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# FIXED POINT THEOREMS OF LOCAL CONTRACTION MAPPINGS ON MENGER SPACES

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**Abstract**

*In this paper, we introduce the concept of $\varepsilon$-chainable PM-space, and give several fixed point theorems of one-valued and multivalued local contraction mapping on the kind of spaces.*

**Key words** probabilistic metric space, Menger space, $\varepsilon$-chainable, local contraction mapping, fixed point

## I. Introduction

In [1], Sehgal and Bharucha-Reid proved a fixed point theorem for one-valued local contraction mapping on $(\varepsilon,\lambda)$-chainable PM-spaces. Later, an important generalization of the theorem was given by Cain, Jr. and Kasriel[2]. However, the restrictive conditions of the theorems given in [1] and [2] are too strong for $t$-norm $\Delta$, where they all require that $\Delta$ satisfy the condition $\Delta(t,t)>t$. It is easy to show that there is only one $t$-norm satisfying the above condition, i.e. $\Delta=\min$. Therefore, the results have bigger limitations. In this paper, we introduce the concept of $\varepsilon$-chainable PM-space, which is a strengthening form of the definition of $(\varepsilon,\lambda)$-chainable PM-space. We only require that $t$-norm $\Delta$ satisfy $\sup\limits_{0<t<1}\Delta(t,t)=1$, Under the condition we give several fixed point theorems for one-valued and multi-valued local contraction mappings on the kind of spaces, which are the generalizations of the fixed point theorems of local contraction mappings on metric spaces given in [3,4].

## II. Preliminaries

Throughout this paper, let $R=(-\infty,\infty)$, $R^+=[0,\infty)$, $Z^+$ be the set of all positive integers. We denote by $\mathscr{D}$ the set of all (left continuous) distribution functions. For the definitions, symbols and related terminologies on probabilistic metric space (for short PM-space) and Menger probabilistic metric space (for short Menger space) one can see [5] or [6].

Let $(X,F,\Delta)$ be a Menger space. Schweizer, Sklar and Thorp[9] pointed that if $t$-norm $\Delta$ satisfies $\sup\limits_{0<t<1}\Delta(t,t)=1$, then $(X,F,\Delta)$ is a Hausdorff topological space and for each $p\in X$

$$\mathscr{U}_p=\{U_p(\varepsilon,\lambda):\varepsilon>0,\ \lambda\in(0,1)\}$$

is a base of neighborhoods of point $p$, where

$$U_p(\varepsilon,\lambda)=\{q\in X:F_{p,q}(\varepsilon)>1-\lambda\}$$

We denote by $\mathscr{T}$ the above topology on $X$ and it is called the $(\varepsilon, \lambda)$-topology of $(X,F,\Delta)$. Thus we can induce a series of concepts in $(X,F,\Delta)$ for the topology $\mathscr{T}$, such as $\mathscr{T}$-convergence, $\mathscr{T}$-Cauchy sequence and $\mathscr{T}$-complete etc.

In the following, we always assume that $t$-norm $\Delta$ satisfies $\sup\limits_{0<t<1}\Delta(t, t)=1$, unless otherwise mentioned.

**Proposition 1** Let $(X,F,\Delta)$ be a Menger space and $\delta$ be a given number in (0,1]. For each $\lambda\in(0, \delta)$, we define a function $d_\lambda: X\times X\to R^+$ as follows:

$$d_\lambda(x, y)=\inf\{t>0: F_{x,y}(t)>1-\lambda\} \tag{2.1}$$

Then $\{d_\lambda:\lambda\in(0, \delta)\}$ have the following properties:

(1) $d_\lambda(x, y)<t$ if and only if $F_{x,y}(t)>1-\lambda$;

(2) $d_\lambda(x, y)=0$, $\forall\lambda\in(0, \delta)$ if and only if $x=y$;

(3) $d_\lambda(x, y)=d_\lambda(y,x)$;

(4) If $\lambda, \mu\in(0, \delta)$, $\mu<\lambda$, then $d_\lambda(x, y)\leqslant d_\mu(x, y)$, $\forall x, y\in X$;

(5) For any $\lambda\in(0, \delta)$, there exists $\mu\in(0, \lambda]$, such that

$$d_\lambda(x, z)\leqslant d_\mu(x, y)+d_\mu(y, z), \qquad \forall x, y, z\in X \tag{2.2}$$

$\{d_\lambda:\lambda\in(0, \delta)\}$ defined by (2.1) is called the family of $L$-pseudo metrics on $X$ induced by the Menger space $(X,F,\Delta)$.

