ON THE FEUERBACH-SPHERES OF AN ORTHOCENTRIC SIMPLEX

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1. Attempting to extend the theorems of the geometry of the triangle to three and more dimensions it is known by experience that strict analogies exist only in the case of an orthocentric tetrahedron or simplex, i. e., such one whose altitudes have a point in common.

Actually it has been recognized rather long ago that the Feuerbach-circle (or nine-point circle) has no analogon in the case of a general tetrahedron, but if the tetrahedron is orthocentric then there are two spheres (the „twelve-point" spheres), each of which can be regarded as the three dimensional extension of the Feuerbach-circle.

Recent researches[1] have shown that with an orthocentric simplex in $n-1$ dimensions n spheres may be associated so that the k-th sphere contains the orthocenters and the barycenters of all the $k-1$ dimensional partial simplices. Consequently, in $n-1$ dimensions there are n extensions of the Feuerbach-circle. These Feuerbach-spheres belong to the pencil which is determined by the circumsphere and the polarsphere.

It is well known that the discussion of the Feuerbach-figure is rather asymmetrical and cumbersome if one uses Cartesian coordinates. Therefore several writers, when dealing with the Feuerbach-circle, made use of triangular coordinates. The discussion in Cartesian coordinates becomes even more clumsy in the case of three and more dimensions.

In the present paper we wish to show that the Feuerbach-figure in any dimension is capable of a symmetrical and very intuitive analytic representation by means of an orthocentric system of coordinates.

The orthocentric system of coordinates is a barycentric system whose fundamental simplex (tetrahedron) is orthocentric. In order to simplify the discussion of metric questions, the barycentric coordinates can be normalized in such a way that their sum should be equal to 1.

[1] E. Egerváry, On Orthocentric Simplexes, *Acta. Scient. Math. Szeged.*, **9** (1940), pp. 218–226.

These orthocentric coordinates proved to be a very useful instrument in the treatment of some metric questions[2]. Their application to the investigation of the Feuerbach-figure is particularly suggested by their close connection with the poly(penta)spherical coordinates.

2. The Feuerbach-circle possesses the following characteristic properties:

I. It is a Carnot-conic of the fundamental triangle belonging to the orthocenter and the barycenter.

II. It is a pedal-conic of the fundamental triangle belonging to the orthocenter and the circumcenter.

III. It touches each of the four tangent-circles of the fundamental triangle.

I shall show first that a simplex in $n-1$ dimensions being given, one can associate with any pair of points n quadrics each of which is an extension of the Carnot-conic.

In agreement with this interpretation, the Feuerbach-spheres $\Phi_{(1)}$, $\Phi_{(2)}, \ldots, \Phi_{(n)}$ of an orthocentric simplex appear as the Carnot-quadrics belonging to the orthocenter and the barycenter.

Moreover I prove that the Feuerbach-sphere $\Phi_{(k)}$ is the pedal-quadric of the orthocenter and of its symmetric with respect to the center of $\Phi_{(\,)}$, i. e., it passes through the orthogonal projections of these points on the $k-1$ dimensional partial simplices.

Before entering into the discussion of the Feuerbach-figure we give a short summary of the properties of the orthocentric system of coordinates.

I. The orthocentric system of coordinates

1. If in the space of $n-1$ dimensions a fixed set of N ($\geqq n$) points $P_1, P_2, \ldots, P_N$ (containing at least one non-degenerate $n-1$ dimensional simplex) is given, then any point X may be represented as the barycenter of masses ξ_i placed in the points P_i $(i = 1, 2, \ldots, N)$. Using the notation of GRASSMANN, a point X can be represented in the form

$$X = \frac{\xi_1 P_1 + \xi_2 P_2 + \ldots + \xi_N P_N}{\xi_1 + \xi_2 + \ldots + \xi_N}, \tag{1}$$

and the (real) numbers ξ_i are the homogeneous (and in the case $N > n$ *supernumerary*) barycentric coordinates of the point X. When dealing with metric problems it is convenient to use normal barycentric coordinates submitted to the restriction $\sum_1^N \xi_i = 1$ (with the exception of points at infinity).

[2] See e. g. E. EGERVÁRY, Über ein räumliches Analogen des Sehnenvierecks, *Journ. f. Math.*, **182** (1940) pp. 122—128.

