ON ABELIAN GROUPS WITH COMMUTATIVE ENDOMORPHISM RING

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§ 1. Introduction

As is well known, the endomorphism ring of an abelian group is in general neither commutative nor without zero divisors. This gives rise to the problem of describing all abelian groups with commutative endomorphism ring and those with endomorphism ring containing no zero-divisors. In a previous paper one of us has considered the latter problem [3][1] and has succeeded in showing that there exists no such group among the mixed groups, while $C(p)$ and $C(p^\infty)$ are the only torsion groups of this property.[2] The present paper is devoted — leaving again the torsion free groups out of consideration — to abelian groups with commutative endomorphism ring. This problem will be solved completely for torsion groups; viz. we shall prove that the endomorphism ring of a torsion group is commutative if and only if the group is isomorphic with a subgroup of the group C of all rotations of finite order of the circle (Theorem 1). Moreover, we can characterize two sufficiently large classes of mixed groups with commutative endomorphism ring (Theorems 2 and 3), but we shall show that these two classes do not exhaust all mixed groups of this property. In describing the structure of the groups belonging to one of these classes we shall need a generalization of the direct sum which has plaid an important role in the theory of rings.

Lemma 1 (§ 3) gives an almost trivial necessary condition for the commutativity of the endomorphism ring of an abelian group. As easily one can show that this condition is necessary, it seems as difficult to prove in general that it also suffices. The results below lead us to conjecture that the condition mentioned above is always sufficient.

[1] The numbers in brackets refer to the Bibliography at the end of this paper.

[2] For the notations and terminology see § 2

We shall see that a torsion group with commutative endomorphism ring is always countable, but there exist mixed as well as torsion free groups of the power of the continuum with the same property. On the other hand, certain facts led us to the conjecture that the endomorphism ring of an abelian group of a cardinal number greater than the power of the continuum is never commutative. If this conjecture will prove to be true, then from the results of the present paper it is easy to conclude that every abelian group with commutative endomorphism ring is isomorphic with a rotation group of the circle.

§ 2. Preliminaries

In what follows by a *group* we shall mean always an additively written abelian group with more than one element. Groups will be denoted by Latin capitals and their elements by $x, a, b, \ldots, g$; the other small Latin letters are reserved for rational integers (in particular p, q for prime numbers). We shall denote the endomorphisms of a group by small Greek letters. A subgroup generated by certain elements $a, b, \ldots$ of a group is denoted by $\{a, b, \ldots\}$. A group, every element of which is of finite order, is called a *torsion group*. In case every non-zero element of the group is of infinite order, the group is called *torsion free*. A group which is neither a torsion group nor torsion free, is said to be a *mixed group*. All elements of finite order of a mixed group form a subgroup which we call the *torsion subgroup* of the group.

Let p be an arbitrary prime number. If the group G contains an element of order p, then p is called an *actual prime* for G. The set of all actual primes for G will be called the *actual prime system* of G. If $pG = G$ for a prime p, then G is called *closed for* p. (Here pG denotes of course the set of all elements pg with $g \in G$.) If H is a subgroup of G and is closed for any actual prime for G, then we say that H is an *actually closed subgroup* of G. If $a \in G$ and the equation $p^n x = a$ is solvable in G for every natural number n, then a is said to be an element of *infinite height for the prime* p *in* G. Clearly, any element of order p^k is an element of infinite height for every prime different from p. The element a of G will be said to be *of actually infinite height in* G, in case a is of infinite height for each actual prime p for G. If G contains no element $\neq 0$ of actually infinite height, we call G a group without elements of actually infinite height.

For an endomorphism $g \rightarrow \varepsilon g$ of G we denote by εG the set of all elements of the form εg $(g \in G)$ and call it an *endomorphic image* of G. The set K of all $x \in G$, for which $\varepsilon x = 0$, is called as usual the *kernel* of the endomorphism ε. If H is a subgroup of G and $\varepsilon H \subseteq H$ for every endomorphism ε of G, then H is a *fully invariant* subgroup in G.

We denote by R the additive group of all rational numbers, by $C(p^k)$ the cyclic group of order p^k for an arbitrary natural number k, and by $C(p^\infty)$

the additive group of all rational numbers mod 1 whose denominators are powers of p. The additive group of all rational numbers mod 1 will be denoted by C. It is clear that C is isomorphic with the group of all rotations of finite order of the circle, and it is the smallest group containing each $C(p^k)\,(p=2,3,5,\ldots;\,k=1,2,\ldots,\infty)$ as its subgroup.

