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ARITHMETICAL RINGS

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1. General arithmetical rings. It is the purpose of this paper to characterize and examine the structure of a class of rings which we according to FUCHS [3] shall call arithmetical. By an arithmetical ring is understood a commutative ring R with identity for which the ideals form a distributive lattice, i. e. for which

$$(\mathfrak{a}+\mathfrak{b})\cap\mathfrak{c}=\mathfrak{a}\cap\mathfrak{b}+\mathfrak{a}\cap\mathfrak{c}\quad \text{for all ideals of } R,$$

or, equivalently,

$$\mathfrak{a}+\mathfrak{b}\cap\mathfrak{c}=(\mathfrak{a}+\mathfrak{b})\cap(\mathfrak{a}+\mathfrak{c})\quad \text{for all ideals of } R.$$

In this first section we are making no assumptions about the zerodivisors in R, while we in the second section shall mainly deal with rings without proper zerodivisors, i. e. integral domains.

We start with the following

THEOREM 1. *Let R be a commutative ring which has an identity element. A necessary and sufficient condition for R to be arithmetical is that, for any maximal ideal $\mathfrak{m}$, the ideals of the local*[1] *generalized quotient ring $R_{\mathfrak{m}}$ should be totally ordered by set inclusion.*

PROOF. Since an ideal $\mathfrak{a}$ of R is uniquely determined by its local components $\mathfrak{a}R_{\mathfrak{m}}$[2] and the formation of sums and intersections of ideals is preserved by extensions of ideals from R to $R_{\mathfrak{m}}$, it suffices to prove that a local ring is arithmetical if and only if its ideals are totally ordered by set inclusion. If the latter of these conditions is fulfilled, the sum of two ideals of R is the set theoretical union, and since the lattice of subsets of a given set is distributive, the „if" part is evident.

To prove the "only if" part it is clearly enough to prove that for two arbitrary elements a and b in an arithmetical local ring will either $a|b$ or $b|a$.

In fact, since the ideals are assumed to form a distributive lattice we have

$$(a)=(a)\cap[(b)+(a-b)]=(a)\cap(b)+(a)\cap(a-b)$$

so that a may be written in the form $a=t+(a-b)c$, where t is an element in $(a)\cap(b)$ and bc is an element of (a). Now, if c is a unit, b is a multiple of bc and thus belongs to (a). If c is not a unit, $(1-c)$ must be a unit, since the ring was supposed to be local. Therefore a is a multiple of $a(1-c)=t-bc$, which is an element of (b). This means that we have either $a|b$ or $b|a$. Q. E. D.

[1] Here and in the following local ring only means that the non-units form an ideal, without any assumption about the ascending chain conditions.

[2] See BOURBAKI [1], p. 112.

From the fact just proved that a local ring is arithmetical if and only if its ideals are totally ordered we immediately obtain the following

COROLLARY 1. If R is an arithmetical ring, the ideals of the quotient ring $R_{\mathfrak{p}}$ are totally ordered for any prime ideal $\mathfrak{p}$ of R.

At this place we shall draw one more conclusion of theorem 1. Since the prime ideals of the generalized quotient ring $R_{\mathfrak{m}}$, $\mathfrak{m}$ maximal, are in $1-1$ correspondence with the prime ideals of R contained in $\mathfrak{m}$, the latter ones are totally ordered when R is arithmetical. Since any two ideals are either comaximal (i. e. their sum is R) or both contained in some maximal ideal, we have

COROLLARY 2. In an arithmetical ring R any two prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ none of which is contained in the other, will be comaximal, i. e. $\mathfrak{p}_1+\mathfrak{p}_2 = R$.

Before stating the next characterization of arithmetical rings we shall prove the quite elementary

LEMMA. *Let S be a multiplicatively closed set, not containing* 0, *of a commutative ring R with identity. If $\mathfrak{b}$ is a finitely generated ideal and $\mathfrak{a}$ an arbitrary ideal of R, we have* $R_S(\mathfrak{a}:\mathfrak{b}) = R_S\mathfrak{a}:R_S\mathfrak{b}$.

