℓ
$\bar{Q}$	is the maximum amount of new production which may be established
Q_ℓ	is the production level of the firm of interest at node $\ell \in N_0$
$\bar{Q}_\ell$	is the production capacity of the facility located at $\ell \in N_0$
M	is a large number, greater than any nodal supply level anticipated

$V_\ell(Q_\ell)$ is the total variable cost of producing Q_ℓ at node ℓ

$L(k)$ is the set of nodes at which firm k has production facilities

$A \backslash B$ is the set of elements of A which are not contained in B.

Note that in the above notation we have assumed a single commodity supply and single commodity demand functions. It is now well understood that multi-commodity spatial price equilibrium problems may be handled in a mathematically rigorous fashion using variational inequalities, and can be solved through the use of diagonalization/relaxation algorithms (Dafermos [1], Friesz et al. [3]). For this reason, we may treat only the single commodity case without loss of generality.

2. Spatial competition models

A prerequisite to building the desired location models are models for determining a network spatial competition equilibrium. The first of these is a model for determining spatial price equilibrium.

2.1. SPATIAL PRICE EQUILIBRIUM

Such an equilibrium is described by the following conditions:

(i) nonnegative flows, demand and supplies:

$$h, D, S \geqslant 0 \tag{1}$$

(ii) trivial flows when delivered price exceeds local price:

$$\pi_\ell + c_p > \pi_m, \; p \in P_{\ell m} \rightarrow h_p = 0 \tag{2}$$

(iii) equality of delivered price and local price for nontrivial flow:

$$h_p > 0, \; p \in P_{\ell m} \rightarrow \pi_\ell + c_p = \pi_m \tag{3}$$

(iv) conservation of flow at all nodes. (4)

These equilibrium conditions may be expressed as a variational inequality as follows (see appendix A):

$(f^\star, D^\star, S^\star)$ is an equilibrium flow if and only if

$$c(f^\star)(f-f^\star) - \theta(D^\star)(D-D^\star) + \psi(S^\star)(S-S^\star) \geqslant 0 \tag{5}$$

for all (f, D, S) that satisfy the following flow conservation and nonnegativity constraints

$$D_\ell - S_\ell + \sum_{a \in T(\ell)} f_a - \sum_{a \in H(\ell)} f_a = 0 \qquad \forall \ell \in N_S \cap N_D \tag{6}$$

$$D_\ell + \sum_{a \in T(\ell)} f_a - \sum_{a \in H(\ell)} f_a = 0 \qquad \forall \ell \in N_D \backslash N_S \tag{7}$$

$$-S_\ell + \sum_{a \in T(\ell)} f_a - \sum_{a \in H(\ell)} f_a = 0 \qquad \forall \ell \in N_S \backslash N_D \tag{8}$$

$$\sum_{a \in T(\ell)} f_a - \sum_{a \in H(\ell)} f_a = 0 \qquad \forall \ell \in N \backslash (N_S \cup N_D) \tag{9}$$

$$f, D, S \geqslant 0\,. \tag{10}$$

For notational simplicity, in the subsequent exposition we will let $\Omega = \{ f, D, S$: (6)–(10) are satisfied$\}$. This formulation of the spatial price equilibrium is similar to that in [3] except in this formualtion, supply and demand functions are not required at every node; some nodes are not market nodes but serve as transshipment nodes.

If it is required to explicitly keep track of path flows, the constraint set Ω can be modified to require

$$D_\ell^\ell - S_\ell + \sum_{a \in T(\ell)} f_a^\ell - \sum_{a \in H(\ell)} f_a^\ell = 0 \qquad \forall \ell \in N_S \cap N_D \tag{11}$$

$$D_\ell^m + \sum_{a \in T(\ell)} f_a^m - \sum_{a \in H(\ell)} f_a^m = 0 \qquad \forall \ell \in N_D,\ \forall m \in N_S,\ m \neq \ell \tag{12}$$

$$-S_\ell + \sum_{a \in T(\ell)} f_a^\ell - \sum_{a \in H(\ell)} f_a^\ell = 0 \qquad \forall \ell \in N_S \backslash N_D \tag{13}$$

$$\sum_{a \in T(\ell)} f_a^m - \sum_{a \in H(\ell)} f_a^m = 0 \qquad \forall \ell \in N \backslash N_D,\ \forall m \in N_S,\ m \neq \ell \tag{14}$$

$$f_a = \sum_{m \in N_S} f_a^m \qquad \forall a \in A \tag{15}$$

$$D_\ell = \sum_{m \in N_S} D_\ell^m \qquad \forall \ell \in N_D \tag{16}$$

$$f_a^m \geqslant 0 \qquad \forall a \in A, \quad \forall m \in N_S \tag{17}$$

$$D_\ell^m \geqslant 0 \qquad \forall \ell \in N_D, \ \forall m \in N_S \tag{18}$$

$$S_\ell \geqslant 0 \qquad \forall \ell \in N_S \ . \tag{19}$$

Again, for notational simplicity, we will let $\Omega' = \ f, D, S: (11)-(19)$ are satisfied , where it is understood that Ω' replaces Ω when it is required to explicitly keep track of path flows. The number of arc flow variables and the number of demand variables are both increased by a factor of $|N_S|$ in this formulation. However, in general, this formulation will require fewer variables than a formulation using path flow variables. Also, since shortest path routines generally find shortest path trees from one origin to all destinations, this formulation is computationally easy to implement.

The equivalence of these variational inequalities and the equilibrium conditions are demonstrated in appendix A.