The distance of two points $X = \Sigma \xi_i P_i$ and $Y = \Sigma \eta_i P_i$ is given by

$$\overline{XY}^2 = - \sum_{\substack{i,j=1 \\ (i>j)}}^{N} \overline{P_i P_j}^2 \, (\xi_i - \eta_i)(\xi_j - \eta_j), \qquad \sum_1^N \xi_i = \sum_1^N \eta_i = 1. \tag{2}$$

2. A general simplex of n points $P_1 P_2 \ldots P_n$ in $n - 1$ dimensions is determined by $\frac{n(n-1)}{2}$ independent parameters, e. g. by the lengths of its edges $\overline{P_i P_j}$. If the simplex is orthocentric, i. e., if its altitudes meet in one point P_0 (the orthocenter), then its parameters have to satisfy $\frac{n(n-3)}{2}$ equations. In order to avoid the necessity of having regard continually to these equations of condition it seems to be desirable to determine an orthocentric simplex by a minimal number of independent and symmetrical parameters.

In one of my previous papers[1] I have shown that an orthocentric simplex $P_1 P_2 \ldots P_n$ in $n - 1$ dimensions can be determined by n independent and symmetrical parameters in such a way, that the measure (length, volume) of its $k - 1$ dimensional partial simplices ($k = 1, 2, \ldots, n - 1$) is immediately given by the elementary symmetric functions of the parameters in the form

$$(k-1)!^2 \, \overline{P_{j_1} P_{j_2} \ldots P_{j_k}}^2 = \lambda_{j_1} \lambda_{j_2} \ldots \lambda_{j_k} \left(\frac{1}{\lambda_{j_1}} + \frac{1}{\lambda_{j_2}} + \ldots + \frac{1}{\lambda_{j_k}} \right). \tag{3}$$

In particular

$$\overline{P_j P_k}^2 = \lambda_j + \lambda_k . \tag{3'}$$

From this representation it is obvious that any partial simplex of an orthocentric simplex is orthocentric too.

For some purposes it is more convenient to consider the set of $n + 1$ points: the orthocenter P_0 and the vertices $P_1, P_2, \ldots, P_n$ as a whole, called an orthocentric set of $n + 1$ points in the $n - 1$ dimensional space, each point of the set being the orthocenter of the simplex formed by the others.

The mutual distances $\overline{P_i P_j}$ of the points of an orthocentric set can be expressed in the same way by $n + 1$ symmetric parameters λ_i ($i = 0, 1, \ldots, n$)

$$\overline{P_i P_j}^2 = \lambda_i + \lambda_j \tag{3''}$$

but the n-dimensional measure of the simplex formed by the points $P_0 P_1 \ldots P_n$ of the $n - 1$ dimensional space is equal to 0, consequently, the parameters λ_i are restricted by the relations

$$\frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda_n} = 0, \qquad (\lambda_i + \lambda_j > 0). \tag{4}$$

According to this there is always one and only one negative value amongst the parameters λ_i, or, what is the same thing, one and only one point of the set $P_0 P_1 \ldots P_n$ is contained in the interior of the convex cover of the set.

The consideration of the $n-1$ dimensional volumes of the simplices contained in the orthocentric set $P_0 P_1 \ldots P_n$ shows immediately that if the masses $\frac{1}{\lambda_i}$ are placed in the points P_i $(i = 0, 1, \ldots, n)$, then each point is the barycenter of the masses placed in the others. In other words, the mass-points $\frac{P_0}{\lambda_0}, \frac{P_1}{\lambda_1}, \ldots, \frac{P_n}{\lambda_n}$ satisfy the equations

$$\frac{P_0}{\lambda_0} + \frac{P_1}{\lambda_1} + \ldots + \frac{P_n}{\lambda_n} = 0, \qquad \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda_n} = 0,$$

i. e. they constitute an indifferent mass-system.

3. In general we shall use in this paper the barycentric simplex-coordinates $x_1 x_2 \ldots x_n$ referring to the basic orthocentric simplex $P_1 P_2 \ldots P_n$. But, for the sake of simplicity and symmetry, at the beginning we shall develop the most important metric relations in terms of the supernumerary barycentric coordinates $\xi_0 \xi_1 \xi_2 \ldots \xi_n$ referring to the orthocentric set $P_0 P_1 P_2 \ldots P_n$. Afterwards we arrive by specialisation at the corresponding relations in terms of the simplex coordinates.