In what follows we shall need a generalization of the concept of the direct sum which coincides with the well-known concept of direct sum in case of a finite number of direct summands. Some denominations relating to this concept are taken from one of JACOBSON's fundamentally important investigations on ring theory [2].

We shall say that the group G is *a direct sum* of its subgroups B_λ if the following requirements are fulfilled (where λ runs over an arbitrary — finite or infinite — set of indices, ordered or not):

There exist endomorphisms ε_λ of G such that

1) $\varepsilon_\lambda G = B_\lambda$;

2) $\varepsilon_\lambda \varepsilon_\mu = \begin{cases} \varepsilon_\lambda & \text{if } \lambda = \mu; \\ 0 & \text{if } \lambda \neq \mu; \end{cases}$

3) *$g \in G$ and $\varepsilon_\lambda g = 0$ for every λ imply $g = 0$.*

Among all direct sums of the groups B_λ there exists a "greatest" one, G_c, satisfying the additional requirement:

4) *For any choice of a representative system of elements $b_\lambda \in B_\lambda$ there exists an element g of G_c such that $\varepsilon_\lambda g = b_\lambda$ holds for each λ.*

Obviously, the group G_c having the properties 1)—4) is uniquely determined (up to an isomorphism) by the groups B_λ; we call it the *complete direct sum* of the B_λ's, in notation:

$$G_c = \sum_\lambda B_\lambda. \tag{1}$$

This group may also be described as the set of all possible "vectors" $\langle \ldots, b_\lambda, \ldots \rangle$ which contain a "component" b_λ from each group B_λ and which are added component-wise. It is easy to see that any direct sum of the groups B_λ is a subgroup of (1).

On the other hand, among all possible direct sums of the groups B_λ there exists always a "smallest" one, denoted by G_d, which is a subgroup of any direct sum. This may be characterized as the direct sum satisfying

4*) *For any element $g \in G_d$, there are only a finite number of λ's with $\varepsilon_\lambda g \neq 0$.*

This group G_d, determined uniquely by the groups B_λ as the group satisfying 1), 2), 3), and 4*), is called the *discrete direct sum* of the B_λ's and will be denoted by

$$G_d = \sum_\lambda{}^{*} B_\lambda. \tag{2}$$

G_d may also be described as the set of all vectors $\langle \ldots, b_\lambda, \ldots \rangle$ having only a finite number of components different from zero. The concept of direct sum used so far in the group theory was this discrete direct sum.

In terms of the complete and discrete direct sums the direct sums of the B_λ's may be characterized as the groups G for which $G_d \subseteq G \subseteq G_c$. For a finite number of groups B_λ always $G_d = G_c$ holds, consequently, in this case there exists only one direct sum. Therefore the concept of the direct summand in the generalized sense is the same as that in the old sense: a certain subgroup H_1 of the group H is a *direct summand* of H if there exists a group $H_2 \subseteq H$ such that $H = H_1 + H_2$.

The definition clearly implies that the complete direct sum of an enumerable infinite set of finite or countable groups has always the power of the continuum.

Let us mention an important example. It is well known that a torsion group T may be represented as the discrete direct sum of its uniquely determined primary components T_p, where T_p is a p-group (i. e. a group containing only elements of p-power order):

(3) $$T = \sum{}^{*} T_p.$$

Therefore the complete direct sum

(4) $$\bar{T} = \sum T_p$$

is uniquely determined by T; it may be called the *complete p-direct sum over T*. In accordance with this, the groups between T and $\bar{T}$ (T and $\bar{T}$ included), in other words, the direct sums of the groups T_p, may be called the *p-direct sums over T*. It is obvious that, if the actual prime system of T contains an infinity of primes, then all of these, except T, are mixed groups and their torsion subgroup is just T.

In case $T = C$ we have obviously

(5) $$C = \sum_p{}^{*} C(p^\infty)$$

where the summation is extended over all distinct prime numbers p. It is not difficult to see that the group

(6) $$\bar{C} = \sum_p C(p^\infty)$$

is isomorphic with the additive group of all real numbers mod 1 (i. e. with the group of all rotations of the circle). In what follows we shall not make use of this fact.

§ 3. Lemmas

We start with some lemmas.

LEMMA 1. *If the ring of endomorphisms of a group is commutative, then every endomorphic image of the group is fully invariant.*

Indeed, if $H=\varepsilon G$ is a certain endomorphic image of G and η denotes an arbitrary endomorphism of G, then by $\eta\varepsilon=\varepsilon\eta$ we have $\eta H=\eta\varepsilon G=$ $=\varepsilon\eta G\subseteqq\varepsilon G=H$.