2.2. COURNOT-NASH OLIGOPOLISTIC EQUILIBRIUM

A second model of spatial competition is a Cournot-Nash oligopolistic model in which a few firms are competing in spatially separated markets. The approach taken here is motivated by that shown by Harker [5,6]. In this case, each of the firms wants to maximize its profits. This profit maximization can be expressed for each firm k which has production facilities at $L(k)$ as follows

$$\text{maximize} \quad \sum_{\ell \in N_D} \theta_\ell \left(\sum_{m \in N_S} D_\ell^m \right) \sum_{i \in L(k)} D_\ell^i$$

$$- \sum_{i \in L(k)} V_i(S_i) - \sum_{a \in A} c_a \left(\sum_{m \in N_S} f_a^m \right) \sum_{i \in L(k)} f_a^i \tag{20}$$

$$\text{subject to } D_\ell^\ell - S_\ell + \sum_{a \in T(\ell)} f_a^\ell - \sum_{a \in H(\ell)} f_a^\ell = 0 \qquad \forall \ell \in L(k) \cap N_D \tag{21}$$

$$D_\ell^i + \sum_{a \in T(\ell)} f_a^i - \sum_{a \in H(\ell)} f_a^i = 0 \qquad \forall \ell \in N_D, \ \forall i \in L(k),\ i \neq \ell \tag{22}$$

$$-S_\ell + \sum_{a \in T(\ell)} f_a^\ell - \sum_{a \in H(\ell)} f_a^\ell = 0 \qquad \forall \ell \in L(k) \setminus N_D \tag{23}$$

$$\sum_{a \in T(\ell)} f_a^i - \sum_{a \in H(\ell)} f_a^i = 0 \qquad \forall \ell \in N \setminus N_D, \ \forall i \in L(k),\ i \neq \ell \tag{24}$$

$$f_a^i \geqslant 0 \qquad \forall a \in A, \quad i \in L(k) \tag{25}$$

$$D_\ell^i \geqslant 0 \qquad \forall \ell \in N_D, \quad i \in L(k) \tag{26}$$

$$S_\ell \geqslant 0 \qquad \forall \ell \in L(k)\,. \tag{27}$$

Let

$$\bar{f}_a^k = \sum_{i \in L(k)} f_a^i \qquad \forall a \in A \tag{28}$$

$$\bar{D}_\ell^k = \sum_{i \in L(k)} D_\ell^i \qquad \forall \ell \in N_D \tag{29}$$

$$\bar{S}_\ell^k = S_\ell \qquad \forall \ell \in L(k) \tag{30}$$

and let $\bar{f}^k$, $\bar{D}^k$ and $\bar{S}^k$ be the vectors of $\bar{f}_a^k$, $\bar{D}_\ell^k$ and $\bar{S}_\ell^k$, respectively. Let

$$\Lambda^k = \{ \bar{f}^k, \bar{D}^k, \bar{S}^k : \text{(21)--(27) are satisfied} \}\,. \tag{31}$$

Furthermore, since

$$f_a = \sum_{i \in N_S} f_a^i \qquad \forall a \in A$$

$$D_\ell = \sum_{i \in N_S} D_\ell^i \qquad \forall \ell \in N_D,$$

the profit maximiazation problem for each firm k can be written

$$\text{minimize} \quad \sum_{i \in L(k)} V_i(\bar{S}_i^k) + \sum_{a \in A} c_a(f_a)\bar{f}_a^k - \sum_{\ell \in N_D} \theta_\ell(D_\ell)\bar{D}_\ell^k \tag{32}$$

$$\text{subject to} \quad (\bar{f}^k, \bar{D}^k, \bar{S}^k) \in \Lambda^k . \tag{33}$$

If the objective function in (32) is convex, then $(\bar{f}^{k\star}, \bar{D}^{k\star}, \bar{S}^{k\star})$ is a minimum if and only if $(\bar{f}^{k\star}, \bar{D}^{k\star}, \bar{S}^{k\star})$ also minimizes the linearized objective function at $(\bar{f}^{k\star}, \bar{D}^{k\star}, \bar{S}^{k\star})$. That is, if and only if $(\bar{f}^{k\star}, \bar{D}^{k\star}, \bar{S}^{k\star})$ minimizes the following linear programming problem

$$\text{minimize} \quad \sum_{i \in L(k)} \nabla V_i(\bar{S}_i^{k\star})\bar{S}_i^k + \sum_{a \in A} [\nabla c_a(f_a^\star)\bar{f}_a^{k\star} + c_a(f_a^\star)]\,\bar{f}_a^k$$

$$- \sum_{\ell \in N_D} [\nabla \theta_\ell(D_\ell^\star)\bar{D}_\ell^{k\star} + \theta_\ell(D_\ell^\star)]\,\bar{D}_\ell^k \tag{34}$$

$$\text{subject to} \quad (\bar{f}^k, \bar{D}^k, \bar{S}^k) \in \Lambda^k , \tag{35}$$

where

$$f_a^\star = \sum_{i \in L(k)} f_a^{i\star} + \sum_{i \in N_S \setminus L(k)} f_a^i$$

$$D_\ell^\star = \sum_{i \in L(k)} D_\ell^{i\star} + \sum_{i \in N_S \setminus L(k)} D_\ell^i ,$$

and therefore if and only if $(\bar{f}^{k\star}, \bar{D}^{k\star}, \bar{S}^{k\star})$ is such that

$$\sum_{i \in L(k)} \nabla V_i(\bar{S}_i^{k\star})\,(\bar{S}_i^k - \bar{S}_i^{k\star})$$

$$+ \sum_{a \in A} [\nabla c_a(f_a^\star)\bar{f}_a^{k\star} + c_a(f_a^\star)]\,(\bar{f}_a^k - \bar{f}_a^{k\star}) \tag{36}$$

$$- \sum_{\ell \in N_D} [\nabla \theta_\ell(D_\ell^\star)\bar{D}_\ell^{k\star} + \theta_\ell(D_\ell^\star)]\,(\bar{D}_\ell^k - \bar{D}_\ell^{k\star}) \geqslant 0$$

for all $(\bar{f}^k, \bar{D}^k, \bar{S}^k) \in \Lambda^k$. This is an equivalent variational inequality formulation. In order to find a Cournot-Nash equilibrium, the profit must be maximized simultaneously for all firms k. This can be accomplished by solving the following variational inequality: Find $(\bar{f}^{k\star}, \bar{D}^{k\star}, \bar{S}^{k\star})$ for all k such that

$$\sum_k \sum_{i \in L(k)} \nabla V_i(\bar{S}_i^{k\star})(\bar{S}_i^k - \bar{S}_i^{k\star})$$