Let $\xi_0 \xi_1 \xi_2 \ldots \xi_n$ be a system of supernumerary barycentric coordinates of a point X, i. e., $X = \sum_0^n P_i \xi_i \Big/ \sum_0^n \xi_i$; $\sum_0^n \xi_i \neq 0$. Then, from the fact that the mass-system P_i/λ_i is indifferent, we infer immediately that the most general system of supernumerary coordinates belonging to the same point X is given by

$$\xi_i' = \xi_i + \frac{\varrho}{\lambda_i}, \ (i = 0, 1, \ldots, n); \ X = \frac{\sum_0^n P_i \xi_i}{\sum_0^n \xi_i} = \frac{\sum_0^n P_i \xi_i'}{\sum_0^n \xi_i} \tag{5}$$

where ϱ denotes an arbitrary real parameter.

Applying the expression (2) of the distance of two points and having regard to the relations (3″), we find immediately that the expression for the distance in terms of the coordinates ξ_i is given by

$$\overline{XY}^2 = -\sum \sum \overline{P_i P_j}^2 (\xi_i - \eta_i)(\xi_j - \eta_j) = \sum_0^n \lambda_i (\xi_i - \eta_i)^2; \quad \sum_0^n \xi_i = \sum_0^n \eta_i = 1. \tag{6}$$

I should not miss here to draw the attention to the analogy between the orthogonal Cartesian coordinates and the orthocentric coordinates. For both system of coordinates (and only in these cases) the expression for the squared distance reduces to a sum of squares.

4. The equation of any sphere, having the center $C = \sum_0^n \gamma_i P_i$; $\sum_0^n \gamma_i = 1$ and the radius $r^2 = \sum_0^n \lambda_i \gamma_i^2$, may be derived immediately from (6)

$$\sum_0^n \lambda_i (\xi_i - \gamma_i)^2 - r^2 \equiv \sum_0^n \lambda_i \xi_i^2 - 2\sum_0^n \lambda_i \gamma_i \xi_i = 0; \qquad \sum_0^n \xi_i = \sum_0^n \gamma_i = 1,$$

or in homogeneous form

$$\sum_0^n \lambda_i \xi_i^2 - 2 \sum_0^n \lambda_i \gamma_i \xi_i \sum_0^n \xi_i = 0. \tag{7}$$

We infer from this that the $n + 1$ spheres Γ_k having their centers in the vertices $P_k = \sum \delta_{ki} P_i$ ($\delta_{ki} = 0$ for $k \neq i$, $\delta_{kk} = 1$) and their radii $r_k^2 = \lambda_k$ are represented by the equations

$$\Gamma_k \equiv \sum_0^n \lambda_i \xi_i^2 - 2 \lambda_k \xi_k \sum_0^n \xi_i = 0, \quad (k = 0, 1, 2, \ldots, n). \tag{8}$$

Any two of these spheres are orthogonal to each other in consequence of the equations $r_k^2 + r_h^2 - \overline{P_k P_h}^2 = \lambda_k + \lambda_h - (\lambda_k + \lambda_h) = 0$.

Comparing (7) with (8) we see that the equation of any sphere (point, plane) may be written as a homogeneous, linear combination of the forms Γ_k

$$\sum_0^n \gamma_i \sum_0^n \lambda_i \xi_i^2 - 2 \sum_0^n \lambda_i \gamma_i \xi_i \sum_0^n \xi_i \equiv \sum_0^n \gamma_i \Gamma_i = 0 \tag{9}$$

and in the case of this representation of a sphere the *supernumerary* coordinates of its center are proportional to the coefficients γ_i, while the radius is given by $r^2 = \sum_0^n \lambda_i \gamma_i^2 / (\sum_0^n \gamma_i)^2$.

The plane resp. the point are obviously characterised by the relations $\sum_0^n \gamma_i = 0$ resp. $\sum_0^n \lambda_i \gamma_i^2 = 0$.