LEMMA 2. *If a group contains an element of order p, then it contains also a direct summand of the form $C(p^m)$ where m is a natural number or ∞.*

For the proof we refer to [4].

LEMMA 3. *If a torsion free group H is closed for the prime p, then $H\sim C(p^\infty)$.*

Let $a\neq 0$ be an element of H. From $pH=H$ we conclude that there exist elements $a_1, a_2, \ldots, a_n, \ldots$ in H such that

$$pa_1=a,\ pa_2=a_1,\ \ldots,\ pa_{n+1}=a_n, \ldots .$$

Therefore $H/\{a\}$ contains a subgroup $C(p^\infty)$, and since this is a direct summand of every group containing it,[5] we obtain

$$H\sim H/\{a\}=C(p^\infty)+H^*\sim C(p^\infty),$$

as desired.

LEMMA 4. *Let A be the set of all elements of actually infinite height of the mixed group G. If G contains only a finite number of elements of order p for any actual prime p, then A is an actually closed subgroup of G.*

It is obvious that A is a subgroup. We have to verify that for any actual prime p_0, $p_0A=A$ holds, i. e. for an arbitrary $a\in A$ among the solutions of the equation $p_0x=a$ in G there exist an element $x=g$ of infinite height for each actual prime p.

First let $p=p_0$ and $d_1, \ldots, d_r$ be the set of all elements of order p_0 in G. Since $a\in A$, the equation $p_0x=a$ has necessarily a solution $x=x_0$ in G. Thus all the solutions of this equation are

$$x_0+d_0,\ x_0+d_1,\ \ldots,\ x_0+d_r \qquad (d_0=0). \tag{7}$$

Let k now be an arbitrary natural integer. Since $a\in A$, the equation $p_0^kx=a$ is also solvable in G and for each solution x we have $p_0^kx=p_0x_0$, i. e. $p_0(p_0^{k-1}x-x_0)=0$. Hence any solution of the equation $p_0^kx=a$ satisfies the equation

$$p_0^{k-1}x=x_0+d_i \qquad (i=0, 1, \ldots, r) \tag{8}$$

for some i. In other words, among the indices $0, 1, \ldots, r$ there is an i such that (8) has a solution for an infinity of k's. But then the element $g=x_0+d^i$

[3] See [1], p. 766.

in (7) is evidently a solution of the equation $p_0 x = 0$ and is of infinite height for p_0.

Secondly let p be an actual prime for G such that $p \neq p_0$. We show that the previous solution $x = g$ of the equation $p_0 x = a$ is an element of infinite height for p too. Since $a \in A$, for each natural number n there exists an element $x_1 \in G$ such that

$$p^n x_1 = a = p_0 g. \tag{9}$$

If u, v are integers with $p_0 u + p^n v = 1$, then by (9) we get

$$g = (p_0 u + p^n v) g = u p^n x_1 + v p^n g = p^n (u x_1 + v g).$$

Thus we have shown that g is an element of infinite height for p. This completes the proof of Lemma 4.

§ 4. Torsion groups

We recall that a group G is called locally cyclic if any two elements of it are contained in some cyclic subgroup of G.[4]

The torsion groups with commutative ring of endomorphisms are characterized in several ways by

THEOREM 1. *For a torsion group T the following statements are equivalent:*

a$_1$) *The ring of endomorphisms of T is commutative.*

b$_1$) *Every endomorphic image of T is fully invariant.*

c$_1$) *T is the discrete direct sum of groups $C(p_k^{m_k})$ $(m_k = 1, 2, \ldots, \infty)$ belonging to different prime numbers p_k.*

d$_1$) *T is a subgroup of the group C.*

e$_1$) *T is locally cyclic.*

f$_1$) *Any finite subgroup of T is a cyclic group.*

g$_1$) *For an arbitrary natural number r the equation $rx = 0$ has at most r solutions $x \in T$.*

h$_1$) *Every subgroup of T is fully invariant.*

REMARKS. According to Theorem 1 a torsion group with commutative endomorphism ring is always countable. Theorem 1 shows in particular that the necessary condition expressed in Lemma 1 is at the same time sufficient for torsion groups in order to have commutative endomorphism ring. Certain statements of Theorem 1 (for example, the equivalence of d$_1$) and e$_1$)) are well-known facts. However, we preferred to enumerate in the theorem all these interesting properties of the group C, because so the proof will be very short.

[4] Obviously this condition is equivalent to the fact that any finite system of elements of G is contained in a cyclic subgroup of G.