**Remark 1** By (4) and (5), we can extend the generalized triangle inequality, i.e. (2.2) as:

(6) For any $n\in Z^+$ and any $\lambda\in(0, \delta)$, there exists $\mu\in(0, \lambda]$ such that

$$d_\lambda(x_1, x_n)\leqslant\sum_{i=1}^{n-1}d_\mu(x_i, x_{i+1}) \tag{2.2$'$}$$

where $x_1, x_2, \ldots, x_n$ are arbitrary $n$ points in $X$.

**Proposition 2** Let $\{d_\lambda:\lambda\in(0, \delta)\}$ be the family of $L$-pseudo metrics induced by Menger space $(X,F,\Delta)$, $\{x_n\}\subset X$, $x\in X$, Then

(i) $\quad x_n\xrightarrow{\mathscr{T}}x\Longleftrightarrow F_{x_n,x}(t)\to 1, \quad \forall t>0\Longleftrightarrow d_\lambda(x_n, x)\to 0, \quad \forall\lambda\in(0, \delta)$;

(ii) $\{x_n\}$ is a $\mathscr{T}$-Cauchy sequence

$\Longleftrightarrow F_{x_n,x_m}(t)\to 1 \quad (n, m\to\infty), \ \forall t>0$

$\Longleftrightarrow d_\lambda(x_n, x_m)\to 0 \quad (n, m\to\infty), \ \forall\lambda\in(0, \delta)$.

**Definition 1** Let $(X,F,\Delta)$ be a Menger space, $A\subset X$ and $x\in X$ The probabilistic distance $F_{x,A}$ between point $x$ and set $A$ is defined as:

$$F_{x,A}(t)=\sup_{y\in A}F_{x,y}(t), \qquad \forall t\in R.$$

We denote by $CB(X)$ the family of nonempty $\mathscr{T}$-closed probabilistically bounded sets and define a mapping $\tilde{F}: CB(X)\times CB(X)\to\mathscr{D}$ as follows (we denote $\tilde{F}(A,B)$ by $\tilde{F}_{A,B}$):

$$\tilde{F}_{A,B}(t)=\sup_{s<t}\Delta\Big(\inf_{a\in A}\sup_{b\in B}F_{a,b}(s), \ \inf_{b\in B}\sup_{a\in A}F_{a,b}(s)\Big)$$

$$\forall A, B\in CB(X), \ t\in R$$

$\tilde{F}_{A,B}$ is called the Menger-Hausdorff metric induced by $F$[7].

**Remark 2** In [7], $F_{x,A}$ is the function defeind by

$$F_{x,A}(t)=\sup_{s<t}\sup_{y\in A}F_{x,y}(s), \quad \forall t\geqslant 0$$

It is not difficult to show that

$$\sup_{s<t}\sup_{y\in A}F_{x,y}(s)=\sup_{y\in A}F_{x,y}(t)$$

So the definition of $F_{x,A}$ in this paper coincides with the definition in [7].

Thus from Proposition 1.3 in [7] we have

**Proposition 3** Let $(X,F,\varDelta)$ be a Menger space, $A\subset X$ and $x, y\in X$. Then

(i) $F_{x,A}(t)=1, \ \forall t>0$ if and only if $x\in\bar{A}$;

(ii) $F_{x,A}(t_1+t_2)\geqslant\varDelta(F_{x,y}(t_1), F_{y,A}(t_2)), \ \forall t_1, t_2>0$;

(iii) for any $A, B\in CB(X)$ and $x\in A$ we have

$$F_{x,B}(t)\geqslant\tilde{F}_{A,B}(t), \ \forall t\geqslant 0.$$

**Proposition 4**[10] Let $(X,F,\varDelta)$ be a Menger space, and $A, B\in CB(X)$. For each $\lambda\in(0, 1)$ we define

$$D_\lambda(A, B)=\inf\{t>0:\tilde{F}_{A,B}(t)>1-\lambda\}$$

Then

(i) $D_\lambda(A, B)<t$ and only if $\tilde{F}_{A,B}(t)>1-\lambda$;

(ii) for any $\lambda\in(0, 1)$, we have

$$d_\lambda(x, B)\leqslant D_\lambda(A, B), \qquad \forall x\in A$$

where $d_\lambda(x, B)=\inf_{y\in B}d_\lambda(x, y)$.

**Definition 2** The PM-space $(X,F)$ is said to be $\varepsilon$-chainable if for given $\varepsilon>0$ and any $x, y\in X$, there is a finite set of points in $X$: $x=x_0, x_1, ..., x_n=y$ such that $F_{x_{i-1},x_i}(\varepsilon)=1$ $i=1,2,\cdots,n$.

**Definition 3** The function $\Phi(t):R^+\to R^+$ is said to satisfy the condition $(\Phi)$, if $\Phi(t)$ is strictly increasing, and the series $\sum_{n=1}^{\infty}\Phi^n(t)$ is convergent for any $t>0$, where $\Phi^n(t)$ denotes the $n$-th iteration of $\Phi(t)$.