PROOF. Since $R_S(\mathfrak{a}:\mathfrak{b})\cdot R_S\mathfrak{b} = R_S((\mathfrak{a}:\mathfrak{b})\mathfrak{b}) \subseteq R_S\mathfrak{a}$, the inclusion $R_S(\mathfrak{a}:\mathfrak{b}) \subseteq R_S\mathfrak{a}:R_S\mathfrak{b}$ is obvious. To prove the converse inclusion $R_S\mathfrak{a}:R_S\mathfrak{b} \subseteq R_S(\mathfrak{a}:\mathfrak{b})$ we consider an element $[x/s]$ in the ring R_S of formal quotients, belonging to the left side. $\mathfrak{b}$ is finitely generated and therefore of the form $b=(b_1, \dots, b_n)$. Since $[x/s]$ is contained in $R_S\mathfrak{a}:R_S\mathfrak{b}$, $[x/s]\cdot[b_i/1] = [xb_i/s]$, for any $b_i (1 \leqq i \leqq n)$ is an element of $R_S\mathfrak{a}$ and thus of the form $[xb_i/s] = [a_i/s_i]$ for suitable $a_i \in \mathfrak{a}$ and $s_i \in S$. By the definition of equality in R_S there exist elements $t_i \in S$ such that $t_i x b_i s_i = t_i a_i s \in \mathfrak{a}$. This means that $x(t_1 \dots t_n \cdot s_1 \dots s_n)$ belongs to $(\mathfrak{a}:\mathfrak{b})$, so that $[x/s] = [xt_1 \dots t_n \cdot s_1 \dots s_n / st_1 \dots t_n \cdot s_1 \dots s_n] \in R_S(\mathfrak{a}:\mathfrak{b})$. Q. E. D.

We are now able to prove

THEOREM 2. *Let R be a commutative ring which has an identity element. A necessary and sufficient condition for R to be arithmetical is that for any pair of ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R, such that*

$$\mathfrak{a} \subseteq \mathfrak{b}, \quad \mathfrak{b} \text{ finitely generated,}$$

there should exist an ideal $\mathfrak{c}$ for which $\mathfrak{a} = \mathfrak{b}\cdot\mathfrak{c}$.

PROOF. To prove the necessity let us consider an arbitrary pair of ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{a} \subseteq \mathfrak{b}$, $\mathfrak{b}$ finitely generated, in an arithmetical ring R. We assert that $(\mathfrak{a}:\mathfrak{b})$ can be used as $\mathfrak{c}$.

In fact, to show that $\mathfrak{b}\cdot(\mathfrak{a}:\mathfrak{b}) = \mathfrak{a}$ it suffices to show that the local components agree. Taking into account that $\mathfrak{b}$ is finitely generated, the preceding lemma tells us that

$$R_{\mathfrak{m}}(\mathfrak{b}\cdot(\mathfrak{a}:\mathfrak{b})) = R_{\mathfrak{m}}\mathfrak{b}\cdot R_{\mathfrak{m}}(\mathfrak{a}:\mathfrak{b}) = R_{\mathfrak{m}}\mathfrak{b}\cdot(R_{\mathfrak{m}}\mathfrak{a}:R_{\mathfrak{m}}\mathfrak{b}). \tag{1. 1}$$

Since $\mathfrak{b}$ is finitely generated and the ideals of $R_{\mathfrak{m}}$ are totally ordered (by theorem 1), $R_{\mathfrak{m}}\mathfrak{b}$ must be a principal ideal, obviously containing $R_{\mathfrak{m}}\mathfrak{a}$; consequently the right side of (1. 1) is equal to $R_{\mathfrak{m}}\mathfrak{a}$. This proves the necessity.

To establish the sufficiency we shall prove that the ideals in the local ring $R_{\mathfrak{m}}$, for any maximal $\mathfrak{m}$, are totally ordered if R satisfies the condition in the theorem.

It is clearly enough to prove that for any two elements $[r_1/s_1]$, $[r_2/s_2]$ in $R_{\mathfrak{m}}$, $r_1, r_2, s_1, s_2 \in R$, $s_1, s_2 \notin \mathfrak{m}$, at least one is a multiple of the other. In R we consider the ideals $\mathfrak{a}=(r_1)$ and $\mathfrak{b}=(r_1, r_2)$. Since $\mathfrak{b}$ is a finitely generated ideal containing $\mathfrak{a}$, the assumption about R involves the existence of an ideal $\mathfrak{c}$ such that $\mathfrak{a}=\mathfrak{b}\mathfrak{c}$. This means that there exist elements x and y in R for which

$$r_1 = r_1x + r_2y \quad \text{with} \quad r_2x \in (r_1) \quad \text{and} \quad r_2y \in (r_1).$$

This implies for $R_{\mathfrak{m}}$

$$[r_1/1] = [r_1/1]\cdot[x/1] + [r_2/1][y/1] \tag{1.2}$$

where

$$[r_2/1]\cdot[x/1] \in ([r_1/1]). \tag{1.3}$$

If $[x/1]$ is a unit in $R_{\mathfrak{m}}$ we have by (1.3)

$$[r_1/s_1] | [r_1/1] | [r_2/1]\cdot[x/1] | [r_2/1] | [r_2/s_2].$$

If $[x/1]$ is a non-unit, $([1/1]-[x/1])$ will be a unit in the local ring $R_{\mathfrak{m}}$, and so we get by (1.2)

$$[r_1/1]\cdot([1/1]-[x/1]) = [r_2/1]\cdot[y/1]$$

which implies

$$[r_2/s_2] | [r_2/1] | [r_1/1]\cdot([1/1]-[x/1]) | [r_1/1] | [r_1/s_1].$$

Thus at least one of the elements $[r_1/s_1]$ and $[r_2/s_2]$ is a multiple of the other. Q. E. D.

