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MENGERIAN THEOREMS FOR PATHS OF BOUNDED LENGTH

by

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Dedicated to the memory of FERNANDO ESCALANTE

1. Introduction

Let u and v be non-adjacent points in a connected graph G. A classical result known to all graph theorists is that called MENGER's theorem. The point version of this result says that the maximum number of point-disjoint paths joining u and v is equal to the minimum number of points whose deletion destroys all paths joining u and v. The theorem may be proved purely in the language of graphs (probably the best known proof is indirect, and is due to DIRAC [3] while a more neglected, but direct, proof may be found in ORE [7]). One may also prove the theorem by appealing to flow theory (e.g. BERGE [1], p. 167).

In many real-world situations which can be modeled by graphs certain paths joining two non-adjacent points may well exist, but may prove essentially useless because they are too long. Such considerations led the authors to study the following two parameters. Let n be any positive integer and let u and v be any two non-adjacent points in a graph G.

Denote by $A_n(u, v)$ the maximum number of point-disjoint paths joining u and v whose length (i.e., number of lines) does not exceed n. Analogously, let $V_n(u, v)$ be the minimum number of points in G the deletion of which destroys all paths joining u and v which do not exceed n in length. A special case would obtain when $n = p = |V(G)|$, and we have by Menger's theorem, the equality $A_n(u, v) = V_n(u, v)$.

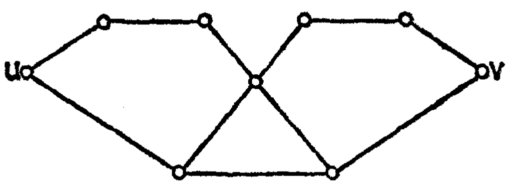

Fig. 1

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In general, however, one does not have equality, but it is trivial that $A_n(u, v) \leq V_n(u, v)$ for any positive integer n. On the other hand, the graph of Fig. 1 has $V_5(u, v) = 2$, but $A_5(u, v) = 1$.

We prefer to formulate our work as a study of the ratio $\frac{V_n(u, v)}{A_n(u, v)}$ or simply $\frac{V_n}{A_n}$ when the points u and v are understood. For any terminology not defined in this paper, the reader is referred to the book by HARARY [4].

2. Bounds for the ratio

As in the introduction we shall assume throughout this paper that u and v are non-adjacent points in the same component of a graph G. It is trivial that $1 \leq \frac{V_n(u, v)}{A_n(u, v)} \leq n - 1$. As usual, $d(u, v)$ denotes the distance between points u and v. Our first result involves this distance.

THEOREM 1. *For every positive integer $n \geq 2$ and for each $m = n - d(u, v) \geq \geq 0$, $\frac{V_n(u, v)}{A_n(u, v)} \leq m + 1$.*

The construction in Section 3 shows that this bound is sharp.

PROOF. The proof proceeds by induction on m. Hence first let $m = 0$, i.e., suppose $n = d(u, v) = n_0$. We orient some of the lines of G according to the following rule: let xy be any line. Then if $d(x, v) > d(y, v)$, orient x to y. Then, clearly, any u-v geodesic (i.e., a shortest u-v path) yields a dipath from u to v. On the other hand, we claim that any u-v dipath must arise from a geodesic u-v path in G, for just consider our rule of orientation. If (x, y) is a directed line in our dipath, $d(x, v) > d(y, v)$ and distance decreases by 1 as we traverse each diline toward v. Hence our dipath cannot have $> n$ lines and hence must have come from a u-v geodesic.

Thus in the oriented subgraph of G, the u-v paths are exactly the geodesics, so by Menger's theorem, $V_n(u, v) = A_n(u, v)$ and the case for $m = 0$ is proved.

Now by induction hypothesis, assume that the theorem holds for some $m_0 \geq 0$ and suppose $m = n - d(u, v) = m_0 + 1$ (and hence that $n > d(u, v)$).