**Proposition 5** If the function $\Phi(t):R^+\to R^+$ satisfies the condition $(\Phi)$, then $\Phi(t)<t, \ \forall t>0$ and $\Phi(0)=0$.

**Proof** The conclusion of Proposition 5 follows from Lemma in [11].

## III. The Fixed Point Theorems of Multi-Valued Local Contraction Mappings

**Theorem 1** Let $(X,F,\varDelta)$ be an $\varepsilon$-chainable (for some $\varepsilon>0$) and $\mathscr{T}$-complete Menger space. Let $T:X\to CB(X)$ be a mapping satisfying the following conditions:

(i) For any number $\beta>1$ and any $x, y\in X$, $u\in Tx$, there exists $v\in Ty$ such that

$$F_{u,v}(\beta t)\geqslant\tilde{F}_{Tx,Ty}(t), \qquad \forall t\in R^+ \tag{3.1}$$

(ii) There exists a right continuous function $\Phi(t)$ satisfying the condition $(\Phi)$ and $\alpha\in(0, 1)$ such that

$$\tilde{F}_{Tx,Ty}(\Phi(t))\geqslant F_{x,y}(t) \tag{3.2}$$

whenever $F_{x,y}(\varepsilon)\neq 0$ and $F_{x,y}(t) > 1-\alpha$ Then $T$ has a fixed point, i.e. there exists $x_*\in X$ such that $x_*\in Tx_*$

**Proof** First we prove that (3.1) can be deduced

$$d_\lambda(u,\ v)\leqslant\beta D_\lambda(Tx,\ Ty),\qquad \forall\lambda\in(0,\ 1) \tag{3.1$'$}$$

and (3.2) can be deduced

$$D_\lambda(Tx,\ Ty)\leqslant\Phi(d_\lambda(x,\ y)),\qquad \forall\lambda\in(0,\ \alpha] \tag{3.2$'$}$$

whenever $F_{x,y}(\varepsilon)\neq 0$ .

Let $D_\lambda(Tx,\ Ty)=t$ . Then for any $s>t$, by Proposition 4 we have $\tilde{F}_{Tx,Ty}(s)>1-\lambda$ . Hence from (3.1) it follows that $F_{u,v}(\beta s)>1-\lambda$ . According to Proposition 1 we have $d_\lambda(u,\ v)<\beta s$ . By the arbitrariness of $s$ we get (3.1)′.

Suppose $F_{x,y}(\varepsilon)\neq 0$ and $d_\lambda(x,y)=t,\ \lambda\in(0,\ \alpha]$ . By Proposition 1 for any $s>t$ we have $F_{x,y}(s)>1-\lambda\geqslant 1-\alpha$ , and from condition (ii) it follows that $\tilde{F}_{Tx,Ty}(\Phi(s))>1-\lambda$ . According to Proposition 4 we have $D_\lambda(Tx,\ Ty)<\Phi(s)$ . By the right continuity of $\Phi(t)$ and letting $s\to t$ it gets (3.2)′.

Next we arbitrarily take $x_0^{(0)}\in X$ and $x_1^{(0)}\in Tx_0^{(0)}$ . Since $(X,F,\Delta)$ is an $\varepsilon$-chainable, there exists a finite set of points in $X$: $x_0^{(0)}$ , $x_0^{(1)},\ \cdots,\ x_0^{(n)}=x_1^{(0)}$ such that $F_{x_0^{(i-1)},x_0^{(i)}}(\varepsilon)=1$ , $i=1,2,\cdots,n$.

Take $\varepsilon_0>\varepsilon,\varepsilon_n\in(\varepsilon,\varepsilon_0)$ and $\varepsilon_n\nearrow\varepsilon_0$. Since $F_{x_0^{(0)},\ x_0^{(1)}}(\varepsilon)=1$ , we have $d_\lambda(x_0^{(0)},\ x_0^{(1)})<\varepsilon,\forall\lambda\in(0,1)$ ·

Putting $d_0(x_0^{(0)},\ x_0^{(1)})=\sup\limits_{\lambda\in(0,1)} d_\lambda(x_0^{(0)},\ x_0^{(1)})$ , obviously we have $d_0(x_0^{(0)},\ x_0^{(1)})\leqslant\varepsilon<\varepsilon_1$. Hence $\Phi(d_0(x_0^{(0)},x_0^{(1)}))<\Phi(\varepsilon_1)$ · Take $\delta_1>0$ such that $\dfrac{\Phi(\varepsilon_1)}{\Phi(d_0(x_0^{(0)},x_0^{(1)}))+\delta_1}>1$ . By condition (i) and (3.1)′, we know that for $x_1^{(0)}\in Tx_0^{(0)}$ there exists $x_1^{(1)}\in Tx_0^{(1)}$ such that

$$d_\lambda(x_1^{(0)},\ x_1^{(1)})\leqslant\frac{\Phi(\varepsilon_1)}{\Phi(d_0(x_0^{(0)},\ x_0^{(1)}))+\delta_1}\cdot D_\lambda\ (Tx_0^{(0)},\ Tx_0^{(1)})$$