We shall now give some more characterizations of arithmetical rings.

THEOREM 3. *For a commutative ring R with identity the following conditions are equivalent*

I. *R is arithmetical.*

II. $(\mathfrak{a}+\mathfrak{b}):\mathfrak{c} = \mathfrak{a}:\mathfrak{c}+\mathfrak{b}:\mathfrak{c}$ *for arbitrary ideals $\mathfrak{a}$ and $\mathfrak{b}$, and any finitely generated ideal $\mathfrak{c}$.*

III. $\mathfrak{c}:(\mathfrak{a}\cap\mathfrak{b}) = \mathfrak{c}:\mathfrak{a}+\mathfrak{c}:\mathfrak{b}$ *for any finitely generated ideals $\mathfrak{a}$ and $\mathfrak{b}$ and arbitrary $\mathfrak{c}$.*

PROOF. I⇒II. Suppose R is arithmetical. To prove the identity in II it suffices to show that the extensions to $R_{\mathfrak{m}}$ agree for any maximal ideal $\mathfrak{m}$. By the previous lemma and the assumption about $\mathfrak{c}$, sums and quotients of ideals are preserved by this extension, so that it will do to show the relation II for the ideals of $R_{\mathfrak{m}}$. But in these rings II surely is true, since the set of ideals is totally ordered.

I⇒III. Suppose R is arithmetical. Before proving the identity in III we notice that in any ring we have

$$\mathfrak{c}:(\mathfrak{a}\cap\mathfrak{b}) \supseteq \mathfrak{c}:\mathfrak{a}+\mathfrak{c}:\mathfrak{b}$$

so that we need only prove the converse inclusion. $\mathfrak{a}$ and $\mathfrak{b}$ are finitely generated, but we do not know if $\mathfrak{a}\cap\mathfrak{b}$ is finitely generated, but anyway the lemma implies

$$R_{\mathfrak{m}}(\mathfrak{c}:(\mathfrak{a}\cap\mathfrak{b})) \subseteq R_{\mathfrak{m}}\mathfrak{c}:R_{\mathfrak{m}}(\mathfrak{a}\cap\mathfrak{b}) = R_{\mathfrak{m}}\mathfrak{c}:(R_{\mathfrak{m}}\mathfrak{a}\cap R_{\mathfrak{m}}\mathfrak{b});$$

$$R_{\mathfrak{m}}(\mathfrak{c}:\mathfrak{a}+\mathfrak{c}:\mathfrak{b}) = R_{\mathfrak{m}}\mathfrak{c}:R_{\mathfrak{m}}\mathfrak{a}+R_{\mathfrak{m}}\mathfrak{c}:R_{\mathfrak{m}}\mathfrak{b} = R_{\mathfrak{m}}\mathfrak{c}:(R_{\mathfrak{m}}\mathfrak{a}\cap R_{\mathfrak{m}}\mathfrak{b}), \tag{1.4}$$

the last equality in (1.4) following from the fact that the set of ideals of $R_{\mathfrak{m}}$ is totally ordered. By the localization principle the desired inclusion is readily obtained.

II⇒I. Let R be a ring for which II holds. We shall prove that for any two elements $[r_1/s_1]$ and $[r_2/s_2]$ in $R_{\mathfrak{m}}$, $\mathfrak{m}$ maximal, at least one is a multiple of the other. If in II we choose $\mathfrak{a}=(r_1)$, $\mathfrak{b}=(r_2)$, $\mathfrak{c}=(r_1, r_2)$, we get

$$R = (r_1 : r_2)+(r_2 : r_1),$$

so that there exist elements x and y for which

$$1 = x+y, \quad r_1|r_2x, \quad r_2|r_1y.$$

At least one of these elements does not belong to $\mathfrak{m}$, for instance $x\notin\mathfrak{m}$. Then $[x/1]$ is a unit in $R_{\mathfrak{m}}$, and consequently

$$[r_1/s_1]|[r_1/1]|[r_2x/1]|[r_2/1]|[r_2/s_2].$$

III⇒I. This may be proved similarly as II⇒I.

As an application of this theorem and corollary 1 we shall prove the following

THEOREM 4. *Let R be an arithmetical ring and K the full ring of quotients of R (i. e. the quotient ring with respect to the set of all elements which are not zero divisors in R). Then any ring R^* between R and K is arithmetical.*

PROOF. The formal quotients from K will be denoted by a/b, and an element $a\in R$ will generally be identified with the quotient $a/1$ so that R may be viewed as a subring of K. To prove that any ring R^* for which $R\subseteqq R^*\subseteqq K$ is arithmetical we consider the generalized quotient ring $R^*_{\mathfrak{m}^*}$ for an arbitrary maximal ideal $\mathfrak{m}^*$ of R^*. The contraction $\mathfrak{m}^*\cap R$ of $\mathfrak{m}^*$ to R is a prime ideal $\mathfrak{p}$ of R. If we can show that the homomorphism, defined by $[r/s]\to[r/1/s/1]$, $r, s\in R$, $s\notin\mathfrak{p}$, of $R_{\mathfrak{p}}$ into $R^*_{\mathfrak{m}^*}$ is "onto", $R^*_{\mathfrak{m}^*}$ is the homomorphic image of $R_{\mathfrak{p}}$ and its set of ideals thus totally ordered (Cor. 1). The proof will then be complete in view of theorem 1.