5. In order to pass from the supernumerary coordinates $(\xi_0 \xi_1 \xi_2 \ldots \xi_n)$ to the simplex-coordinates $(x_1 x_2 \ldots x_n)$ referring to one of the simplices, e. g. to $P_1 P_2 \ldots P_n$, we have only to impose on the supernumerary coordinates the restriction that the point P_0 is deprived of mass. This involves that the arbitrary parameter ϱ in the expression (5) of the supernumerary coordinates must be chosen so that $x_0 = \xi_0 - \varrho/\lambda_0 = 0$, i. e. $\varrho = \xi_0 \lambda_0$. Hence the passage from ξ_i to x_j is mediated by the equations (and similarly the passage from γ_i to c_j)

$$x_j = \xi_j - \frac{\lambda_0}{\lambda_j} \xi_0 \quad (j = 1, 2, \ldots, n). \tag{10}$$

By means of these equations $\sum_0^n \lambda_i (\xi_i - \gamma_i)^2$ will be transformed into $\sum_1^n \lambda_j (x_j - c_j)^2$ and similarly $\sum_0^n \xi_i$ into $\sum_0^n x_j$; consequently, the general equation of a sphere in orthocentric simplex-coordinates takes the homogeneous form

$$(11) \qquad \sum_1^n \lambda_j x_j^2 - 2 \sum_1^n a_j x_j \sum_1^n x_j = 0$$

with arbitrary coefficients a_j.

Comparing this with (9), excluding the case of a plane and assuming $\sum_0^n \gamma_i = 1$, we get for the *supernumerary* coordinates γ_i and the radius r of the sphere the following expressions:

$$(12) \quad \gamma_0 = 1 - \sum_1^n \frac{a_j}{\lambda_j}; \quad \gamma_j = \frac{a_j}{\lambda_j}; \quad r^2 = \sum_1^n \lambda_i \gamma_i^2 = \lambda_0 \left(1 - \sum_1^n \frac{a_j}{\lambda_j}\right)^2 + \sum_1^n \frac{a_j^2}{\lambda_j}.$$

The corresponding formulae in terms of the orthocentric simplex-coordinates x_j follow from here immediately by substituting these values in the equations (10).

II. Representation of the Feuerbach-spheres as Carnot-quadrics

1. Two points

$$U = \frac{u_1 P_1 + u_2 P_2 + u_3 P_3}{u_1 + u_2 + u_3} \quad \text{and} \quad V = \frac{v_1 P_1 + v_2 P_2 + v_3 P_3}{v_1 + v_2 + v_3}$$

in the plane of the basic-triangle $P_1 P_2 P_3$ should be projected from the vertices on the opposite sides. According to a theorem of CARNOT, these projections $(0\, u_2 u_3)$ $(u_1\, 0\, u_3)$ $(u_1 u_2\, 0)$; $(0\, v_2 v_3)$ $(v_1\, 0\, v_3)$ $(v_1 v_2\, 0)$ lie on a conic represented by the equation

$$2\left(\frac{x_1^2}{u_1 v_1} + \frac{x_2^2}{u_2 v_2} + \frac{x_3^2}{u_3 v_3}\right) - \left(\frac{x_1}{u_1} + \frac{x_2}{u_2} + \frac{x_3}{u_3}\right)\left(\frac{x_1}{v_1} + \frac{x_2}{v_2} + \frac{x_3}{v_3}\right) = 0.$$

Consider now two points U, V in the $n-1$ dimensional space, whose barycentric coordinates are $(u_1\, u_2 \ldots u_n)$ resp. $(v_1\, v_2 \ldots v_n)$.

These points U, V can now be projected from any $n-k-1$ dimensional partial simplex $P_{k+1} P_{k+2} \ldots P_n$ on the $k-1$ dimensional complementary partial simplex $P_1 P_2 \ldots P_k$. The projection of U (being the point of intersection of the $n-k$ dimensional plane $P_{k+1} P_{k+2} \ldots P_n U$ with the $k-1$ dimensional plane $P_1 P_2 \ldots P_k$) has the coordinates

$$(u_1\, u_2 \ldots u_{k-1}\, u_k\, 0\, 0 \ldots 0)$$

and the coordinates of the projection of V are

$$(v_1 \, v_2 \ldots v_{k-1} \, v_k \, 0 \, 0 \ldots 0).$$

Obviously all these projections of the pair of points U, V on the $k-1$ dimensional partial simplices lie on a quadric whose equation is easily found to be

$$k \sum_1^n \frac{x_j^2}{u_j v_j} - \sum_1^n \frac{x_j}{u_j} \sum_1^n \frac{x_j}{v_j} = 0. \tag{13}$$

Consequently, with any pair of points and with an $n-1$ dimensional (general) simplex n quadrics may be associated each of which can be interpreted as an extension of the Carnot-conic, i. e., the quadrics given by (13) for the values $k = 1, 2, \ldots, n$.