From (3.2)′ any noting $\Phi(d_\lambda(x_0^{(0)},\ x_0^{(1)}))\leqslant\Phi(d_0(x_0^{(0)},\ x_0^{(1)}))$ we get

$$d_\lambda(x_1^{(0)},\ x_1^{(1)})<\Phi(\varepsilon_1),\qquad \forall\lambda\in(0,\ \alpha].$$

Similarly, from $F_{x_0^{(1)},x_0^{(2)}}(\varepsilon)=1$ and $x_1^{(1)}\in Tx_0^{(1)}$ we know that there exists $x_1^{(2)}\in Tx_0^{(2)}$ such that

$$d_\lambda(x_1^{(1)},\ x_1^{(2)})<\Phi(\varepsilon_1),\qquad \forall\lambda\in(0,\ \alpha].$$

Continuing in this way we can obtain a finite set of points: $x_1^{(0)},\ x_1^{(1)},\ \cdots,\ x_1^{(n)}=x_2^{(0)}$ such that $x_1^{(i)}\in Tx_0^{(i)}$ and

$$d_\lambda(x_1^{(i-1)},\ x_1^{(i)})<\Phi(\varepsilon_1),\qquad (i=1,\ 2,\ \cdots,\ n)\ \forall\lambda\in(0,\ \alpha].$$

Since $d_\lambda(x_1^{(0)},\ x_1^{(1)})<\Phi(\varepsilon_1),\ \forall\lambda\in(0,\ \alpha]$ , we have

$$d_0(x_1^{(0)},\ x_1^{(1)})=\sup_{\lambda\in(0,\alpha]} d_\lambda(x_1^{(0)},\ x_1^{(1)})\leqslant\Phi(\varepsilon_1)<\Phi(\varepsilon_2).$$

Hence $\Phi(d_0(x_1^{(0)},x_1^{(1)}))<\Phi^2(\varepsilon_2)$ . Take $\delta_2>0$ such that

$$\frac{\Phi^2(\varepsilon_2)}{\Phi(d_0(x_1^{(0)},\ x_1^{(1)}))+\delta_2}>1.$$

By (3.1)′ and (3.2)′, for $x_2^{(0)}\in Tx_1^{(0)}$ there exists $x_2^{(1)}\in Tx_1^{(1)}$ such that

$$d_\lambda(x_2^{(0)}, x_2^{(1)}) \leqslant \frac{\Phi^2(\varepsilon_2)}{\Phi(d_0(x_1^{(0)}, x_1^{(1)}))+\delta_2} \cdot D_\lambda(Tx_1^{(0)}, Tx_1^{(1)})$$

$$\leqslant \frac{\Phi^2(\varepsilon_2)}{\Phi(d_0(x_1^{(0)}, x_1^{(1)}))+\delta_2} \cdot \Phi(d_\lambda(x_1^{(0)}, x_1^{(1)})) < \Phi^2(\varepsilon_2) \qquad \forall \lambda \in (0, a].$$

Continuing in this way we can get a finite set of points: $x_2^{(0)}, x_2^{(1)}, \cdots, x_2^{(n)} = x_3^{(0)}$ such that $x_2^{(i)} \in Tx_1^{(i)}$ and

$$d_\lambda(x_2^{(i-1)}, x_2^{(i)}) < \Phi^2(\varepsilon_2) \qquad (i=1, 2, \cdots, n) \ \forall \lambda \in (0, a].$$

Using the mathematical induction, it is not difficult to show that for any natural number $m$, there exists a finite set of points: $x_m^{(0)}, x_m^{(1)}, \cdots, x_m^{(n)} = x_{m+1}^{(0)}$ such that $x_m^{(i)} \in Tx_{m-1}^{(i)}$ and

$$d_\lambda(x_m^{(i-1)}, x_m^{(i)}) < \Phi^m(\varepsilon_m) < \Phi^m(\varepsilon_0) \qquad (i=1, 2, \cdots, n) \quad \forall \lambda \in (0, a].$$

Now we prove that $\{x_m^{(0)}\}_{m=1}^{\infty}$ is a $\mathscr{T}$-Cauchy sequence of $X$.