The above homomorphism is easily seen to map a unit of $R_{\mathfrak{p}}$ on a unit of $R^*_{\mathfrak{m}^*}$ and a non-unit of $R_{\mathfrak{p}}$ on a non-unit of $R^*_{\mathfrak{m}^*}$ ($\mathfrak{p}R_{\mathfrak{p}}$ is mapped into $\mathfrak{m}^*R^*_{\mathfrak{m}^*}$). To see that the homomorphism is surjective, we consider an arbitrary element in $R^*_{\mathfrak{m}^*}$:

$$[r_1/s_1/r_2/s_2] \qquad (r_1, r_2\in R, s_1, s_2 \text{ are not zero divisors of } R, r_2/s_2\notin\mathfrak{m}^*).$$

In the arithmetical ring R we put $\mathfrak{a}=(r_2s_1)$ and $\mathfrak{b}=(r_1s_2)$, and by applying condition II of theorem 3 we get

$$(r_2s_1 : r_1s_2)+(r_1s_2 : r_2s_1) = R,$$

so that there exist elements x and y in R for which

$$(1.5) \qquad x+y = 1, \quad xr_1s_2 = r_2s_1u, \quad yr_2s_1 = r_1s_2v, \quad u\in R, \ v\in R.$$

At least one of the elements x and y does not belong to $\mathfrak{p}=\mathfrak{m}^*\cap R$; thus there will be two cases to go through.

If $x\notin\mathfrak{p}$, $[u/x]$ is an element in $R_{\mathfrak{p}}$ which is mapped on $[u/1/x/1]=[r_1/s_1/r_2/s_2]$.

If $y\notin\mathfrak{p}$, $[v/y]$ is an element in R whose image $[v/1/y/1]$ is a unit in $R^*_{\mathfrak{m}^*}$, since (1.5) involves

$$[v/1/y/1]\cdot[r_1/s_1/r_2/s_2]=[vr_1/s_1/yr_2/s_2]=[1/1/1/1].$$

Therefore $[v/y]$ is a unit in $R_{\mathfrak{p}}$ too. This means that $v \notin \mathfrak{p}$, so that $[y/v]$ is an element of $R_{\mathfrak{p}}$, which, because of (1. 5) is mapped on $[r_1/s_1 / r_2/s_2]$. Thus in either case $[r_1/s_1 / r_2/s_2]$ belongs to the image of the above homomorphism which is consequently a homomorphism onto. Q. E. D.

In view of theorem 2 it is easily seen that any Bézout ring, i. e. a commutative ring with identity in which any finitely generated ideal is principal, is arithmetical. The converse, of course, is generally not true. In the next theorem, however, we shall show that the converse will hold if some restriction is imposed on the ring. In fact, it turns out that in the semi-local case, i. e. provided the ring has only finitely many maximal ideals, we can prove the converse.

THEOREM 5. *A semi-local arithmetical ring is a Bézout ring.*

PROOF. Let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ be the finitely many maximal ideals of the arithmetical ring R. We have to show that any finitely generated ideal is principal, but it is clearly enough to show that any ideal $\mathfrak{a}=(a, b)$ generated by two elements of R is principal.

We shall do this by constructing two elements α and β in R such that the local components of the principal ideal $(\alpha a+\beta b)$ satisfy the conditions

$$(1.6) \qquad (\alpha a+\beta b)R_{\mathfrak{m}_i} = aR_{\mathfrak{m}_i}+bR_{\mathfrak{m}_i} \quad \text{for} \quad 1 \leqq i \leqq n;$$

because this by the localization principle ensures that $(\alpha a+\beta b) = (a, b)$. Since the ideals of $R_{\mathfrak{m}_i}$ form a totally ordered set, for each $\mathfrak{m}_i$ we have either $aR_{\mathfrak{m}_i} \subseteqq bR_{\mathfrak{m}_i}$ or $bR_{\mathfrak{m}_i} \subseteqq aR_{\mathfrak{m}_i}$.