PROOF OF THEOREM 1.

a_1) *implies* b_1). See Lemma 1.

b_1) *implies* c_1). T is a discrete direct sum of p-groups. If one of these primary components of T were not of type $C(p^m)$, then by repeated application of Lemma 2 we would conclude that T might be represented in the orm

$$T = C(p^m) + C(p^n) + T' \qquad (1 \leqq m \leqq \infty;\ 1 \leqq n \leqq \infty).$$

But this would imply that T has an endomorphism such that one of the subgroups $C(p^m)$, $C(p^n)$ is mapped onto a subgroup $\neq 0$ of the other. This would contradict b_1), since every direct summand is an endomorphic image.

c_1) *implies* d_1). See (5).

d_1) *implies* e_1). This is clear if one takes into account the representation of C as the additive group of all rational numbers mod 1.

e_1) *implies* f_1). This is obvious. See [4].

f_1) *implies* g_1). If the equation $rx = 0$ had $r+1$ solutions in T, then these would generate a finite subgroup which is not cyclic.

g_1) *implies* h_1). Obviously it is sufficient to show that, if g_1) holds for T, then any cyclic subgroup of T is fully invariant. Let $a \in T$, and let ε be an arbitrary endomorphism of T. If the order of a is r, then $r \cdot \varepsilon a = 0$. On the other hand, by hypothesis, the solutions of the equation $rx = 0$ are exhausted by the elements of the cyclic group $\{a\}$. Hence $\varepsilon a \in \{a\}$.

h_1) *implies* a_1). For let $a \in T$, further ε, η denote two arbitrary endomorphisms of T. Then by h_1) $\varepsilon a = ma$, $\eta a = na$, hence $\varepsilon\eta a = nma = mna = \eta\varepsilon a$ Therefore $\varepsilon\eta = \eta\varepsilon$.

§ 5. Mixed groups

LEMMA 5. *Let G be a mixed group with the torsion subgroup T. Then each of the following three statements is a consequence of its predecessor:*

a) *The ring of endomorphisms of G is commutative.*

b) *Every endomorphic image of G is fully invariant.*

c) *T is a locally cyclic group without subgroups of type $C(p^\infty)$* [5] *and the factor group G/T is closed for any prime p which is actual for G.*

PROOF. By Lemma 1, a) implies b). Consequently it is sufficient to show that b) implies c).

First of all we note that by b) there exists no endomorphism ε of G in case $G = D + E$, $D \neq 0$, $E \neq 0$, for which $0 \neq \varepsilon E \subseteq D$ holds.

At first we shall show that if b) holds for G, then T is locally cyclic. Indeed, if T were not locally cyclic, then by Theorem 1 there would exist a prime number p such that T contains more than one subgroup of type $C(p)$.

[5] This requirement can obviously be expressed also in the following manner: T is a ocally cyclic group without elements of actually infinite height.

Then, by applying Lemma 2, we get

$$G = C(p^m) + C(p^n) + G' \qquad (1 \leqq m \leqq \infty;\ 1 \leqq n \leqq \infty),$$

which is impossible according to our previous remark.

Now we are going to prove that G/T is closed for any actual prime number p. For let p be an arbitrary actual prime for G. Then, by Lemma 2,

$$G = C(p^m) + G_0 \qquad (1 \leqq m \leqq \infty). \tag{10}$$

If here $pG_0 = G_0$, then by $C(p^m) \subseteq T$ we have

$$G_0 \cong G/C(p^m) \sim G/T,$$

consequently $p(G/T) = G/T$. In the contrary case, i. e. if $pG_0 \neq G_0$, then the factor group G_0/pG_0 is an elementary p-group and hence

$$G_0 \sim G_0/pG_0 = \sum^* C(p) \sim C(p).$$

Therefore G has an endomorphism ε such that, in view of (10), $0 \neq \varepsilon G_0 \subseteq C(p^m)$ holds, in contradiction to b).

Finally we show that the locally cyclic group T contains no subgroup of type $C(p^\infty)$,[5] i. e.

$$T = \sum^* C(p_k^{m_k}) \qquad (1 \leqq m_k < \infty;\ p_i \neq p_j \text{ for } i \neq j). \tag{11}$$

In fact, assuming $C(p^\infty) \subseteq T$ we get

$$G = C(p^\infty) + G_1, \tag{12}$$

$C(p^\infty)$ being a direct summand of every group containing it.[3] Then

$$G_1 \cong G/C(p^\infty) \sim G/T.$$

Further on account of what has been said above we have $p(G/T) = G/T$; therefore if we apply Lemma 3 to the group $H = G/T$, we get $G/T \sim C(p^\infty)$. Hence

$$G_1 \sim C(p^\infty)$$

which leads by (12) again to a contradiction.