For any $i, j \in Z^+$, $i<j$ and $\lambda \in (0, a]$. By Remark 1, there exists $\mu \in (0, \lambda]$ such that

$$d_\lambda(x_i^{(0)}, x_j^{(0)}) \leqslant \sum_{m=i}^{j-1} d_\mu(x_m^{(0)}, x_{m+1}^{(0)}).$$

Similarly, for the above $\mu$ there exists $\nu \in (0, \mu]$ such that

$$d_\mu(x_m^{(0)}, x_{m+1}^{(0)}) = d_\mu(x_m^{(0)}, x_m^{(n)})$$

$$\leqslant \sum_{i=1}^{n} d_\nu(x_m^{(i-1)}, x_m^{(i)}) < n\Phi^m(\varepsilon_0).$$

Thus we have

$$d_\lambda(x_i^{(0)}, x_j^{(0)}) \leqslant n\sum_{m=i}^{j-1} \Phi^m(\varepsilon_0), \qquad \forall \lambda \in (0, a].$$

Letting $i \to \infty$ and noting that the series $\sum_{m=1}^{\infty} \Phi^m(\varepsilon_0)$ is convergent, we have $d_\lambda(x_i^{(0)}, x_j^{(0)}) \to 0$, $\forall \lambda \in (0, a]$. Therefore from Proposition 2 we know that $\{x_m^{(0)}\}_{m=1}^{\infty}$ is a $\mathscr{T}$-Cauchy sequence in $X$. Since $(X, F, \Delta)$ is $\mathscr{T}$-complete, there exists $x_* \in X$ such that $x_m^{(0)} \xrightarrow{\mathscr{T}} x_*$

Lastly, we prove that $x_*$ is a fixed point of $T$, i.e. $x_* \in Tx_*$.

Since $x_m^{(0)} \xrightarrow{\mathscr{T}} x_*$, we have $F_{x_m^{(0)}, x_*}(t) \to 1$, $\forall t > 0$. Hence for any $t>0$ there exists $N \in Z^+$ such that $F_{x_m^{(0)}, x_*}(\varepsilon) \neq 0$ and $F_{x_m^{(0)}, x_*}(t) > 1-a$, whenever $m>N$. Thus from condition (ii) it follows that

$$\tilde{F}_{Tx_m^{(0)}, Tx_*}(\Phi(t)) \geqslant F_{x_m^{(0)}, x_*}(t) \to 1 \quad (m \to \infty), \ \forall t > 0$$

Noting $\Phi(t) < t$, from the above expression we have

$$\tilde{F}_{Tx_m^{(0)}, Tx_*}(t) \to 1 \qquad (m \to \infty), \ \forall t > 0$$

Since $x_m^{(0)} = x_{m-1}^{(n)} \in Tx_{m-2}^{(n)} = Tx_{m-1}^{(0)}$, by Proposition 3 we get

$$F_{x_*, Tx_*}(t) \geqslant \Delta(F_{x_*, x_m^{(0)}}(t/2), F_{x_m^{(0)}, Tx_*}(t/2))$$

$$\geqslant \Delta(F_{x_*, x_m^{(0)}}(t/2), \tilde{F}_{T_{m-1}^{(0)}, Tx_*}(t/2)) \tag{3.3}$$

Letting $m\to\infty$ on the right of (3.3) and noting that $\Delta(t,t)$ is continuous at $t=1$, since the $t$-norm $\Delta$ satisfies $\sup\limits_{0<t<1}\Delta(t,t)=1$, we have $F_{x_*,Tx_*}(t)=1$, $\forall t>0$, and therefore $x_*\in Tx_*$ by Proposition 3. This completes the proof.

In Theorem 1 taking $\Phi(t)=kt$, $0<k<1$, we have the following result.

**Corollary 1** Let $(X,F,\Delta)$ be an $\varepsilon$-chainable and $\mathscr{T}$-complete Menger space. Let the mapping $T$: $X\to CB(X)$ satisfy condition (i) and the following condition:

(ii)′ there exists a positive number $k<1$ and $\alpha\in(0,1)$ such that

$$\tilde{F}_{Tx,Ty}(kt)\geqslant F_{x,y}(t) \tag{3.4}$$

whenever $F_{x,y}(\varepsilon)\neq 0$ and $F_{x,y}(t)>1-\alpha$. Then $T$ has a fixed point.

In the following, we shall use Theorem 1 to prove the fixed point theorem for multi-valued local contraction mapping on metric spaces.

Let $\varepsilon$ be a positive number. The metric space $(X,d)$ is said to be $\varepsilon$-chainable if for any two points $x$ and $y$ in $X$, there exists a finite set of points in $X$: $x=x_0, x_1, \cdots, x_n=y$ such that $d(x_{i-1},x_i)<\varepsilon$, for $i=1,2,\cdots,n$.