Let X be a minimum set of points covering all u-v geodesics. By the case for $m = 0$,

$$|X| = V_{d(u,v)}(u, v) = A_{d(u,v)}(u, v) \leq A_n(u, v).$$

Consider the graph $G - X$. If $d_{G-X}(u, v) > n$, X has covered all u-v paths of length $\leq n$ and we have, $V_n(u, v) = |X| \leq A_n(u, v) \leq mA_n(u, v)$ and we are done. So suppose $d_{G-X}(u, v) \leq n$, say $d_{G-X}(u, v) = n - t$ for some t, $0 \leq t < m$. (Note that $t < m$ for X destroys all u-v geodesics and thus $t = n - d_{G-X}(u, v) < n - d(u, v) = m$).

So by the induction hypothesis applied to points u and v in graph $G-X$, we have

$$V_n^{G-X}(u, v) \leq (t + 1) A_n^{G-X}(u, v).$$

But we can then cover all n-paths in G joining u and v with a set Y where

$$|Y| = |X| + (t + 1) A_n^{G-X}(u, v) \leq |X| + (t + 1) A_n(u, v).$$

So

$$V_n(u, v) \leq |X| + (t + 1) A_n(u, v) \leq (t + 2) A_n(u, v) \leq (m + 1) A_n(u, v)$$

and the proof is complete.

The next theorem shows that we can do better as far as a bound depending solely upon n is concerned.

THEOREM 2. *For any graph G, any non-negative integer n, and any two non-adjacent points u and v, $V_n(u, v) \leq \left[\frac{n}{2}\right] A_n(u, v)$.*

PROOF. If $d(u, v) \geq n/2 + 1$, we are done by Theorem 1. So suppose $d(u, v) \leq (n + 1)/2$. Choose D such that $d(u, v) \leq D \leq n$ and let P_0 be a u-v geodesic in G. Form a new graph G_1 from G by removing all interior points of P_0. Clearly $d_{G_1}(u, v) \geq d_G(u, v)$. Now remove any u-v geodesic in G_1, say P_1, to obtain G_2. Continue in this manner until we obtain a graph G_r containing a u-v geodesic P_r such that $l(P_r) \leq D$, but the length of any u-v geodesic in $G_{r+1} > D$. For convenience let us denote G_{r+1} by G' and similarly for parameters of this graph. Thus $d_{G_{r+1}}(u, v) = d'(u, v) \geq D + 1$.

Since we have removed r disjoint u-v paths from G to get G', we have

$$A_n \geq A'_n + r, \tag{1}$$

for all discarded paths had length no greater than the length of a u-v geodesic in G'.

Also

$$V_n \leq V'_n + r(D - 1). \tag{2}$$

Moreover, if G' is connected, we have by Theorem 1 that

$$V'_n \leq (n - d'(u, v) + 1) A'_n \leq (n - D - 1 + 1) A'_n = (n - D) A'_n. \tag{3}$$

The combining (2) and (3), we obtain by (1)

$$V_n \leq (n - D) A'_n + r(D - 1) \leq (n - D)(A_n - r) + r(D - 1) =$$
$$= (n - D) A_n + r(2D - n - 1).$$

Since r is non-negative, choose D to be the greatest integer so that $2D - n - 1 \leq 0$. Hence $D \leq \left[\frac{n+1}{2}\right]$ and since D is integral, $D = \left[\frac{n+1}{2}\right]$. Hence $n - D = n - \left[\frac{n+1}{2}\right] = \left[\frac{n}{2}\right]$ and thus $V_n \leq \left[\frac{n}{2}\right] A_n$.

If G' is not connected between u and v, we have $A'_n = V'_n = 0$ and conclude similarly.

The bound in this theorem is sharp for $n = 2$, 3 and 5 (for $n = 5$, see Fig. 1). It is, however, not sharp for $n = 4$.

THEOREM 3. *For any graph G with non-adjacent points u and v, $V_4(u, v) = A_4(u, v)$.*

PROOF. Partition the points of $G - u - v$ into disjoint classes (i, j) as follows: $w \in (i, j)$ iff $d(u, w) = i$ and $d(w, v) = j$. Clearly we may ignore classes $(1, 1)$ and all (i, j) for $i + j > 4$. So the remaining graph $\hat{G}$ has the appearance of Figure 2.