From Lemma 5 thus having been proved we easily obtain the following theorems, of which Theorem 2 throws light on all mixed groups with commutative endomorphism ring and without elements of actually infinite height, while Theorem 3 characterizes the mixed groups with commutative endomorphism ring in which T is a direct summand.

THEOREM 2. *For a mixed group G without elements of actually infinite height the following statements are equivalent:*

a$_2$) *The ring of endomorphisms of G is commutative.*

b$_2$) *Every endomorphic image of G is fully invariant.*

c$_2$) *The torsion subgroup T of G is locally cyclic and contains no subgroup of type $C(p^\infty)$; further G is a p-direct sum over T such that G/T is closed for each actual prime.*

REMARKS. First of all we note that there exist in fact groups G described in c_2) of Theorem 2 and we can get an oversight on them. Indeed, the complete p-direct sum over the group (11), i. e. the group

$$\bar{T}=\sum C(p_k^{m_k}) \qquad (1\leq m_k<\infty;\, p_i\neq p_j \text{ for } i\neq j) \tag{13}$$

has the property that $\bar{T}/T$ is closed for any prime number p. To prove this we must show that if the "vector" $c=\langle\ldots, c_k, \ldots\rangle$ $(c_k\in C(p_k^{m_k}))$ is an arbitrary element of the group (13) and p is an arbitrary prime number, then there exists an $x\in\bar{T}$ such that $c-px\in T$. This is obvious, since one may plainly construct a "vector" x with $c-px=0$ or $c-px\in C(p_j^{m_j})$, according as $p\neq p_k\,(k=1,2,3,\ldots)$ or $p=p_j$. Hence the group (13) corresponding to any prescribed group (11) has always commutative endomorphism ring, and according to Theorem 2 all mixed groups of this property without elements of actually infinite height are exhausted by those groups G for which $T\subset G\subseteq\bar{T}$ and $p_k(G/T)=G/T$ for every k. For a given T the determination of all G's of this kind is naturally equivalent to giving all those subgroups of the factorgroup $\bar{T}/T$ which are closed for every p_k. Since the group $\bar{T}/T$ is torsion free, this process becomes easier by taking into account that if S is an arbitrary subgroup of $\bar{T}/T$ and if we adjoin to S all those elements e of $\bar{T}/T$ for which $re\in S$ with some natural number r divisible only by primes in (11), then we get a subgroup S_0 of $\bar{T}/T$ such that $p_kS_0=S_0$ for every k.

The results below will show that in a group G characterized by Theorem 2 the torsion subgroup T is never a direct summand.

Theorem 2 implies the existence of mixed groups of the power of the continuum with commutative ring of endomorphisms. We have to point out the fact that the necessary condition of Lemma 1 also suffices for mixed groups without elements of actually infinite height in order to have commutative endomorphism ring.

PROOF OF THEOREM 2.

In view of Lemma 5 it is sufficient to show that, if G is a mixed group without elements of actually infinite height, then c) of Lemma 5 implies c_2); further if G is an arbitrary mixed group, then c_2) implies a_2) besides the fact that G is a group without elements of actually infinite height.

Now we consider the first assertion. According to c), T is a group of the form (11). First of all we show that in (11) there is an infinity of primes p_k. In the contrary case, by a repeated application of Lemma 2, we would have $G=T+U$, and here, by c) and $U\cong G/T$, the torsion free group U would be an actually closed subgroup of G. This is, however, impossible, since by hypothesis G contains no element $\neq 0$ of actually infinite height. (In the same way we can prove on basis of Theorem 2 that T is never a direct summand of the groups G described by Theorem 2.)

Now let p_k be an arbitrary actual prime number for G. Then by Lemma 2

$$G = C(p_k^{m_k}) + G_k \qquad (1 \leqq m_k < \infty) \tag{14}$$

where $C(p_k^{m_k})$ is the same direct summand as that occurring in (11). As a matter of fact, since the group $C(p_k^{m_k})$ in (11) includes all those elements of G whose order is some power of p_k, obviously G has no other direct summand of type $C(p_k^n)$. Thus in the representation (14) of G the direct summand $C(p_k^{m_k})$ is uniquely determined. On the other hand we show that also *G_k is uniquely determined as the set of all elements of infinite height for p_k in G.* A part of this assertion, viz. that $g \in G$ and $g \notin G_k$ imply that g is not of infinite height for p_k, is obvious. Consequently, it is enough to show that if $g_k \in G_k$, then the equation $p_k x = g_k$ has always a solution $x \in G_k$. Since by c) G/T is closed for p_k, there exists an $x \in G$ such that $p_k x - g_k = d \in T$. Let $C(p_k^{m_k}) = \{c_k\}$ and let the elements x and d be represented in the form according to (14)