$$+ \sum_k \sum_{a \in A} [\nabla c_a(f_a^\star)\bar{f}_a^{k\star} + c_a(f_a^\star)](\bar{f}_a^k - \bar{f}_a^{k\star}) \tag{37}$$

$$- \sum_k \sum_{\ell \in N_D} [\nabla \theta_\ell(D_\ell^\star)\bar{D}_\ell^{k\star} + \theta_\ell(D_\ell^\star)](\bar{D}_\ell^k - \bar{D}_\ell^{k\star}) \geq 0$$

for all $(\bar{f}^k, \bar{D}^k, \bar{S}^k) \in \Lambda^k$ for all k. Clearly, if (36) is satisfied for all k, then (37) will be satisfied. To see that solving (37) implies that (34), (35) is solved for each k, note that the objective value of the dual linear program to (34), (35) is zero, since the right-hand sides of the constraints forming Λ^k are zero. Therefore, the minimum value of (34) is zero for any feasible value of $f^{i\star}$, $S^{i\star}$, $D^{i\star}$ for $i \in N_S \backslash L(k)$. Therefore, the sum of the objective functions (34) over all k is also zero. This implies that the sum of terms for each k in (37) is nonnegative. Since $(0, 0, 0) \in \Lambda^k$ for all k, if (37) is satisfied, then the sum of terms for each k is zero when (0, 0, 0) is substituted for $(\bar{f}^k, \bar{D}^k, \bar{S}^k)$. However, for each k, this sum of terms is equivalent to (34). Therefore, if (37) is satisfied, the value of (34) for each k will be zero and hence, optimal. If each V_i is strictly convex, each c_a strictly convex and monotonically increasing, and each θ_ℓ strictly concave and monotonically decreasing and, in addition, each D_ℓ can be bounded so that $\Lambda = \cup_k \Lambda^k$ is compact, then a unique solution exists for variational inequality (37) (see Kinderlehrer and Stampacchia [7]).

3. The location models

The models presented here locate a firm's production facilities and determine production levels at these facilities so as to maximize the firm's profit. These models account for the changes in prices at each of the spatially separated markets that would result from the increase in supply provided by the new facilities and also from the responses of the competing firms.

Either of the two different models of spatial competition presented in sect. 2 can be used to represent the competitive market situation in which the firm's production facilities are being located. In the case of the spatial price equilibrium model, it is assumed that the locating firm is a large firm entering an industry with a large number of small firms. In this case and in the case of the Cournot-Nash oligopoly model, the entrant knows that its policy will have an impact on market prices. Therefore, the locating firm anticipates the reaction of the incumbents before choosing its optimal policy. This means that the locating firm behaves like the leader of a Stackelberg game, while the established firms are the followers.

It is assumed that the firm of interest wishes to establish production facilities at a set of eligible nodes N_0 so as to maximize its profits. The firm's profit at a node $\ell \in N_0$ is

$$Z_\ell = \pi_\ell Q_\ell - V_\ell(Q_\ell) - F_\ell , \tag{38}$$

where π_ℓ is the market price at ℓ, Q_ℓ is the production level of the facility located at ℓ, $V_\ell(Q_\ell)$ is the total variable cost of production at ℓ, and F_ℓ is the fixed cost of locating at ℓ. Note that using the market price π_ℓ to determine revenue does not require that all the production Q_ℓ is sold at ℓ. If some is sold at a remote market, it is assumed that the selling price at that market will be increased by the transportation costs. The costs of supplying that market will also increase by the transportation costs, so the profit is the same. The strategy of the firm is to determine the locations ℓ in N_0 and production levels Q_ℓ which will maximize their profits taking into account the impact that these production levels will have on the spatial competitive equilibrium and hence on the price π_ℓ. The firm's location problem can be stated as

$$\text{maximize} \quad Z(y, Q, \pi) = \sum_{\ell \in N_0} [\pi_\ell Q_\ell - V_\ell(Q_\ell) - y_\ell F_\ell] \tag{39}$$

$$\text{subject to} \quad Q_\ell \leqslant \bar{Q}_\ell y_\ell \qquad \forall \ell \in N_0 \tag{40}$$

$$\sum_{\ell \in N_0} Q_\ell \leqslant \bar{Q} \tag{41}$$

$$Q_\ell \geqslant 0 \qquad \forall \ell \in N_0 \tag{42}$$

$$y_\ell = (0, 1) \qquad \forall \ell \in N_0 \tag{43}$$

$$\pi = \phi(Q) . \tag{44}$$

Constraint (40) requires the production level at ℓ to be less than the capacity $\bar{Q}_\ell$ if the facility is located at ℓ ($y_\ell = 1$) or zero if not ($y_\ell = 0$). Constraint (41) imposes a limitation on the total level of production, constraints (42) require the production variables to be nonnegative, and constraints (43) require choice variables y_ℓ to be zero or one. Additionally, it is required that the market is in an equilibrium given production levels Q_ℓ for all $\ell \in N_0$ and the resulting equilibrium price is π_ℓ. The implicit relationship between the market prices and production levels Q is given by (44).

Let

$$\Gamma(y) = \{Q\colon (40)-(42) \text{ are satisfied}\}. \tag{45}$$

For any given $\bar{y} \geqslant 0$, $\Gamma(\bar{y})$ is a non-empty convex set. The location problem is then given as

$$\text{maximize} \quad Z(y, Q, \pi) \tag{46}$$

$$\text{subject to} \quad Q \in \Gamma(y) \tag{47}$$

$$\pi = \phi(Q) \tag{48}$$

$$y \text{ a zero-one vector.}$$

We refer to the optimization problem (46)–(48) as the discrete spatial competition location model.

One major difficulty with this formulation is that the constraint (48) is not known explicitly, but is implicit in the spatial competition model. The only way to determine a price vector π given a production vector Q is to solve the spatial competition model. However, given a solution to the spatial competition model, sensitivity analysis methods can be used to relate changes in production to changes in price. This relationship can then be used to determine which locations are likely to produce the greatest profits.