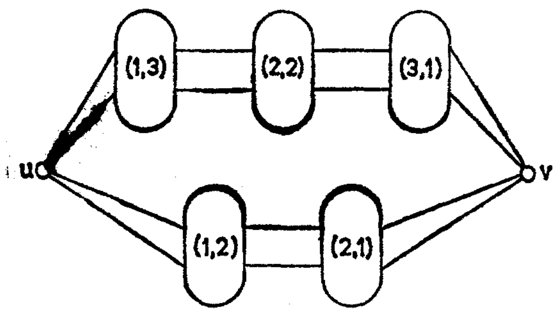

Fig. 2

Now construct a di-graph $\hat{D}$ as follows. Let $V(\hat{D}) = V(\hat{G})$ and $(x, y) \in E(\hat{D})$ iff (1) $xy \in E(\hat{G})$ and (2) $d(u, y) > d(u, x)$.
Hence $\hat{D}$ has the appearance of Figure 3.

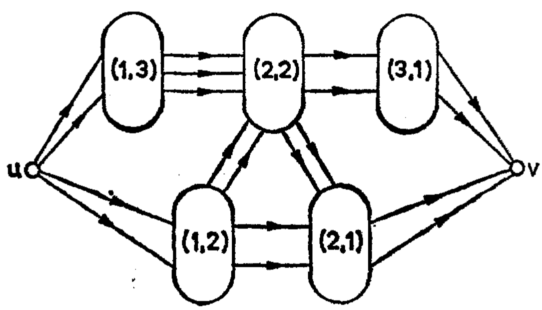

Fig. 3

Observe that

(a) each dipath in $\hat{D}$ has length ≤ 4 and
(b) each *chordless* path of $\hat{G}$ of length ≤ 4 corresponds to a dipath in $\hat{D}$.

Let S be a set of V_4 points in $\hat{G} - u - v$ whose deletion destroys all u-v paths of length ≤ 4. But then in $\hat{D} - u - v$ all dipaths from u to v are also destroyed, so $V_4 \geq \vec{H}(u, v)$ where $\vec{H}(u, v)$ denotes the minimum number of points whose deletion separates u and v in $\hat{D}$. But by Menger's theorem applied to $\hat{D}$, $\vec{H}(u, v)$ (= the maximum number of point-disjoint dipaths from u to v) $\leq A_4$, since each set of point-disjoint dipaths from u to v in $\hat{D}$ corresponds to a set of point-disjoint u-v paths in $\hat{G}$ of the same cardinality.

Thus it will suffice to prove $V_4 \leq \vec{H}(u, v)$. Let L be any set of $\vec{H}(u, v)$ points in $\hat{D} - u - v$ whose removal separates u and v. We now claim L meets all u-v paths in $\hat{G}$ of length ≤ 4. If not, there is a path P joining u and v with length ≤ 4 and $(V(P) - u - v) \cap L = \emptyset$. We may assume P is chordless. But, then it translates into a dipath from u to v in $\hat{D}$ on the same points. L does not meet this dipath, which is a contradiction.

In the construction of the next section we will have $\frac{V_n}{A_n} = \left[\sqrt{\frac{n}{2}}\right]$ or $\left[\sqrt{\frac{n}{2}}\right] + 1$. It is unknown to us where for a fixed n, the value of $\sup \frac{V_n}{A_n}$ lies in the interval $\left(\left[\sqrt{\frac{n}{2}}\right], \left[\frac{n}{2}\right]\right)$.

3. A Construction

We will construct a graph $G(n, t)$ such that given $t (> 0)$, there is an n and a graph $G(n, t)$ which has 2 distinct non-adjacent points u and v such that $A_n(u, v) = 1$, but $V_n(u, v) = t + 1$. Moreover, we will show in addition that given any integer $k (\geq 1)$, we can construct a $G(n, t, k)$, which is k-connected.

For the moment, suppose t is a given positive integer. Choose any $n > t + 1$ and fix it. Construct a path L of length $s = n - t$ joining u and v. As is customary, we shall refer to paths having at most their endpoints in common as *openly disjoint*. Now for each i, $2 \leq i \leq t + 1$, take every pair of points a, b on L which are at a distance $= i$ on L and attach a path of length $i + 1$ at a and b which is openly disjoint from L. Such paths we shall call *ears*. (See Figure 4).