$$x = ic_k + g'_k, \quad d = jc_k + g''_k \qquad (g'_k, g''_k \in G_k).$$

Here g''_k being an element of finite order in G_k, the order of g''_k is not divisible by p_k. Therefore $g''_k = p_k g'''_k \quad (g'''_k \in G_k)$. Hence the equation $p_k x - g_k = d$ can be written in the form

$$p_k(ic_k + g'_k) - g_k = jc_k + p_k g'''_k,$$

whence we get $p_k(g'_k - g'''_k) = g_k$ on account of the direct representation in (14). Consequently the element $g'_k - g'''_k$ is, indeed, a solution in G_k of the equation $p_k x = g_k$.

By the uniqueness, thus proved, of both terms on the right hand of (14), we conclude that each element g of G may be written in exactly one way as the sum of an element $\varepsilon_k g$ in $C(p_k^{m_k})$ and of an element in G_k. It is clear that the mapping $g \to \varepsilon_k g$ is an endomorphism of G. The endomorphisms thus defined possess obviously the following properties:

1) $\varepsilon_k G = C(p_k^{m_k})$;

2) $\varepsilon_i \varepsilon_k = \begin{cases} \varepsilon_k & \text{if} \quad i = k; \\ 0 & \text{if} \quad i \neq k; \end{cases}$

3) If $g \in G$ and $\varepsilon_k g = 0$ for every k, then $g = 0$.

Indeed, 3) is a consequence of the fact that if $\varepsilon_k g = 0$ for every k, then $g \in G_k$ for every k, i. e. g is an element of infinite height for each p_k, so that, by hypothesis, $g = 0$. Thus we have shown that G is a p-direct sum over T in the sense of § 2.

In order to complete the proof of Theorem 2 we have only to show that if c_2) holds for the mixed group G, then G contains no element of actually infinite height and the endomorphism ring of G is commutative. The previous part follows from that by (13)

$$p_1^{m_1}\bar{T} \cap p_2^{m_2}\bar{T} \cap \cdots \cap p_k^{m_k}\bar{T} \cap \cdots = 0,$$

and hence a fortiori for $G \subseteq \overline{T}$

$$p_1^{m_1} G \cap \cdots \cap p_k^{m_k} G \cap \cdots = 0.$$

Now let ε and η be any two endomorphisms of G and let us consider the endomorphism $\delta = \varepsilon\eta - \eta\varepsilon$. Since any endomorphism of G induces an endomorphism in T and since, by c_2) and Theorem 1, the endomorphism ring of T is commutative, we obtain $\delta T = 0$. Therefore T is contained in the kernel K of the endomorphism δ. But then, by $G/T \sim G/K \cong \delta G$ and by the fact that $p_k(G/T) = G/T$ for every k, δG is an actually closed subgroup of G. Hence $\delta G = 0$. Thus we have shown that $\delta = \varepsilon\eta - \eta\varepsilon = 0$, and this completes the proof of Theorem 2.

THEOREM 3. *Suppose the mixed group G can be represented as $G = T + U$, where T is the torsion subgroup of G. Then the endomorphism ring of G is commutative if and only if T is a locally cyclic group containing no subgroup of type $C(p^\infty)$ and U is an actually closed subgroup of G with commutative endomorphism ring.*

REMARKS. It is clear that in the groups described by Theorem 3 the set of all elements of actually infinite height is just the subgroup U. Hence Theorem 2 and Theorem 3 exhaust two classes of mixed groups which have no groups in common, since Theorem 2 concerns groups without elements of actually infinite height.

It is easy to give examples for groups satisfying the conditions of Theorem 2. An instance for a group of this kind is the direct sum of a group T of the form (11) and the group $U = R$. We shall show in § 6 that also among the groups described by Theorem 3 exist groups of the power of the continuum.

We may expect to obtain further informations of the structure of the groups given by Theorem 3 only in case one would succeed in getting some further details of the structure of torsion free groups with commutative endomorphism ring. Only in this case one can answer the question whether or not the groups satisfying the conditions of Theorem 3 are all the mixed groups whose torsion group is a direct summand and which satisfy the necessary condition of Lemma 1.

PROOF OF THEOREM 3.

The necessity of the conditions of Theorem 3 follows obviously from Lemma 5, as well as from the fact that if $G = T + U$, then $U \cong G/T$ and the commutativity of the endomorphism ring of G implies the same for U.