**Lemma 1** Let $(X,d)$ be a metric space. We define the mapping $F$: $X\times X\to\mathscr{D}$ as follows (we denote $F(x,y)$ by $F_{x,y}$):

$$F_{x,y}(t)=H(t-d(x,y)),\qquad \forall x,y\in X \tag{3.5}$$

and take $\Delta=\min$. Then $(X,F,\Delta)$ is a Menger space and the topology $\mathscr{T}_d$ on $X$ induced by $d$ coincides with the $(\varepsilon,\lambda)$-topology $\mathscr{T}$ of $(X,F,\Delta)$. $(X,F,\Delta)$ will be called the Menger space induced by $(X,d)$.

**Proof** From Theorem 3 in [1] we know that $(X,F,\min)$ is a Menger space. By (3.5) it is easy to show that for any $\varepsilon>0$ and any $\lambda\in(0,1)$

$$d(x,y)<\varepsilon\Longleftrightarrow F_{x,y}(\varepsilon)=1\Longleftrightarrow F_{x,y}(\varepsilon)>1-\lambda$$

Therefore $\mathscr{T}_d$ and $\mathscr{T}$ are coincident.

**Lemma 2** Let $(X,F,\Delta)$ be a Menger space induced by the metric space $(X,d)$. Then

(i) $(X,d)$ is complete if and only if $(X,F,\Delta)$ is $\mathscr{T}$-complete.

(ii) $A$ is a bounded closed set in $(X,d)$ if and only if $A$ is a probabilistically bounded $\mathscr{T}$-closed set in $(X,F,\Delta)$.

Hence, we may also denote the family of all nonempty bounded closed sets in $X$ by $CB(X)$.

**Lemma 3** Let $(X,F,\Delta)$ be a Menger space induced by the metric space $(X,d)$. Then for any $A,B\in CB(X)$,

$$\tilde{F}_{A,B}(t)=H(t-D(A,B)) \tag{3.6}$$

where $D(A,B)=\inf\{r: A\subset S(B,r)$, and $B\subset S(A,r)\}$ is a Hausdorff metric between $A$ and $B$, $S(A,r)=\{x\in X: d(x,A)<r\}$

**Proof** By (3.5) it is easy to show that

$$A\subset S(B,r)\Longleftrightarrow \inf_{x\in A}\sup_{y\in B}F_{x,y}(r)=1\Longleftrightarrow \inf_{x\in A}\sup_{y\in B}F_{x,y}(r)>0.$$

From the conclusion we can prove that if $D(A,B)<t$, then

$$\tilde{F}_{A,B}(t)=\sup_{s<t}\min\{\inf_{x\in A}\sup_{y\in B}F_{x,y}(s),\ \inf_{x\in B}\sup_{y\in A}F_{x,y}(s)\}=1;$$

if $D(A,B)>t$, then $\tilde{F}_{A,B}(t)=0$. The former is obvious. In order to prove the latter, we suppose $\tilde{F}_{A,B}(t)>0$, then there exists $r\in(0, t)$ such that

$$\inf_{x\in A}\sup_{y\in B}F_{x,y}(r)>0 \text{ and } \inf_{x\in B}\sup_{y\in A}F_{x,y}(r)>0,$$

so that $D(A, B)\leqslant r<t$. This is a contradiction. Hence (3.6) is proved.

**Theorem 2** Let $(X,d)$ be an $\varepsilon$-chainable and complete metric space. Let the mapping $T$: $X \to CB(X)$ satisfy the following condition: there exists a right continuous function $\Phi(t)$ satisfying the condition $(\Phi)$, such that

$$D(Tx, Ty)\leqslant\Phi(d(x, y)), \tag{3.7}$$

whenever $d(x, y)<\varepsilon$. Then $T$ has a fixed point.

**Proof** Suppose that $(X,F,\min)$ is a Menger space induced by $(X,d)$. Since $(X,d)$ is an $\varepsilon$-chainable and complete metric space, by Lemma 2 and (3.5), it is easy for us know that $(X,F,\min)$ is $\varepsilon$-chainable and $\mathscr{T}$-complete Menger space.

In the following, we only need to show that $T$ satisfies conditions (i) and (ii) of Theorem 1.

(i) for any $\beta>1$ and any $x, y\in X$, $u\in Tx$, by Nadler Lemma[12], there exists $v\in Ty$ such that

$$d(u, v)\leqslant\beta D(Tx, Ty)$$

From the above expression we can prove that (3.1) is true. In fact, we might as well suppose $\tilde{F}_{Tx,Ty}(t)>0$. By (3.6) we have $D(Tx,Ty)<t$, so that $d(u,v)<\beta t$. It follows from (3.5) that $F_{u,v}(\beta t)=1$. Hence $F_{u,v}(\beta t)\geqslant\tilde{F}_{Tx,Ty}(t)$.