4. Sensitivity analysis of spatial competition

Recent results on sensitivity analysis for variational inequalities provide a means to relate changes in production to changes in price in the above models by determining the derivatives of prices with respect to production.

4.1. SENSITIVITY ANALYSIS OF VARIATIONAL INEQUALITIES

The following results are from Tobin [9].

Let $F:R^n \to R^n$ be continuous, $g:R^n \to R^m$ be differentiable, and let $h:R^n \to R^p$ be linear affine. Define

$$K = \{x \in R^n \mid g(x) \geqslant 0,\ h(x) = 0\}. \tag{49}$$

We then want to find a solution $x^\star$ to the variational inequality

$$F(x^\star)^T (x - x^\star) \geqslant 0 \qquad \text{for all } x \in K. \tag{50}$$

THEOREM 1 (Necessary conditions for solution)

If the vector $x^\star \in K$ is a solution to the variational inequality (50) and the gradients $\nabla g_i(x^\star)$ for i such that $g_i(x^\star) = 0$ and $\nabla h_i(x^\star)$ for $i = 1, \ldots, p$ are linearly independent, then there exists $\lambda \in R^m$, $\mu \in R^p$ such that

$$F(x^\star) - \nabla g(x^\star)^T \lambda - \nabla h(x^\star)^T \mu = 0 \tag{51}$$

$$\lambda^T g(x^\star) = 0 \tag{52}$$

$$\lambda \geqslant 0. \tag{53}$$

Proof: See [9] □

THEOREM 2 (Sufficient conditions for solution)

If $g_i(x)$ for $i = 1, \ldots, m$ are concave and $x^\star \in K$, $\lambda^\star \in R^m$ and $\mu^\star \in R^p$ satisfy (51), (52) and (53), then $x^\star$ is a solution to the variational inequality (50).

Proof: See [9] □

THEOREM 3 (Sufficient conditions for a locally unique solution)

If the conditions of theorem 2 hold and in addition if F is differentiable and

$$y^T \nabla F(x^\star) y > 0 \qquad \text{for all } y \neq 0 \tag{54}$$

such that

$$\nabla g_i(x^\star) y \geqslant 0 \qquad \text{for all } i \text{ such that } g_i(x^\star) = 0 \tag{55}$$

$$\nabla g_i(x^\star) y = 0 \qquad \text{for all } i \text{ such that } \lambda_i^\star > 0 \tag{56}$$

$$\nabla h_i(x^\star) y = 0 \qquad \text{for } i = 1, \ldots, p, \tag{57}$$

then $x^\star$ is a locally unique solution to variational inequality (50).

Proof: See [9] □

Let $F(x)$ be once continuously differentiable, $g(x)$ be concave and twice continuously differentiable, and $h(x)$ linear affine. Define

$$K(\epsilon) = \{x : g(x) \geqslant 0,\ h(x) + \epsilon = 0\}, \tag{58}$$

where $\epsilon = [\epsilon_1, \ldots, \epsilon_p]^t$. We then say $x^\star \in K(\epsilon)$ is a solution to $VI(\epsilon)$ if

$$F(x^\star)^t (x - x^\star) \geqslant \qquad \text{for all } x \in K(\epsilon). \tag{59}$$

The following result is a special case of a more general sensitivity analysis result. Here, it is specialized to right-hand side perturbations of the equality constraints only.

THEOREM 4 (Implicit function)

Let $x^\star$ be a solution to $VI(0)$ with the gradients $\nabla g_i(x^\star)$ for i such that $g_i(x^\star) = 0$, and $\nabla h_i(x^\star)$ for $i = 1, \ldots, p$ linearly independent, with the conditions of theorem 3 satisfied and, in addition, the strict complementary slackness condition

$$\lambda_i^\star > 0 \text{ when } g_i(x^\star) = 0.$$

Then $\lambda^\star$ and $\mu^\star$ are unique, and in a neighborhood of $\epsilon = 0$ there exists a differentiable function $[x(\epsilon), \lambda(\epsilon), \mu(\epsilon)]$, where $x(\epsilon)$ is a locally unique solution to $VI(\epsilon)$ and $\lambda(\epsilon)$, $\mu(\epsilon)$ are the unique associated multipliers satisfying strict complementarity and with

$$[x(\epsilon), \lambda(\epsilon), \mu(\epsilon)] = [x^\star, \lambda^\star, \mu^\star].$$

Proof: See [9] □

For $\epsilon = 0$ and $(x, \lambda, \mu) = (x^*, \lambda^*, \mu^*)$ by theorem 1

$$F(x) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x)^T - \sum_{i=1}^{p} \mu_i \nabla h_i(x)^T = 0 \tag{60}$$

$$\lambda_i g_i(x) = 0 \quad \text{for } i = 1, \ldots, m \tag{61}$$

$$h_i(x) + \epsilon = 0 \quad \text{for } i = 1, \ldots, p\,. \tag{62}$$

Let ϕ represent the left-hand side of (60). Then the Jacobian matrix of the system (60), (61), (62) with respect to x, λ, μ is

$$J_{x,\lambda,\mu} = \begin{bmatrix} \nabla\phi & -\nabla g(x)^t & -\nabla h(x)^t \\ \operatorname{diag}(\lambda_i)\nabla g(x) & \operatorname{diag}(g_i(x)) & 0 \\ \nabla h(x) & 0 & 0 \end{bmatrix}. \tag{63}$$

The Jacobian matrix of the same system with respect to ϵ is

$$J_\epsilon = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}. \tag{64}$$

We then have the following:

COROLLARY 1 (First-order approximation of solution to $VI(\epsilon)$ for ϵ near 0)

Under the assumptions of theorem 4, a first-order approximation of $[x(\epsilon), \lambda(\epsilon), \mu(\epsilon)]$ in a neighborhood of $\epsilon = 0$ is given by

$$\begin{bmatrix} x(\epsilon) \\ \lambda(\epsilon) \\ \mu(\epsilon) \end{bmatrix} = \begin{bmatrix} x^* \\ \lambda^* \\ \mu^* \end{bmatrix} - J_y^{-1} J_\epsilon\, \epsilon\,. \tag{65}$$