Let us assume that the $\mathfrak{m}_i$'s are numbered such that $aR_{\mathfrak{m}_i} \subseteqq bR_{\mathfrak{m}_i}$ for $1 \leqq i \leqq k$ and $bR_{\mathfrak{m}_i} \subseteqq aR_{\mathfrak{m}_i}$ for $k<i \leqq n$. Since a maximal ideal is not contained in any other maximal ideal different from itself, it follows from NORTHCOTT [8] 1. Prop. 6 that there exists for each i an element c_i such that $c_i \in \mathfrak{m}_i$, but $c_i \notin \mathfrak{m}_j$ for $i \neq j$. Set $\alpha = c_1 \ldots c_k$ and $\beta = c_{k+1} \ldots c_n$, then $\alpha \in \mathfrak{m}_i$ for $1 \leqq i \leqq k$, but $\alpha \notin \mathfrak{m}_i$ for $k<i \leqq n$, and $\beta \in \mathfrak{m}_i$ for $k<i \leqq n$, but $\beta \notin \mathfrak{m}_i$ for $1 \leqq i \leqq k$. With this choice of α and β we have for the principal ideal $(\alpha a+\beta b)$

$$(\alpha a+\beta b)R_{\mathfrak{m}_i} = bR_{\mathfrak{m}_i} \quad \text{for} \quad 1 \leqq i \leqq k,$$

$$(\alpha a+\beta b)R_{\mathfrak{m}_i} = aR_{\mathfrak{m}_i} \quad \text{for} \quad k<i \leqq n.$$

Since in the first case $aR_{\mathfrak{m}_i} \subseteqq bR_{\mathfrak{m}_i}$ and in the second $bR_{\mathfrak{m}_i} \subseteqq aR_{\mathfrak{m}_i}$, in either case we have obtained (1. 6). Q. E. D.

It is a well-known fact that in a Noetherian ring any irreducible ideal is primary, while a primary ideal need not be irreducible. It might be worth while noticing that for an arithmetical ring the situation is just the opposite. In fact, in a valuation ring of rank 2 all ideals are irreducible, since they are totally ordered by set inclusion, but not all of its ideals are primary. That a primary ideal, however, is irreducible is stated in

THEOREM 6. *In an arithmetical ring R any primary ideal is irreducible.*

PROOF. Let $\mathfrak{q}$ be a primary ideal of R with the prime ideal $\mathfrak{p}$ as its radical. Let us further assume that $\mathfrak{q}$ is represented as an intersection $\mathfrak{q}=\mathfrak{a} \cap \mathfrak{b}$. We have to show that $\mathfrak{q}=\mathfrak{a}$ or $\mathfrak{q}=\mathfrak{b}$. By passage to the generalized quotient ring $R_{\mathfrak{p}}$ we get $\mathfrak{q}R_{\mathfrak{p}}=\mathfrak{a}R_{\mathfrak{p}}\cap\mathfrak{b}R_{\mathfrak{p}}$. By corollary 1 the set of ideals of $R_{\mathfrak{p}}$ is totally ordered, so that $\mathfrak{q}R_{\mathfrak{p}}=\mathfrak{a}R_{\mathfrak{p}}$ or $\mathfrak{q}R_{\mathfrak{p}}=\mathfrak{b}R_{\mathfrak{p}}$. Suppose $\mathfrak{q}R_{\mathfrak{p}}=\mathfrak{a}R_{\mathfrak{p}}$. In that case we shall finish the proof by showing that $\mathfrak{q}=\mathfrak{a}$. Since $\mathfrak{q}$ is $\mathfrak{p}$-primary, the contraction of $\mathfrak{q}R_{\mathfrak{p}}=\mathfrak{a}R_{\mathfrak{p}}$ to R is $\mathfrak{q}$. The contraction of $\mathfrak{a}R_{\mathfrak{p}}$ to R is the S-component $\mathfrak{a}_s$ of $\mathfrak{a}$, S denoting the complement of $\mathfrak{p}$ in R. Now, clearly $\mathfrak{a}\subseteqq\mathfrak{a}_s$ so that $\mathfrak{a}_s=\mathfrak{q}$ implies $\mathfrak{a}\subseteqq\mathfrak{q}$. The converse of this inclusion being obvious, we have $\mathfrak{a}=\mathfrak{q}$. Q. E. D.

By the way, if R is an integral domain whose non-zero prime ideals are all of them maximal, theorem 6 may be reversed:

THEOREM 6'. *Let R be an integral domain for which any non-zero prime ideal is maximal. Then R is arithmetical if and only if any primary ideal is irreducible.*

PROOF. The "only if" follows from theorem 6. To obtain the "if" part we have to show that the set of ideals in any $R_{\mathfrak{m}}$, $\mathfrak{m}$ maximal, is totally ordered by set inclusion. Now, $\mathfrak{m}R_{\mathfrak{m}}$ is the only non-zero prime ideal of $R_{\mathfrak{m}}$ and is therefore the radical of any non-zero proper ideal in $R_{\mathfrak{m}}$. All non-zero ideals of $R_{\mathfrak{m}}$ are $\mathfrak{m}R_{\mathfrak{m}}$ primary, and consequently in a $1-1$ correspondence with the $\mathfrak{m}$-primary ideals of R. The intersection of two $\mathfrak{m}$-primary ideals is itself an $\mathfrak{m}$-primary ideal, so that the irreducibility of the $\mathfrak{m}$-primary ideals means that these are totally ordered by set inclusion. Since the above $1-1$ correspondence is order-preserving, the set of ideals in $R_{\mathfrak{m}}$ is totally ordered. Q. E. D.