(ii) Suppose $F_{x,y}(\varepsilon)\neq0$ and $F_{x,y}(t)>1-a$. Then by (3.5) we have $d(x, y)<\varepsilon$ and $d(x, y)<t$. Hence

$$D(Tx, Ty)\leqslant\Phi(d(x, y))<\Phi(t).$$

It follows from (3.6) that

$$\tilde{F}_{Tx,Ty}(\Phi(t))=1\geqslant F_{x,y}(t).$$

All the conditions of Theorem 1 are satisfied, and therefore $T$ has a fixed point.

**Remark 3** Theorem 1 in [4] and Theorem 5.4.5 in [8] are all the special cases of Theorem 2.

## IV. The Fixed Point Theorems of One-Valued Local Contraction Mappings

**Theorem 3** Let $(X,F,\Delta)$ be an $\varepsilon$-chainable and $\mathscr{T}$-complete Menger space. Let the mapping $T$: $X \to X$ satisfy the following condition: there exists a $\delta\in(0, 1)$, so that for each $a\in(0, \delta)$ there is a function $\Phi_a(t)$ satisfying the condition $(\Phi)$ such that

$$F_{Tx,Ty}(\Phi_a(t))\geqslant F_{x,y}(t) \tag{4.1}$$

whenever $F_{x,y}(\varepsilon)\neq0$ and $F_{x,y}(t)>1-a$. Then $T$ has a unique fixed point $x_*$ in $X$ and $T^nx_0\xrightarrow{\mathscr{T}}x_*$ for any $x_0\in X$.

**Proof** For any $x_0\in X$, we might as well suppose $Tx_0\neq x_0$. Since $(X,F,\Delta)$ is $\varepsilon$-chainable, there exists a finite set of points in $X$: $x_0, x_1, \cdots, x_n=Tx_0$, such that $F_{x_{i-1},x_i}(\varepsilon)=1$, $i=1,2,\cdots,n$.

Hence by the condition of Theorem 3, we have

$$F_{Tx_{i-1}Tx_i}(\Phi_a(\varepsilon))=1 \quad (i=1, 2, \cdots, n)\ \forall a\in(0, \delta),$$

Using the mathematical induction, it is easy to prove that for any $m\in Z^+$ ,

$$F_{T^m x_{i-1},T^m x_i}(\Phi_\alpha^m(t))=1 \qquad (i=1,\ 2,\ \cdots,\ n),\ \forall \alpha\in(0,\ \delta),$$

hence it follows from Proposition 1 that

$$d_\alpha(T^m x_{i-1}T^m x_i)<\Phi_\alpha^m(t) \qquad (i=1,\ 2,\ \cdots,\ n)\ \ \forall \alpha\in(0,\ \delta).$$

Define $z_m=T^m x_0\,(m=1,2,\cdots)$. For any $i,\ j\in Z^+,\ i<j$ and any $\alpha\in(0,\ \delta)$ , by Remark 1, there exists $\lambda\in(0,\ \alpha]$ such that

$$d_\alpha(z_i,z_j)\leqslant\sum_{m=i}^{j-1}d_\lambda(z_m,\ z_{m+1})$$

Also, for the given $\lambda$ there exists $\mu\in(0,\ \lambda]$ such that

$$d_\lambda(z_m,\ z_{m+1})=d_\lambda(T^m x_0,\ T^m x_n)$$

$$\leqslant\sum_{i=1}^{n}d_\mu(T^m x_{i-1},\ T^m x_i)<n\Phi_\mu^m(\varepsilon)$$

Thus

$$d_\alpha(z_i,\ z_j)<n\sum_{m=i}^{j-1}\Phi_\mu^m(\varepsilon), \qquad \forall \alpha\in(0,\ \delta)$$

Note that the series $\sum_{m=1}^{\infty}\Phi_\mu^m(\varepsilon)$ is convergent, we have $d_\alpha(z_i,\ z_j)\to 0(i\to\infty),\ \ \forall \alpha\in(0,\ \delta)$. Hence $\{z_m\}$ is a $\mathscr{T}$-Cauchy sequence by Proposition 2. As $(X,F,\Delta)$ is $\mathscr{T}$-complete, $z_m\xrightarrow{\mathscr{T}}x_*\in X$.