4.2. SENSITIVITY ANALYSIS OF SPATIAL PRICE EQUILIBRIUM

We need to slightly modify the spatial price equilibrium models described in subsect. 2.1 to put them in a form required by the location models of sect. 3. In particular, the equilibrium model must include the production vector Q. The constraints (6)–(9) have the form

$$h_\ell(f, D, S) = 0 \qquad \text{for all } \ell \in N. \tag{66}$$

These are replaced by

$$h_\ell(f, D, S) = \begin{cases} Q_\ell & \text{if } \ell \in N_0 \\ 0 & \text{otherwise.} \end{cases} \tag{67}$$

In the following, we assume that a solution to the equilibrium model satisfies the conditions of theorem 4. The system of equations equivalent to the system (60)–(62) for the spatial price equilibrium variational inequality (5) over $\bar{\Omega}(Q)$ where

$$\bar{\Omega}(Q) = \{f, D, S : (67) \text{ is satisfied}\}$$

are

$$\begin{aligned} c(f) - \lambda_f - A^T \mu &= 0 \\ -\theta(D) - \lambda_D - E_D^T \mu &= 0 \\ \psi(S) - \lambda_S + E_S^T \mu &= 0 \end{aligned} \tag{68}$$

$$\begin{aligned} \operatorname{diag}(\lambda_f) f &= 0 \\ \operatorname{diag}(\lambda_D) D &= 0 \\ \operatorname{diag}(\lambda_S) S &= 0 \end{aligned} \tag{69}$$

$$Af + E_D D - E_S S - Q = 0, \tag{70}$$

where λ_f is the vector of multipliers associated with the nonnegativity constraints on f, and similarly for λ_D and λ_S; A is the node-arc incidence matrix; E_D is the demand/

node incidence matrix and E_S the supply/node incidence matrix, and Q is a node vector with entries equal to Q_ℓ for $\ell \in N_0$ and zero elsewhere. The Jacobian matrix of the system (68), (69) and (70) with respect to f, D, S, λ, μ is

$$\begin{bmatrix} \nabla c(f) & 0 & 0 & -I & 0 & 0 & -A^T \\ 0 & -\nabla\theta(D) & 0 & 0 & -I & 0 & -E_D^T \\ 0 & 0 & \psi(S) & 0 & 0 & -I & +E_S^T \\ \operatorname{diag}(\lambda_f) & 0 & 0 & \operatorname{diag}(f) & 0 & 0 & 0 \\ 0 & \operatorname{diag}(\lambda_D) & 0 & 0 & \operatorname{diag}(D) & 0 & 0 \\ 0 & 0 & \operatorname{diag}(\lambda_S) & 0 & 0 & \operatorname{diag}(S) & 0 \\ A & E_D & -E_S & 0 & 0 & 0 & 0 \end{bmatrix} = J. \quad (71)$$

Since non-binding constraints do not affect the solution, the system (69) can be reduced to include only binding constraints. Let $\hat{I}_f$, $\hat{I}_D$ and $\hat{I}_S$ be the matrices remaining when columns corresponding to non-binding nonnegativity constraints are deleted from the identity matrices of order $|f|$, $|D|$ and $|S|$, respectively. The reduced Jacobian can then be written (note that the remaining entries of diag(f), diag(D) and diag(S) are zero)

$$\begin{bmatrix} \nabla c(f) & 0 & 0 & -\hat{I}_f & 0 & 0 & -A^T \\ 0 & -\nabla\theta(D) & 0 & 0 & -\hat{I}_D & 0 & -E_D^T \\ 0 & 0 & \psi(S) & 0 & 0 & -\hat{I}_S & +E_S^T \\ \hat{I}_f^T\operatorname{diag}(\lambda_f) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{I}_D^T\operatorname{diag}(\lambda_D) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{I}_S^T\operatorname{diag}(\lambda_S) & 0 & 0 & 0 & 0 \\ A & E_D & -E_S & 0 & 0 & 0 & 0 \end{bmatrix} = \bar{J}. \quad (72)$$

This matrix can be written as a product ΠZ, where

$$Z = \begin{bmatrix} \nabla c(f) & 0 & 0 & -\hat{I}_f & 0 & 0 & -A^T \\ 0 & -\nabla\theta(D) & 0 & 0 & -\hat{I}_D & 0 & -E_D^T \\ 0 & 0 & \psi(S) & 0 & 0 & -\hat{I}_S & +E_S^T \\ -\hat{I}_f^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\hat{I}_D^T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\hat{I}_S^T & 0 & 0 & 0 & 0 \\ -A & -E_D & E_S & 0 & 0 & 0 & 0 \end{bmatrix} \tag{73}$$

and

$$\Pi = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\operatorname{diag}(\lambda_f) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\operatorname{diag}(\lambda_D) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\operatorname{diag}(\lambda_S) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} \tag{74}$$

Let

$$\nabla\phi = \begin{bmatrix} \nabla c(f) & 0 & 0 \\ 0 & -\nabla\theta(D) & 0 \\ 0 & 0 & \nabla\psi(S) \end{bmatrix} \tag{75}$$

and

$$M = \begin{bmatrix} -\hat{I}_f^T & 0 & 0 \\ 0 & -\hat{I}_D^T & 0 \\ 0 & 0 & -\hat{I}_S^T \\ -A & -E_D & E_S \end{bmatrix} \tag{76}$$

Then

$$Z = \begin{bmatrix} \nabla\phi & M^T \\ M & 0 \end{bmatrix} \tag{77}$$

Suppose

$$Z^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{78}$$

and that $\nabla\phi^{-1}$ exists. (This will be the case if f, $-\theta$, and ψ are strictly monotone. Strict monotonicity of these functions also guarantees that the solution to the spatial price equilibrium variational inequality is globally unique.) It is easily shown that

$$B_{11} = \nabla\phi^{-1}\,[I - M^T\,[M\nabla\phi^{-1}M^T]^{-1}\,M\nabla\phi^{-1}\,] \tag{79}$$

$$B_{12} = \nabla\phi^{-1}M^T[M\nabla\phi^{-1}M^T]^{-1} \tag{80}$$

$$B_{21} = [M\nabla\phi^{-1}M^T]^{-1}M\nabla\phi^{-1} \tag{81}$$

$$B_{22} = -\,[M\nabla\phi^{-1}M^T]^{-1}\,. \tag{82}$$

The matrix $[M\nabla\phi^{-1}\,M^T]^{-1}$ exists since by the conditions of theorem 3, the rows of M are linearly independent.