We shall conclude this section by pointing out some examples of arithmetical rings. For instance, it can be shown that a commutative ring R with identity has w. gl. dim. $R\leqq 1$ (i. e. $\mathrm{Tor}_2^R(A, B)=0$ for all R-modules A and B) if and only if R is an arithmetical ring with no proper nilpotent elemets [7]. Combining this result with one of ENDO [2] it follows that R is semi-hereditary (which is a stronger property than w. gl. dim. $R\leqq 1$) if and only if R is an arithmetical ring for which the full ring of quotients is regular (in the sense of von Neumann, i. e. $axa=a$ solvable in x for all a).

In this connection it is of some interest to examine the structure of arithmetical rings R with Rad $R=(0)$. We shall here only prove the following

THEOREM 7. *A semi-local arithmetical ring R with* Rad $R=(0)$ *is a direct sum of semi-local arithmetical domains.*

PROOF. The radical Rad $R=(0)$ is the intersection of all minimal prime ideals of R. If $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two different minimal prime ideals, neither of them can be contained in the other. By corollary 2 they are therefore comaximal, i. e. $\mathfrak{p}_1+\mathfrak{p}_2=R$. In particular, this involves that a maximal ideal cannot contain more than one minimal prime ideal, so that the number of minimal prime ideals in R is finite. R is thus a ring for which (0) is a finite intersection of pairwise comaximal prime ideals. By a well-known argument (ZARISKI—SAMUEL [9], III, Theorem 32) this implies that R is isomorphic to the direct sum of the corresponding residue class rings. These are integral domains which as homomorphic images of a semi-local arithmetical ring are themselves semi-local and arithmetical.

REMARK. Since the properties, R semi-hereditary and w. gl. dim. $R\leqq 1$ are equivalent for integral domains (first proved by HATTORI [5]) theorem 7 shows that a semi-local ring R is semi-hereditary if and (trivially) only if w. gl. dim. $R\leqq 1$.

2. Prüfer rings. In this section we shall restrict ourselves to consider integral domains. If $\mathfrak{b}$ is a finitely generated ideal $\neq (0)$ and $a \neq 0$ an element of $\mathfrak{b}$, theorem 2 shows that there exists an ideal $\mathfrak{c}$ such that $(a) = \mathfrak{b} \cdot \mathfrak{c}$, provided R is arithmetical. This means that any finitely generated ideal $\neq (0)$ is invertible (or equivalently, projective if viewed as an R-module). Conversely, if any finitely generated ideal $\neq (0)$ is invertible the condition of theorem 2 is readily seen to be fulfilled so that we have the following

COROLLARY 3. An integral domain is arithmetical if and only if it is a Prüfer ring.

Consequently for an integral domain R the following properties are identical: R is semi-hereditary, w. gl. dim. $R \leqq 1$, R is a Prüfer ring, R is arithmetical.

Since the set of ideals of an integral domain is totally ordered if and only if it is a valuation ring, corollary 3 combined with theorem 1 yields the well-known result that the integral domain R is a Prüfer ring if and only if $R_{\mathfrak{m}}$ is a valuation ring for any maximal ideal $\mathfrak{m}$.

Consequently, to any Prüfer ring R there is attached a set of valuations of the quotient field of R, whose corresponding valuation rings are the quotient rings $R_{\mathfrak{m}}$. In the following we shall refer to these valuations as associated valuations of R.

We shall now consider and characterize those Prüfer rings for which the associated valuations are independent. (For the notion of independent valuations see for instance ZARISKI—SAMUEL [10], VI, § 10.) In our case this means that no two of the quotient rings $R_{\mathfrak{m}}$ are contained in one and the same non-trivial valuation ring. Since it is implicitly contained in the proof of theorem 4 that any local ring between an arithmetical domain R and its quotient field K is of the form $R_{\mathfrak{p}}$ for a suitable prime ideal $\mathfrak{p}$,[3] the definition of independence of valuations shows that the associated valuations of a Prüfer ring are independent if and only if the intersection of any two different maximal ideals contains no prime ideals apart from (0).