In order to prove the sufficiency of the conditions, let us consider a group $G = T + U$ satisfying the hypotheses of Theorem 3. It is obvious that both T and U are fully invariant subgroups of G (the latter being the set of all elements of actually infinite height of G). Consequently any endomorphism of G induces an endomorphism both in T and U. On the other hand, as the endomorphism ring both of T (see Theorem 1) and of U is commutative, T and U are contained in the kernel of the endomorphism $\varepsilon\eta-\eta\varepsilon$ for any two endomorphism ε, η of G. Then the kernel of $\varepsilon\eta-\eta\varepsilon$ contains also $T+U=G$, i. e. $\varepsilon\eta-\eta\varepsilon=0$.

Now the question arises as to whether the groups given by Theorems 2 and 3 exhaust all mixed groups with commutative endomorphism ring. We have to answer this question in the negative. More exactly:

If the actual prime system of the mixed group G with commutative endomorphism ring contains all the prime numbers, then G is covered by Theorem 2. If the actual prime system of G consists only of a finite number of primes, then G is covered by Theorem 3. In all other cases — i. e. when the actual prime system of G contains infinitely many prime numbers, but not all of them — there is a group G with commutative endomorphism ring which is not covered neither by Theorem 2, nor by Theorem 3.

In order to prove this, let G be a mixed group with commutative endomorphism ring, and first let us consider the case when the actual prime system of G consists of all primes. Then by Lemmas 5 and 4 all elements of actually infinite height of G form a torsion free subgroup A closed for every prime. Therefore, according to a well-known theorem [3] A is a direct summand of G:

$$G=G_0+A \tag{15}$$

where G_0 is already a group without elements of actually infinite height, i. e. G_0 is covered by Theorem 2. But by Theorem 2, G_0/T is a torsion free group closed for every prime and thus, it is a discrete direct sum of rational groups R. Hence $G_0 \sim G_0/T \sim R$. On the other hand, $A \neq 0$ would imply $R \subseteq A$. Thus, by (15) one might find an endomorphism ε of G such that $0 \neq \varepsilon(G_0) \subseteq A$ contradicting Lemma 1. Therefore only $A=0$ is possible, completing the proof that in this case G is covered by Theorem 2.

Let us proceed to the case if the actual prime system of G contains but a finite number of primes. Then by Lemma 5, T is a finite cyclic group and a repeated application of Lemma 2 leads to the representation $G=T+U$ which shows that now G is covered by Theorem 3.

Finally let us consider the case when the actual prime system of G contains an infinity of prime numbers $p_1, p_2, \ldots$, but not all of them. Let q be a prime not actual for G and denote by R_q the additive group of all rational numbers whose denominator is relatively prime to q. Then

$$G=R_q+\sum_k C(p_k) \tag{16}$$

is a mixed group covered neither by Theorem 2, nor by Theorem 3, considering that the set of elements of actually infinite height of it is R_q, further neither $R_q=0$ nor $G=R_q+T$ holds. That the endomorphism ring of the group (16) is commutative, we shall show below. (See Theorem 5.)

It is worth while having a look at the consequences of our results in the most general case. We get a necessary condition as well as a sufficient one for the endomorphism ring of a mixed group G to be commutative. The previous condition is contained in Theorem 4 and is an immediate consequence of Lemmas 4 and 5 as well as of the first part of the proof of Theorem 2.

THEOREM 4. *If the endomorphism ring of a mixed group G is commutative, then the torsion subgroup T of G is a group of type* (11), *G/T is closed for every actual prime number, the elements of actually infinite height of G form an actually closed torsion free subgroup A of G and G/A is a group without elements of actually infinite height with commutative endomorphism ring (consequently, it is a group of the type given by Theorem* 2).

The following example shows that the conditions of Theorem 4 are not always sufficient for ensuring the commutativity of the endomorphism ring:

$$G = R + \sum_p C(p).$$

The complete direct sum on the right side is to be extended over all distinct prime numbers p. By $\sum_p C(p) \sim R$, G does not fulfils the requirement of Lemma 1, so that the endomorphism ring of G is not commutative.

A sufficient condition is given by the following

THEOREM 5. *If a mixed group G satisfying the conditions of Theorem* 4 *has the property that there exists a prime number q such that $q(G/A) = G/A$, further A contains no element $(\neq 0)$ of infinite height for q,*[6] *and the endomorphism ring of A is commutative, then the endomorphism ring of G is commutative.*

PROOF. Obviously T and A are fully invariant subgroups of G. Hence any endomorphism of G induces an endomorphism both in T and in A. But the endomorphism ring of T and that of A are commutative, so that T and A are both contained in the kernel K of the endomorphism $\delta = \varepsilon\eta - \eta\varepsilon$ for any two endomorphisms ε and η. Therefore

$$G/T \sim G/K \cong \delta G \tag{17}$$

and

$$G/A \sim G/K \cong \delta G. \tag{18}$$

Since G/T is closed for every actual prime, (17) means that δG is an actually closed subgroup of G, i. e. $\delta G \subseteq A$. On the other hand, from $q(G/A) = G/A$ and from (18) we may conclude that $q(\delta G) = \delta G$. However, the only subgroup of A closed for q is 0, hence $\delta G = 0$ and $\delta = \varepsilon\eta - \eta\varepsilon = 0$.