The Jacobian of (68), (69), (70) with respect to Q_ℓ for $\ell \in N_0$ is

$$J_q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -E_{N_0} \end{bmatrix} \tag{83}$$

where E_{N_0} is the diagonal matrix with diagonal entries of one corresponding to $\ell \in N_0$ and zero elsewhere. Let

$$\hat{E}_{N_0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -E_{N_0} \end{bmatrix} \tag{84}$$

and

$$y = [f^T, D^T, S^T, \lambda^T, \mu^T]^T . \tag{85}$$

Then

$$\nabla_Q y = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \hat{\Pi}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \hat{E}_{N_0} \end{bmatrix}, \tag{86}$$

where

$$\hat{\Pi}^{-1} = \begin{bmatrix} -\operatorname{diag}(1/\lambda_f) & 0 & 0 & 0 \\ 0 & -\operatorname{diag}(1/\lambda_D) & 0 & 0 \\ 0 & 0 & -\operatorname{diag}(1/\lambda_S) & 0 \\ 0 & 0 & 0 & -I \end{bmatrix} \tag{87}$$

Therefore,

$$\begin{bmatrix} \nabla_Q f \\ \nabla_Q D \\ \nabla_Q S \end{bmatrix} = B_{12} \hat{\Pi}^{-1} \hat{E}_{N_0} \tag{88}$$

or

$$\begin{bmatrix} \nabla_Q f \\ \nabla_Q D \\ \nabla_Q S \end{bmatrix} = \phi^{-1} M^T [M \quad \phi^{-1} M^T]^{-1} \hat{\Pi}^{-1} \hat{E}_{N_0} . \tag{89}$$

It can be seen from (68) that, in this case, μ is the spatial price equilibrium price vector. Therefore, the derivative of price with respect to Q can be calculated as

$$\begin{bmatrix} \nabla_Q \lambda \\ \nabla_Q \mu \end{bmatrix} = B_{22} \hat{\Pi}^{-1} \hat{E}_{N_0} \tag{90}$$

or

$$\begin{bmatrix} \nabla_Q \lambda \\ \nabla_Q \mu \end{bmatrix} = - [M \nabla \phi^{-1} M^T]^{-1} \hat{\Pi}^{-1} \hat{E}_{N_0} . \tag{91}$$

4.3. SENSITIVITY ANALYSIS OF COURNOT-NASH OLIGOPOLISTIC EQUILBRIUM

The Cournot-Nash oligopolistic equilibrium model described in subsect. 2.2 also needs to be modified to put it in a form required by the location model. In particular, the representation of the firm making the location decisions is different from the other firms. Let this firm be $\hat{k}$. The objective function (32) for $\hat{k}$ is modified to be

$$\text{minimize} \sum_{a \in A} c_a(f_a) \bar{f}_a^{\hat{k}} - \sum_{\ell \in N_D} \theta(D_\ell) \bar{D}_\ell^{\hat{k}} \tag{92}$$

and constraints (21) and (23) are modified as follows

$$D_\ell^\ell + \sum_{a \in T(\ell)} f_a^\ell - \sum_{a \in H(\ell)} f_a^\ell = Q_\ell \qquad \forall \ell \in L(\hat{k}) \cap N_D \tag{93}$$

$$\sum_{a \in T(\ell)} f_a^\ell - \sum_{a \in H(\ell)} f_a^\ell = Q_\ell \qquad \forall \ell \in L(\hat{k}) \backslash N_D . \tag{94}$$

Therefore, the firm $\hat{k}$ is maximizing profit given that production levels Q_ℓ for $\ell \in L(\hat{k})$ are fixed. Let the modified constraint set be denoted as $\Lambda^{\hat{k}}(Q)$. The variational inequality (36) for $\hat{k}$ is to find $\bar{f}^{\hat{k}*}$ and $\bar{D}^{\hat{k}*}$ such that

$$\sum_{a \in A} [\nabla c_a(f_a^*)\bar{f}_a^{\hat{k}*} + c_a(f_a^*)]\,(\bar{f}_a^{\hat{k}} - \bar{f}_a^{\hat{k}*})$$

$$- \sum_{\ell \in N_D} [\nabla \theta_\ell(D_\ell^*)\bar{D}_\ell^{\hat{k}*} + \theta_\ell(D_\ell^*)]\,(\bar{D}_\ell^{\hat{k}} - \bar{D}_\ell^{\hat{k}*}) \geqslant 0 \tag{95}$$

for all $\bar{f}^{\hat{k}}, \bar{D}^{\hat{k}} \in \Lambda^{\hat{k}}$. This variational inequality can be combined with the variational inequalities for the other firms for simultaneous solution as in (37).

The sensitivity of the solution to changes in Q can be determined using theorem 4 and corollary 1. However, in this case, it is not as straightforward to determine the sensitivity of prices to changes in Q. As in the previous case, $\nabla_Q f$, $\nabla_Q D$ and $\nabla_Q S$ can be calculated. Using this information, the prices can be computed. The revenue at node $i \in L(\hat{k})$ is

$$R^i = \sum_{\ell \in N_D} \theta_\ell(D_\ell)D_\ell^i - \sum_{a \in A} c_a(f_a)f_a^i\,. \tag{96}$$

Therefore,

$$\nabla_Q R^i = \sum_{\ell \in N_D} [\theta'_\ell(D_\ell)D_\ell^i\,\nabla_Q D_\ell + \theta_\ell(D_\ell)\,\nabla_Q D_\ell^i]$$

$$- \sum_{a \in A} [c'_a(f_a)f_a^i\,\nabla_Q f_a + c_a(f_a)\,\nabla_Q f_a^i]\,, \tag{97}$$

where " $'$ " denotes the derivative with respect to the argument. The revenue at i is equal to $Q_i\pi_i$, where π_i is the price at i. Therefore,

$$\nabla_Q R^i = e_i\pi_i + Q_i\,\nabla_Q \pi_i\,, \tag{98}$$

where e_i is a vector of length $|Q|$ with a one in the ith position and zeros elsewhere. Then

$$\nabla_Q \pi_i = (\nabla_Q R^i - e_i\pi_i)/Q_i\,, \tag{99}$$

where

$$\pi_i = R^i/Q_i \,. \tag{100}$$

These equations provide the information required by the location models.