2. Let us now assume that the basic simplex $P_1 P_2 \ldots P_n$ is an orthocentric one, specified by the parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ and consider the Carnot-quadrics $\Phi_{(1)}, \Phi_{(2)}, \ldots, \Phi_{(n)}$ associated with the orthocenter $O = P_0 = \sum_1^n \frac{P_j}{\lambda_j} \Big/ \sum_1^n \frac{1}{\lambda_j}$ and the barycenter $B = \frac{1}{n} \sum_1^n P_j$.

Their equation follows at once from (13):

$$\Phi_{(k)} \equiv k \sum_1^n \lambda_j x_j^2 - \sum_1^n \lambda_j x_j \sum_1^n x_j = 0 \quad (k = 1, 2, \ldots, n), \tag{14}$$

and comparing this equation with (11) we notice that they represent n spheres belonging to the pencil which is determined by the polarsphere ($k = \infty$) and the circumsphere ($k = 1$). But the projections of O and B are obviously identical with the orthocenters and barycenters of the partial simplices, consequently, each of the spheres $\Phi_{(1)}, \Phi_{(2)}, \ldots \Phi_{(n)}$ can be interpreted as an $n-1$ dimensional extension of the Feuerbach-circle. The sphere $\Phi_{(k)}$ ($k = 2, 3, \ldots, n-1$) passes through the orthocenters and barycenters of all the $k-1$ dimensional partial simplices, $\Phi_{(1)}$ is the circumsphere; $\Phi_{(n)}$ is the orthocentroidal sphere, because it has the join of the orthocenter O and barycenter B as diameter.

Applying (12) we get for the radius $r_{(k)}$ and the *supernumerary* coordinates $\gamma_i^{(k)}$ of the center $C_{(k)}$ of the Feuerbach-sphere $\Phi_{(k)}$

$$\begin{aligned} &\gamma_0^{(k)} = 1 - \frac{n}{2k}; \quad \gamma_j^{(k)} = \frac{1}{2k} \quad (j = 1, 2, \ldots, n); \\ &r_{(k)} = \sqrt{(n-2k)^2 \lambda_0 + \lambda_1 + \ldots \lambda_n}/2k. \end{aligned} \tag{15}$$

According to (15) the center $C_{(k)}$ of $\Phi_{(k)}$ is given by

$$C_{(k)} = \frac{(2k-n) O + n B}{2k} = O + \frac{n}{2k}(B - O), \tag{16}$$

i. e., $C_{(k)}$ is collinear with O and B and coincides with the terminal-point of the vector $\frac{n}{2k}\overrightarrow{OB}$ issuing from O. Thus all the centers lie on the „Euler-line" joining O and B.

3. Having regard to the following discussions it is convenient to distribute the Feuerbach-spheres into complementary pairs $\Phi_{(k)}$, $\Phi_{(n-k)}$ (which coincide only for $n = 2m$, $k = m$). The radii of a complementary pair are connected by the simple relation

$$k\, r_{(k)} = (n-k)\, r_{(n-k)},$$

while their centers satisfy the equations

$$2\,k C_{(k)} = nB + (2\,k - n)\, O; \quad 2\,(n-k)\, C_{(n-k)} = n\,B - (2\,k - n)\, O,$$

i. e., the barycenter B, the orthocenter O and the centers $C_{(k)}$, $C_{(n-k)}$ of a pair of complementary Feuerbach-spheres form a harmonic quadruple of points.