The group G in (16) satisfies obviously the conditions of Theorem 5, so that its endomorphism ring is commutative.

[6] Consequently q cannot be an actual prime for G.

§ 6. Final remarks and some conjectures

In order to construct groups of the power of the continuum with commutative endomorphism ring we need the following

THEOREM 6. *If* $H_1, H_2, \ldots$ *are countably many groups such that*

(I) *The endomorphism ring of* H_n *is commutative* $(n=1,2,\ldots)$,

(II) H_n *is a fully invariant subgroup of the complete direct sum* $G=\sum H_n$,

(III) *The only homomorphic image of* $\sum H_n/\sum^* H_n$ *in* G *is* 0,

then the endomorphism ring of $G=\sum H_n$ *is commutative.*

PROOF. *Let* ε, η be arbitrary endomorphisms of G. By (II) and (I), each H_n, and hence also $\sum^* H_n$ is contained in the kernel of the endomorphism $\delta=\varepsilon\eta-\eta\varepsilon$. Therefore

$$\sum H_n/\sum^* H_n \sim \delta G \subseteq G.$$

By (III) we have $\delta G=0$, consequently $\varepsilon\eta-\eta\varepsilon=0$.

Using Theorem 6 one can easily construct a torsion free group of the power of the continuum with commutative endomorphism ring. In order to do this, let $p_1, p_2, \ldots$ be an infinity of distinct prime numbers and denote again by R_{p_n} the additive group of all rational numbers whose denominator is relatively prime to p_n. Then the complete direct sum $G=\sum R_{p_n}$ is a group having the required property. (II) is fulfilled, since R_{p_n} contains all the elements of G which are of infinite height for each prime $\neq p_n$. (III) also holds, for $G/\sum^* R_{p_n}$ is now a group closed for every prime number p_n, while the only subgroup of G with the same property is obviously 0.

If q is a prime number different from each prime number p_n, then

$$C(q)+\sum_n R_{p_n}$$

is obviously a group satisfying the conditions of Theorem 3. Thus we have shown that among the groups covered by Theorem 3 there exist groups of the power of the continuum.

In conclusion we formulate some conjectures.

CONJECTURE 1. *If every endomorphic image of a group is fully invariant, then the endomorphism ring of the group is commutative.*

CONJECTURE 2. *Every group with commutative endomorphism ring is at most of the power of the continuum.*

If Conjecture 2 will prove to be true, then on basis of Lemma 5 it is easy to show that even the following conjecture will hold:

CONJECTURE 3. *Any group with commutative endomorphism ring is isomorphic with a subgroup of the group of all rotations of the circle.*

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О ГРУППАХ АБЕЛЯ, КОЛЬЦО ЭНДОМОРФИЗМА КОТОРЫХ КОММУТАТИВНО

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(Резюме)

В настоящей работе авторы изучают такие группы Абеля, кольцо эндоморфизма которых коммутативно. Доказывают, что торзиогруппа тогда и только тогда обладает этим свойством, если локально цикличка, т. е. изоморфна какой-нибудь подгруппе группы, состоящей из вращений окружности конечной степени. После этого они переходят к исследованию смешанных групп, обладающих указаным свойством. Среди этих групп удаётся описать группы, которые могут быть представленны в виде прямой суммы группы с торзией и группы без торзии и группы, несодержащей такого отличного от нуля элемента, который бесконечно высок относительно любого такого простого числа, который является порядком какого-либо элемента группы. Во втором случае используют некоторое обобщение понятия прямой суммы. Из результатов следует, что существует смешанная группа, мощность которой есть мощность континуума, обладающая вышеуказанным свойствам.

Авторам удаётся построить и группу без торзии, мощность которой есть мощность континуума, обладающую этим свойством. В заключеним они выдвигают гипотезу, согласно которой мощность группы, обладающей вышеуказанным свойствам, не может быть более мощности континуума. Эта гипотеза может быть сформулигованиа и так: любая группа, обладающая этим свойствам, изоморфна какой то подгруппе группы, состоящей из всех вращений окружности.