5. Solution of the discrete spatial competition location model

5.1. A HEURISTIC SOLUTION ALGORITHM

The following heuristic solution approach uses the results of sect. 4 to determine the sensitivity of profits to production changes and uses this information to select locations and production levels likely to maximize total profits. The competitive equilibrium model can be either the spatial equilibrium or the Cournot-Nash oligopolistic equilibrium model. The algorithm is as follows:

Step 0

Set $j = 0$, $Q^j = 0$, $y^j = 0$.

Step 1

Solve the competitive equilibrium model for $Q = Q^j$. Denote the resulting prices π_ℓ for $\ell \in N_0$ as π_ℓ^j. Evaluate $Z^j = (y^j, Q^j, \pi^j)$. If $j = 0$ or if $Z^j > Z^{j-1}$, go to step 2. Otherwise, take y^{j-1} and Q^{j-1} as a solution to the location model – stop.

Step 2

Obtain the matrix $\nabla_Q \pi$ and the linear constraints

$$\pi - \nabla_Q \pi Q = \pi^j \tag{101}$$

using the results of sect. 4. Substitute these linear constraints for (48) in the discrete spatial competition location model given by (46)–(48) in sect. 3. Solve the resulting linearly constrained, nonlinear integer programming problem

$$\begin{aligned} &\text{maximize} && Z(y, Q, \pi) \\ &\text{subject to} && Q \in \Gamma(y) \\ & && \pi - \nabla_Q \pi Q = \pi^j \\ & && y \text{ a zero-one vector.} \end{aligned}$$

Let the solution be denoted as y^{j+1}, Q^{j+1}, π^{j+1}. Set $j = j + 1$ and go to step 1.

In step 1, the spatial competitive equilibrium model, represented as a variational inequality, can be solved by any of the many solution methods for variational inequalities (see [3,1,8]). Once a solution vector $(f^\star, S^\star, D^\star)$ to the spatial competitive equilibrium model is found, the system of equations (60)–(62) can be solved for $\lambda^\star$ and $\mu^\star$. Then the appropriate Jacobian matrices can be evaluated at the point $(f^\star, S^\star, D^\star, \lambda^\star, \mu^\star)$ and the derivatives of prices with respect to productions calculated.

The integer programming problem to be solved in step 2 is still a difficult problem to optimize. It has a nonlinear objective function with linear and integer constraints. However, the number of locations under consideration, and therefore the number of integer variables y will not, in general, be very large in comparison to the number of variables f, S, and D.

This problem is a special type of capacitated plant location problem and in many cases, the efficient solution methods that have been developed for simple or capacitated plant location problems can be adapted to solve this problem. In general, this problem can be solved by using exact integer programming methods such as branch and bound generalized Bender's decomposition or cutting plane techniques, or by using heuristic methods. Since the overall solution method is a heuristic method, the effort of finding exact optimal solutions to the integer programming sub-problems may or may not prove worth the effort in terms of the resulting quality of the solution of the location model. Which type of solution method is most effective will depend on the size of the fixed costs relative to the variable costs, and whether or not the variable costs are convex or concave.

References

[1] S. Dafermos, An iterative scheme for variational inequalities, Mathematical Programming 26 (1983)40.

[2] D. Erlenkotter, Facility location with price-sensitive demands: Private, public and quasi-public, Management Science 24, 4(1977)378.

[3] T.L. Friesz, P.T. Harker and R.L. Tobin, Alternative algorithms for the general network spatial price equilibrium problem, J. of Regional Science 24, 4(1984)475.

[4] P. Hansen and J.-F. Thisse, Multiplant location for profit maximization, Environment and Planning A9, 1(1977)63.

[5] P.T. Harker, A variational inequality approach for the determination of oligopolistic market equilibrium, Mathematical Programming 30(1984)105.

[6] P.T. Harker, Alternative models of spatial competition, Department of Decision Sciences, The Wharton School, University of Pennsylvania, Working Paper 85-01-04 (1984).

[7] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications* (Academic Press, 1980).

[8] J. Pang and D. Chan, Iterative methods for variational and complemetarity problems, Mathematical Programming 24(1982)284.

[9] R.L. Tobin, Sensitivity analysis for variational inequalities, J. of Optimization Theory and Applications 48, 1(1986)191.

Appendix A

EQUIVALENCE BETWEEN SPATIAL PRICE EQUILIBRIUM CONDITIONS AND VARIATIONAL INEQUALITY FORMULATION

First, to show that variational inequality (5) with constraint set Ω is a necessary condition for equilibrium conditions (1)–(4), note that conditions (1) and (4) are equivalent to the definition of Ω, using the relationship $f_a = \Sigma_p \delta_{ap} h_p$. Therefore, it needs to be shown that (2) and (3) imply (5). Let $h^\star$, $\pi^\star$ satisfy (1)–(4). From (2) and (3), $c_p(h^\star) \geqslant \pi_k^\star - \pi_\ell^\star$ for $p \in P_{\ell k}$ and if $h_p^\star > 0$, then $c_p(h^\star) = \pi_k^\star - \pi_\ell^\star$ for $p \in P_{\ell k}$. Therefore, $c_p(h^\star)\,(h_p - h_p^\star) \geqslant (\pi_j^\star - \pi_i^\star)\,(h_p - h_p^\star)$ for $(h, \pi) \in \Omega$. (Note that if $(h_p - h_p^\star) < 0$, then $h_p^\star > 0$ and $c_p(h^\star) = \pi_k^\star - \pi_\ell^\star$.) If this inequality is summed over all paths from supply nodes to demand nodes,

$$\sum_{i \in N_S} \sum_{j \in N_D} \sum_{p \in P_{ij}} c_p(h^\star)\,(h_p - h_p^\star) \geqslant \sum_{i \in N_S} \sum_{j \in N_D} \sum_{p \in P_{ij}} (\pi_j^\star - \pi_i^\star)\,(h_p - h_p^\star).$$