In the next theorem we shall give another criterion for the independence of the associated valuations

THEOREM 8. *A necessary and sufficient condition for the associated valuations of a Prüfer ring R to be independent is that any ideal $\mathfrak{a}$ whose radical is a prime ideal should be irreducible.*

PROOF. Suppose first that the associated valuations of R are independent. Let $\mathfrak{a}$ be an ideal for which $\text{Rad}\,\mathfrak{a} = \mathfrak{p}$, $\mathfrak{p}$ being a prime ideal of R. We may assume that $\mathfrak{p} \neq (0)$, since otherwise $\mathfrak{a}$ would be (0) and thus trivially irreducible. Because of the assumption about the independence of the valuations, $\mathfrak{p}$ is contained in exactly one maximal ideal $\mathfrak{m}^*$. Since $\mathfrak{p} = \text{Rad}\,\mathfrak{a}$, the same holds true for $\mathfrak{a}$. If $\mathfrak{a}$ is represented as an intersection $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ we obtain by passage to the local ring $R_{\mathfrak{m}^*}$: $\mathfrak{a}R_{\mathfrak{m}^*} = \mathfrak{b}R_{\mathfrak{m}^*} \cap \mathfrak{c}R_{\mathfrak{m}^*}$. $R_{\mathfrak{m}^*}$ is a valuation ring, therefore $\mathfrak{a}R_{\mathfrak{m}^*}$ must be equal to one of the containing ideals, say $\mathfrak{a}R_{\mathfrak{m}^*} = \mathfrak{b}R_{\mathfrak{m}^*}$. For any maximal ideal $\mathfrak{m} \neq \mathfrak{m}^*$ we have $\mathfrak{a} \subseteqq\!\!\!\!/\ \mathfrak{m}$, so that $\mathfrak{a}R_{\mathfrak{m}} = R_{\mathfrak{m}}$ and therefore also $\mathfrak{b}R_{\mathfrak{m}} = R_{\mathfrak{m}}$. This means that $\mathfrak{a} = \mathfrak{b}$ as they have the same local components.

[3] Combining this result with Theorem 2.5 in [4] one sees that an integral domain R is a Prüfer ring, if and only if any local ring between R and K has the form $R_{\mathfrak{p}}$ for a suitable prime ideal $\mathfrak{p}$ in R.

Conversely, let R be a Prüfer ring for which $\mathrm{Rad}\,\mathfrak{c}=\mathfrak{p}$, $\mathfrak{p}$ being a prime ideal, implies that $\mathfrak{c}$ is irreducible. We will assume that there exists a non-zero prime ideal $\mathfrak{p}$ contained in two different maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ and derive a contradiction from this.

We choose elements a and b in R such that $a\in\mathfrak{m}_1$, $a\notin\mathfrak{m}_2$, $b\in\mathfrak{m}_2$, $b\notin\mathfrak{m}_1$, and a non-zero element $c\in\mathfrak{p}\subseteqq\mathfrak{m}_1\cap\mathfrak{m}_2$. Let S be the complement of the union of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ in R. S is multiplicatively closed and the corresponding quotient ring R_S is an arithmetical semi-local ring with $\mathfrak{m}_1R_S$ and $\mathfrak{m}_2R_S$ as its only maximal ideals. The radical of the extension ideal $(c)R_S\subseteqq\mathfrak{p}R_S$ is the intersection of all prime ideals in R_S containing $(c)R_S$. Any such prime ideal is contained in either $\mathfrak{m}_1R_S$ or $\mathfrak{m}_2R_S$, and consequently cannot be comaximal to $\mathfrak{p}R_S\subseteqq\mathfrak{m}_1R_S\cap\mathfrak{m}_2R_S$. By corollary 2 it must contain or be contained in $\mathfrak{p}R_S$. If it contains $\mathfrak{p}R_S$ it is superfluous by the formation of the above intersection and may be omitted. Hence, $\mathrm{Rad}\,((c)R_S)$ is the intersection of all prime ideals contained in $\mathfrak{p}R_S$ and containing $(c)R_S$. Again by corollary 2 these prime ideals are totally ordered by set inclusion. By a well-known argument $\mathrm{Rad}\,((c)R_S)$ must therefore be a prime ideal. The radical of the contraction of $(c)R_S$ is the contraction of the radical and thus a prime ideal; in other words, the radical of the S-component $(c)_S$ is a prime ideal $\mathfrak{p}^*$ in R, where $\mathfrak{p}^*\subseteqq\mathfrak{p}$. Now, the radicals of $\mathfrak{a}=((a)+\mathfrak{m}_1\cap\mathfrak{m}_2)\cdot(c)_S$ and $\mathfrak{b}=((b)+\mathfrak{m}_1\cap\mathfrak{m}_2)\cdot(c)_S$ are also $\mathfrak{p}^*$, and thereby $\mathrm{Rad}\,(\mathfrak{a}\cap\mathfrak{b})=\mathfrak{p}^*$. By the assumption about the Prüfer ring R, $\mathfrak{a}\cap\mathfrak{b}$ must be irreducible, so that $\mathfrak{a}\subseteqq\mathfrak{b}$ or $\mathfrak{a}\supseteqq\mathfrak{b}$. Suppose, for instance, that $\mathfrak{a}\subseteqq\mathfrak{b}$. Then $\mathfrak{a}R_{\mathfrak{m}_2}\subseteqq$ $\subseteqq\mathfrak{b}R_{\mathfrak{m}_2}$. It is readily seen that $(c)_SR_{\mathfrak{m}_2}=(c)R_{\mathfrak{m}_2}$, $((a)+\mathfrak{m}_1\cap\mathfrak{m}_2)R_{\mathfrak{m}_2}=R_{\mathfrak{m}_2}$ and $((b)+\mathfrak{m}_1\cap\mathfrak{m}_2)R_{\mathfrak{m}_2}=\mathfrak{m}_2R_{\mathfrak{m}_2}$. Consequently, we should have $(c)R_{\mathfrak{m}_2}=\mathfrak{a}R_{\mathfrak{m}_2}\subseteqq\mathfrak{b}R_{\mathfrak{m}_2}=$ $=(c)\mathfrak{m}_2R_{\mathfrak{m}_2}$ which is obviously impossible and we have arrived at the desired contradiction. Q. E. D.