Particularly in the case of the orthocentric tetrahedron ($n = 4$) we get from (15)

$$C_{(1)} = 2\,B - O,\ r_{(1)} = \sqrt{4\,\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}/2 \quad \text{(circumsphere)}$$

$$C_{(2)} = B \quad ,\ r_{(2)} = \sqrt{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}/4 \quad \text{(first twelve-point-sphere)}$$

$$3\,C_{(3)} = 2\,B + O,\ r_{(3)} = \sqrt{4\,\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}/6 \quad \text{(second twelve-point-sphere)}$$

in agreement with the well known results in the geometry of the tetrahedron.

It has been observed that the four circumcircles, resp. the four Feuerbach-circles of triangles contained in a planar orthocentric quadruple of points are equal, while in the three dimensional space no such relations exist. The expression (15) of $r_{(k)}$ shows clearly that similar relations exist only in a space of an even number of dimensions. In the case of $2\,m$ dimensions we have from (15)

$$r_{(k)} = \sqrt{\lambda_0 + \lambda_1 + \ldots + \lambda_{2m+1}}/2\,k \qquad (k = m,\ m+1),$$

i. e., the Feuerbach-spheres $\Phi_{(m)}$ resp. $\Phi_{(m+1)}$ of the orthocentric simplices contained in an orthocentric set of $2\,m + 1$ points in $2\,m$ dimensions are equal.

III. Representation of the Feuerbach-spheres as pedal quadrics

1. An other characteristic feature of the Feuerbach-circle is that it passes through the orthogonal projections of the orthocenter and circumcenter on the sides of the basic triangle (it is the pedal-circle of O and $C_{(1)}$).

In order to establish the corresponding property of the Feuerbach-spheres in $n-1$ dimensions let us consider the intersection of the Feuerbach-sphere

$$\Phi_{(k)} \equiv k \sum_1^n \lambda_j\, x_j^2 - \sum_1 \lambda_j\, x_j \sum_1^n x_j = 0$$

with one of „its" partial-simplices, e. g. with

$$x_{k+1} = x_{k+2} = \ldots = x_n = 0.$$

This intersection

$$\Phi^* \equiv k \sum_1^k \lambda_j x_j^2 - \sum_1^k \lambda_j x_j \sum_1^k x_j = 0$$

s obviously the orthocentroidal-sphere of the partial-simplex $P_1 P_2 \ldots P_k$. According to one of our former results **(II. 2.)** the center C^* of Φ^* is collinear with and equidistant to the orthocenter O^* and the barycenter B^* of the partial simplex $P_1 P_2 \ldots P_k$.

Erect now through O^*, C^* and B^* $n-k-1$ dimensional planes which are perpendicular to the $k-1$ dimensional plane of $P_1 \ldots P_k$. These parallel planes meet the Euler-line of the basic orthocentric simplex in O, $C_{(k)}$ and in a point $B_{(k)}$, this latter being obviously the symmetric of O with respect to $C_{(k)}$.

Starting with any other $k-1$ dimensional partial-simplex, we arrive evidently at the same point $B_{(k)}$. Hence we have the theorem:

The Feuerbach-sphere $\Phi_{(k)}$ is the pedal-sphere of the orthocenter O and of its symmetric $B_{(k)}$ with respect to the center $C_{(k)}$ of $\Phi_{(k)}$. In other words, $\Phi_{(k)}$ passes through the orthogonal projections of O and $B_{(k)}$ on all the $k-1$ dimensional partial-simplices.

The point $B_{(k)}$, being the symmetric of O with respect to $C_{(k)}$, is represented by

$$B_{(k)} = 2\,C_{(k)} - O = O + \frac{n}{k}\,(B - O),$$

i. e., $B_{(k)}$ coincides with the terminal-point of the vector $\frac{n}{k}\overrightarrow{OB}$ issuing from O.

All these centers of projection lie on the Euler-line, moreover $C_{(2k)}$ coincides with $B_{(k)}$.

IV. On the intersections of the Feuerbach-spheres and the altitudes

1. In the case of three and more dimensions the notion of the altitude may be viewed from a more general point of view, i. e., as the common perpendicular of a pair of complementary partial-simplices. This extension is justified by the fact that the normal-transversal of any pair of complementary partial-simplices passes through the orthocenter O as well as through the orthocenters $O_{(k)}$ and $O_{(n-k)}$ of the complementary partial-simplices.