The left-hand side may be written

$$\sum_{i \in N_S} \sum_{j \in N_D} \sum_{p \in P_{ij}} \sum_a \delta_{ap} c_a(f^\star)\,(h_p - h_p^\star) = \sum_a c_a(f^\star)\,(f_a - f_a^\star).$$

The right-hand side becomes

$$\sum_{i \in N_S} \sum_{j \in N_D} \sum_{p \in P_{ij}} \pi_j(h_p - h_p^\star) - \sum_{i \in N_S} \sum_{j \in N_D} \sum_{p \in P_{ij}} \pi_i(h_p - h_p^\star)$$

or

$$\sum_{j \in N_D} \pi_j \sum_{i \in N_S} \sum_{p \in P_{ij}} (h_p - h_p^\star) - \sum_{i \in N_S} \pi_i \sum_{j \in N_D} \sum_{p \in P_{ij}} (h_p - h_p^\star),$$

which is equivalent to

$$\sum_{j \in N_D} \pi_j(D_j - D_j^\star) - \sum_{i \in N_S} \pi_i(S_i - S_i^\star).$$

Since $\pi_j = \theta_j(D^\star)$ and $\pi_i = \psi_i(S^\star)$, the right-hand side becomes

$$\theta(D^\star)(D-D^\star) - \psi(S^\star)(S-S^\star)$$

and therefore, the inequality is equivalent to

$$c(f^\star)(f-f^\star) - \theta(D^\star)(D-D^\star) - \psi(S^\star)(S-S^\star) \geqslant 0$$
$$\text{for all } (f, D, S) \in \Omega .$$

To show necessity for the case in which path flows are kept track of, note that

$$f_a^\ell = \sum_{j \in N_D} \sum_{p \in P_{\ell j}} \delta_{ap} h_p$$

and

$$f_a = \sum_{\ell \in N_S} f_a^\ell = \sum_{\ell \in N_S} \sum_{j \in N_D} \sum_{p \in P_{\ell h}} \delta_{ap} h_p .$$

Similarly

$$D_\ell^k = \sum_{p \in P_{k\ell}} h_p$$

and

$$D_\ell = \sum_{k \in N_S} D_\ell^k = \sum_{k \in N_S} \sum_{p \in P_{k\ell}} h_p .$$

Therefore, the relationships between f and D, and h are the same as in the previous case, and the same argument follows.

To show sufficiency of (5), note that (5) implies that $(f^\star, D^\star, S^\star)$ is an optimal solution to the linear program

$$\text{minimize} \quad c(f^\star)f - \theta(D^\star)D + \psi(S^\star)S$$

$$\text{subject to} \quad (f, D, S) \in \Omega . \qquad (\pi)$$

The dual is

$$\text{maximize} \quad \pi \cdot 0$$

$$\pi_j - \pi_i \leqslant c_a(f^\star) \qquad \forall a \, (a = (i,j)) \qquad (f_a)$$

$$-\pi_\ell \leqslant -\theta(D^\star) \qquad \forall \ell \in N_D \qquad (D_\ell)$$

$$\pi_\ell \leqslant \psi(S^\star) \qquad \forall \ell \in N_S \qquad (S_\ell)$$

π unrestricted.

By complementary slackness, if $f_a^\star \geqslant 0$, then $c_a(f^\star) = \pi_j - \pi_i$, if $D_\ell^\star > 0$, then $\pi_\ell = \theta(D^\star)$, and if $S_\ell^\star > 0$, then $\pi_\ell = \psi(S^\star)$. Therefore, for a path $p \in P_{km}$ for $k \in N_S$ and $m \in N_D$ with $h_p > 0$, it follows that

$$c_p(h^\star) = \sum_a \delta_{ap} c_a(f^\star) = \pi_m - \pi_k = \theta_m(D^\star) - \psi_k(S^\star).$$

Equivalently, if

$$c_p(h^\star) > \theta_m(D^\star) - \psi_k(S^\star)$$

then $h_p^\star = 0$. Therefore, (2) and (3) are satisfied, and (1) and (4) are satisfied since $(f^\star, D^\star, S^\star) \in \Omega$.

For the case using Ω', the argument is the same. In this case, the dual variables are $\pi = [\pi_\ell^k]$ for $k \in N_S$ and $\ell \in N$. The dual is

$$\text{maximize} \quad \pi \cdot 0$$

$$\text{subject to} \quad \pi_j^k - \pi_i^k \leqslant c_a(f^\star) \qquad \forall a \text{ and } k \in N_S \qquad (f_a^k)$$

$$-\pi_\ell^k \leqslant -\theta(D^\star) \qquad \forall \ell \in N_D \text{ and } k \in N_S \qquad (D_\ell^k)$$

$$\pi_\ell^\ell \leqslant \psi(S^\star) \qquad \forall \ell \in N_S \qquad (S_\ell).$$

By complementary slackness, if $f_a^{k\star} > 0$, then $c_a(f^\star) = \pi_j^k - \pi_i^k$, if $D_\ell^{k\star} > 0$, then $\pi_\ell^k = \theta_\ell(D^\star)$, and if $S_\ell > 0$, then $\pi_\ell^\ell = \psi_\ell(S^\star)$. Therefore, for a path $p \in P_{km}$ with $h_p > 0$

$$c_p(h^\star) = \sum_a \delta_{ap} c_a(f^\star) = \pi_m^k - \pi_k^k = \theta_m(D^\star) - \psi_k(S^\star).$$