If R is a Prüfer ring and $\mathfrak{a}$ an arbitrary ideal of R, the radical of $\mathfrak{a}R_{\mathfrak{m}}$ in the quotient ring $R_{\mathfrak{m}}$ is a prime ideal, because $R_{\mathfrak{m}}$ is a valuation ring. Hence the radical of the contraction $\mathfrak{a}R_{\mathfrak{m}}\cap R$ is also a prime ideal. If the associated valuations of R are independent, $\mathfrak{a}R_{\mathfrak{m}}\cap R$ is irreducible.

It is a well-known fact that any ideal in any commutative ring is a finite or infinite intersection of irreducible ideals. The above remark allows us to give an explicit representation of this kind for Prüfer rings with independent valuations. By the localization principle (slightly transformed) we have for any integral domain R and any ideal $\mathfrak{a}$ of R

$$\mathfrak{a}=\bigcap_{\mathfrak{m}}(\mathfrak{a}R_{\mathfrak{m}}\cap R) \tag{2. 1}$$

$\mathfrak{m}$ running through the maximal ideals of R.

We formulate this in

COROLLARY 4. If R is a Prüfer ring for which the associated valuations are independent, (2. 1) gives a representation of the arbitrary ideal $\mathfrak{a}$ as an intersection of irreducible ideals.

A more special class of Prüfer rings R, "R of Dedekind type" has been considered by JAFFARD [6] which actually, in the terminology used here, means that R is a Prüfer ring with independent valuations for which any infinite intersection of different maximal ideals is (0). For such rings only finitely many ideals of the intersection in (2. 1) are $\neq R$. Moreover, if we delete the ideals which are R we get an irredundant decomposition of $\mathfrak{a}$ into irreducible ideals, provided that $\mathfrak{a}\neq(0)$.

Indeed, let us assume that a component $\mathfrak{a}R_{\mathfrak{m}_1} \cap R$ were superfluous in (2. 1). In that case we should have

$$\mathfrak{a}R_{\mathfrak{m}_1} \cap R \supseteqq \bigcap_{\mathfrak{m} \neq \mathfrak{m}_1} (\mathfrak{a}R_{\mathfrak{m}} \cap R), \tag{2. 2}$$

the intersection of the right side being finite. By a theorem due to FUCHS [3] an irreducible ideal of an arithmetical ring has the characteristic property that if it contains the intersection of two ideals, it must contain at least one of them. Applying this result to the finite intersection in (2. 2), we see that $\mathfrak{a}R_{\mathfrak{m}_1} \cap R \supseteqq \mathfrak{a}R_{\mathfrak{m}_2} \cap R$ for a suitable $\mathfrak{m}_2 \neq \mathfrak{m}_1$. Hence $\mathfrak{a}R_{\mathfrak{m}_2} \cap R \subseteqq \mathfrak{m}_1 \cap \mathfrak{m}_2$. But the radical of $\mathfrak{a}R_{\mathfrak{m}_2} \cap R$ is a non-zero prime ideal contained in $\mathfrak{m}_1 \cap \mathfrak{m}_2$, contradicting the independence of the associated valuations of R.

Furthermore, any irreducible ideal $\mathfrak{a} \neq (0)$ is equal to exactly one of its components $\mathfrak{a}R_{\mathfrak{m}} \cap R$. This involves that any representation of an ideal $\mathfrak{a} \neq (0)$ as a finite irredundant intersection of irreducible ideals must coincide with the one just constructed. In other words:

COROLLARY 5. Let R be a Prüfer ring for which the associated valuations are independent and for which any infinite intersection of different maximal ideals is (0). The intersection (2. 1) is a finite irredundant decomposition in irreducible ideals of the arbitrary ideal $\mathfrak{a} \neq (0)$, provided we delete the components equal to R. This is the only finite irredundant decomposition of $\mathfrak{a}$ into irreducible ideals.

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