Indeed, the orthocenter of $P_1 P_2 \ldots P_n$ is $O = \sum P_j/\lambda_j : \sum_1^n 1/\lambda_j$, the orthocenters of $P_1 P_2 \ldots P_k$ resp. $P_{k+1} P_{k+2} \ldots P_n$ are $O_{(k)} = \sum_1^k P_j/\lambda_j : \sum_1^k 1/\lambda$

resp. $O_{(n-k)} = \sum_{k+1}^{n} P_j/\lambda_j : \sum_{k+1}^{n} 1/\lambda_j$. Hence

$$O_{(k)} \sum_{1}^{k} 1/\lambda_j + O_{(n-k)} \sum_{k+1}^{n} 1/\lambda_j = O \sum_{1}^{n} 1/\lambda_j ,$$

i. e., O, $O_{(k)}$, $O_{(n-k)}$ are collinear.

We have further $\overline{OO_{(k)}} \perp \overline{P_1 P_2 \ldots P_k}$ and $\overline{OO_{(n-k)}} \perp \overline{P_{k+1} \ldots P_n}$. consequently the altitude $\overline{O_{(k)}\, O_{(n+k)}}$ belonging to $P_1 P_2 \ldots P_k$ and $P_{k+1} P_{k+2} \ldots P_n$ is the normal-transversal of these partial-simplices.

In this wider sense the number of the altitudes of an orthocentric simplex $P_1 P_2 \ldots P_n$ in the $n-1$ dimensional space is obviously

$$2^{2m} - 1 \quad \text{for} \quad n = 2m + 1 \quad \text{and} \quad 2^{2m-1} + \frac{1}{2}\binom{2m}{m} - 1 \quad \text{for} \quad n = 2m.$$

We shall prove that the products of the segments, into which any of these altitudes is divided by the orthocenter, has the same value, this common value being equal to $|\lambda_0|$.

The segment $\overline{OO_{(k)}}$ is the altitude (in the narrower sense) of the partial-simplex $P_0 P_1 \ldots P_k$, therefore its length is given by

$$\overline{OO}^2_{(k)} = k^2 \frac{\overline{P_0 P_1 P_2 \ldots P_k}^2}{\overline{P_1 P_2 \ldots P_k}^2} \text{ and similarly } \overline{OO}^2_{(n-k)} = (n-k)^2 \frac{\overline{P_0 P_{k+1} \ldots P_n}^2}{\overline{P_{k+1} \ldots P_n}^2}$$

Applying (3) we get from here

$$\overline{OO}^2_{(k)} \cdot \overline{OO}^2_{(n-k)} =$$

$$= \frac{\lambda_0 \lambda_1 \ldots \lambda_k \left(\frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda_k}\right)}{\lambda_1 \ldots \lambda_k \left(\frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda_k}\right)} \cdot \frac{\lambda_0 \lambda_{k+1} \ldots \lambda_n \left(\frac{1}{\lambda_0} + \frac{1}{\lambda_{k+1}} + \ldots + \frac{1}{\lambda_n}\right)}{\lambda_{k+1} \ldots \lambda_n \left(\frac{1}{\lambda_{k+1}} + \ldots + \frac{1}{\lambda_n}\right)}$$

or, having regard to $\sum_{0}^{n} 1/\lambda_i = 0$,

$$\overline{OO}^2_{(k)} \cdot \overline{OO}^2_{(n-k)} = \left(\lambda_0 + \frac{1}{\frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda_k}}\right)\left(\lambda_0 + \frac{1}{\frac{1}{\lambda_{k+1}} + \ldots + \frac{1}{\lambda_n}}\right) = \lambda_0^2 ,$$

q. e. d.

2. The denomination „nine-point circle" is justified by the fact, that this circle passes not only through the projections of O and B on the sides, but it passes also through the middle-point of the „upper" segments of the altitudes. A corresponding theorem is known in the case of an orthocentric tetrahedron, i. e., the sphere which contains the orthocenters and barycenters of the faces, divides the „upper" segments of the altitudes in the ratio 1 : 2.