Now we prove that $x_*$ is a fixed point of $T$. Since $z_m\xrightarrow{\mathscr{T}}x_*$ , for any $t>0$ and any $\alpha\in(0,\ \delta)$ we have $F_{z_m,x_*}(\varepsilon)\neq 0$ and $F_{z_m,x_*}(t)>1-\alpha$ , whenever $m$ is big enough. Hence by the condition of the theorem we get

$$F_{Tz_m,Tx_*}(\Phi_\alpha(t))\geqslant F_{z_m,x_*}(t)$$

Owing to $\Phi_\alpha(t)<t$ and $F_{z_m,x_*}(t)\to 1(m\to\infty)$ , we have

$$F_{Tz_m,Tx_*}(t)\to 1\ \ (m\to\infty), \qquad \forall t>0.$$

Note that $\Delta(t,t)$ is continuous at $t=1$. Therefore

$$F_{x_*,Tx_*}(t)\geqslant\Delta(F_{x_*,z_m}(t/2),\ F_{z_m,Tx_*}(t/2))\to 1\ .(m\to\infty).$$

This implies that $Tx_*=x_*$ , i.e. $x_*$ is a fixed point of $T$.

Lastly we prove that $x_*$ is the unique fixed point of $T$. Suppose $y_*\in X$ is also a fixed point of $T$, i.e. $Ty_*=y_*$ and $x_*\neq y_*$ . Since $(X,F,\Delta)$ is $\varepsilon$-chainable, there exists a finite set of points in $x_*=y_0,\ y_1,\ \cdots,\ y_p=y_*$, such that $F_{y_{i-1},y_i}(\varepsilon)=1$ , $i=1,2,\cdots,p$. By (4.1), for any $m\in Z^+$ we have

$$F_{T^m y_{i-1},T^m y_i}(\Phi_\alpha^m(\varepsilon))=1 \qquad (i=1,\ 2,\ \cdots,\ p)\ \forall \alpha\in(0,\ \delta)$$

Hence

$$d_\alpha(T^m y_{i-1},\ T^m y_i)<\Phi_\alpha^m(\varepsilon) \qquad (i=1,\ 2,\ \cdots,\ p)\ \ \forall \alpha\in(0,\ \delta)$$

For any $\alpha\in(0,\ \delta)$ there exists $\lambda\in(0,\ \alpha]$ such that

$$d_\alpha(x_*,\ y_*)=d_\alpha(T^m x_*,\ T^m y_*)$$
$$\leqslant\sum_{i=1}^{p} d_\lambda(T^m y_{i-1} T^m y_i)<\sum_{i=1}^{p}\Phi_\lambda^m(\varepsilon)\to 0\quad (m\to\infty)$$

This implies that $d_\alpha(x_*,\ y_*)=0$, $\forall\alpha\in(0,\ \delta)$ . Thus $x_*=y_*$ , a contradiction. Therefore the fixed point of $T$ is unique.

**Corollary 2** Let $(X,F,\Delta)$ be an $\varepsilon$ -chainable and $\mathscr{T}$-complete Menger space. Let $T$ be a mapping of $X$ onto itself having the property that there exists $\delta\in(0,\ 1)$ , so that for each $\alpha\in(0,\ \delta)$ there is a corresponding positive nunmber $k_\alpha\in(0,1)$ such that

$$F_{Tx,Ty}(k_\alpha t)\geqslant F_{x,y}(t) \tag{4.2}$$

whenever $F_{x,y}(\varepsilon)\neq 0$ and $F_{x,y}(t)>1-\alpha$ . Then $T$ has a unique fixed point $x_*$ in $X$ and $T^n x_0\xrightarrow{\mathscr{T}} x_*$ for any $x_0\in X$ .

**Theorem 4** Let $(X,d)$ be an $\varepsilon$ -chainable and complete metric space. Let $T$ be a mapping of $X$ onto itself. If there exists a function $\Phi(t)$ satisfying the condition $(\Phi)$, such that

$$d(Tx,\ Ty)\leqslant\Phi(d(x,y)), \tag{4.3}$$

whenever $d(x,\ y)<\varepsilon$ , then $T$ has a unique fixed point $x_*$ in $X$ and $T^n x_0\xrightarrow{d} x_*$ for any $x_0\in X$.

**Proof** Suppose $(X,F,\min)$ is a Menger space reduced by $(X,d)$. Then $(X,F,\min)$ is $\varepsilon$-chainable and $\mathscr{T}$-complete. Arbitrarily take $\delta\in(0,\ 1)$ and set $\Phi_\alpha(t)=\Phi(t)$ , for any $\alpha\in(0,\ \delta)$. Suppose $F_{x,y}(\varepsilon)\neq 0$ and $F_{x,y}(t)>1-\alpha$ , then by (3.5) we have $d(x,\ y)<\varepsilon$ and $d(x,\ y)<t$ . Hence it follows from (4.3) that

$$d(Tx,\ Ty)\leqslant\Phi_\alpha(d(x,\ y))<\Phi_\alpha(t)$$

so that $F_{Tx,\ Ty}(\Phi_\alpha(t))=1\geqslant F_{x,y}(t)$ . This implies that all the conditions of Theorem 3 are satisfied. This completes the proof.

**Remark 4** The fixed point theorem given in [3] is a special case of Theorem 4.

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