These statements can be generalised as follows. Each extremity of an altitude is lying on a Feuerbach-sphere. E. g. the extremities of an altitude of type $\overline{O_{(k)}O_{(n-k)}}$ (belonging to a $k-1$ dimensional and to an $n-k-1$ dimensional partial-simplex) are lying on $\Phi_{(k)}$ and $\Phi_{(n-k)}$. Using our former results we can easily determine the second point of intersection of a Feuerbach-sphere with its corresponding altitude.

Retaining the previously used notations and denoting the second point of intersection of $\Phi_{(k)}$ with its altitude $\overline{O_{(k)}O_{(n-k)}}$ by $O'_{(k)}$, we inquire after the value of the ratio

$$\frac{\overline{OO'_{(k)}}}{\overline{OO_{(n-k)}}}=\frac{\overline{OO'_{(k)}}\cdot\overline{OO_{(k)}}}{\overline{OO_{(n-k)}}\cdot\overline{OO_{(k)}}}.$$

It has been previously proved that $\overline{OO_{(k)}}\cdot\overline{OO_{(n-k)}}=\lambda_0$, thus we have only to calculate $\overline{OO'_{(k)}}\cdot\overline{OO_{(k)}}$, i. e., the power of the orthocenter O with respect to the Feuerbach-sphere $\Phi_{(k)}$. Using our former results (15) we have

$$\overline{OO'_{(k)}}\cdot\overline{OO_{(k)}}=OC^2_{(k)}-r^2_{(k)}=\frac{n^2}{k^2}r^2_{(n)}-r^2_{(k)}=\lambda_0\frac{n-k}{k}.$$

Hence we infer that $\overline{OO'_{(k)}}:\overline{OO_{(n-k)}}=(n-k):k$ and this result can be stated as follows.

Each altitude of the type $\overline{O_{(k)}O_{(n-k)}}$ joins a point $O_{(k)}$ of the Feuerbach-sphere $\Phi_{(k)}$ to a point $O_{(n-k)}$ of the complementary Feuerbach-sphere $\Phi_{(n-k)}$. Consequently, this altitude cuts both of $\Phi_{(k)}$, $\Phi_{(n-k)}$ once more, and the positions of these intersections $O'_{(k)}$, $O'_{(n-k)}$ are determined by the ratios

$$\frac{\overline{OO'_{(k)}}}{\overline{OO_{(n-k)}}}=\frac{n-k}{k};\quad\frac{\overline{OO'_{(n-k)}}}{\overline{OO_{(k)}}}=\frac{k}{n-k}.$$

This theorem contains the above mentioned results in the geometry of the triangle, resp. of the tetrahedron, if we adopt the circumcircle, resp. sphere as the first member $\Phi_{(1)}$ in the Feuerbach-sequence.

If $n=2\,m$ and $k=m$, then the two complementary Feuerbach-spheres coincide. This is realized in three dimensions ($n=4$) for $k=2$, i. e., in the case of the first twelve-point sphere.

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ШАРЫ ФЕЙЕРБАХА ОРТОЦЕНТРИЧНОГО СИМПЛЕКСА

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(Резюме)

Если ребра симплекса $P_1 P_2 \ldots P_n$ размерности $n-1$ могут быть выражены с помощью n параметров в виде $\overline{P_i P_j}^2 = \lambda_i + \lambda_j$, то симплекс является ортоцентричным и в относящейся к нему барицентричной системе координат $(x_1, x_2, \ldots, x_n)$ координаты ортоцентра суть $\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\right)$ и общее уравнение шара будет

$$a_0 \sum_1^n \lambda_i x_i^2 - \sum_1^n a_i x_i \sum_1^n x_i = 0.$$

Уравнение

$$k \sum_1^n \lambda_i x_i^2 - \sum_1^n \lambda_i x_i \sum_1^n x_i = 0$$

получающееся из соответствующей специализации коэффициентов $a_0 a_1 \ldots a_n$ означает такой шар, который содержит ортоцентры и барицентры всех частных симплексов $P_{\nu_1} P_{\nu_2} \ldots P_{\nu_k}$ всех размерностей $k-1$, то есть все шары соответствующие $k = 1, 2, \ldots, n$ могут считаться обобщением окружности Фейербаха. Использованием вышеприведенного уравнения шаров Фейербаха многие теоремы тригонометрии и геометрии тетраэдров обобщаются на ортоцентричные симплексы.