Now let P be any u-v path of length $= s' (\leq n)$. P has at least $n - t$ lines since L is a u-v geodesic.

Suppose P uses r ears. Since replacing an ear by the corresponding segment of L shortens the length by ≥ 1, we have $s' \geq n - t + r$. Hence

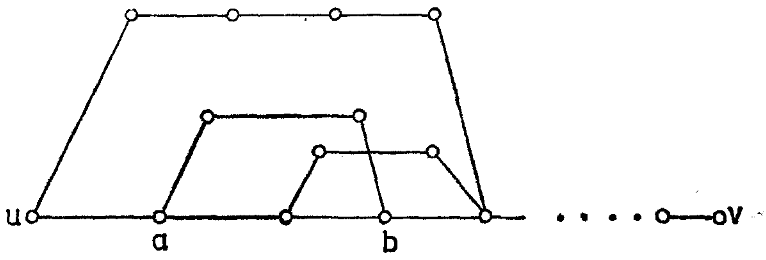

Fig. 4

$r \leq t$. Since each ear has $\leq t + 1$ interior points, P has $\leq r(t + 1)$ points not on L. So the number of points of P on L is (not including u and v)

$$\geq (s' - 1) - r(t + 1) \geq n - t + r - 1 - r(t + 1) =$$
$$= n - (r + 1)t - 1 \geq n - (t + 1)t - 1.$$

If $n - (t + 1)t - 1 > \frac{1}{2}$ (the number of inner points of L), then any two such paths P must have an interior point in common. Note that the number of inner points of $L = n - t - 1$. Thus what we need is that $n - (t + 1)t - 1 > > \frac{1}{2}(n - t - 1)$, i.e., $n \geq 2t^2 + t + 2$. If n is given, the best t satisfying this inequality is either $\left[\sqrt{\frac{n}{2}}\right] - 1$ or $\left[\sqrt{\frac{n}{2}}\right]$. Then with such an n, any two u-v paths of length $\leq n$ must have some inner point of L in common; i.e., $A_n(u, v) = 1$.

We now proceed to show that $V_n(u, v) \geq t + 1$. Suppose there is a set T of t points which cover all u-v paths of length $\leq n$. We may assume all points of T lie on L, for otherwise move right on the "offending ear" until L is reached and use the point of L thus encountered in place of the original T-point. If the ear ends at v take the left-hand end point on L. Note also that u, v are joined by no one ear by our choice of n.

Let us call the sets of points of T which are consecutive on L the *blocks* of T. There are no more than t such blocks. Recall that L contains $n - t + 1$ points where $n - t + 1 = (n + 1) - t \geq 3$ and hence $n - t \geq 2$. Thus we can form a new u-v path Q by jumping each block of T with an ear. This new path Q then misses T and we have added exactly one to the length of L for each block jumped. It follows that Q has length $\leq s + t = n - t + t = n$. Hence, there is a u-v path Q of length $\leq n$ which misses T contradicting the definition of T. Thus $V_n(u, v) \geq t + 1$.

We know at this point that $G(n, t)$ is at least 2-connected. Let k be any integer ≥ 2. We now proceed to modify the graph $G(n, t)$ constructed above so that the resulting graph $G(n, t, k)$ retains the properties that $A_n(u, v) = 1$, $V_n \geq t + 1$ and in addition is k-connected.

The idea is to construct a new graph H, join it to $G(n, t)$ by suitably chosen lines so that the resulting graph is k-connected, but also so that no new "short" u-v paths are introduced.

Let the points of $G(n, t)$ be $w_1, \ldots, w_N$. Further, let $M = k + n$. Form a path of MN points $p_1p_2 \ldots p_{MN}$ and then replace each p_i with a clique, K_k^i, on k points where each point of K_k^i is joined to each point of K_k^{i+1}. Now join w_1 to exactly one point of each of $K_k^1, \ldots, K_k^k$; w_2 to exactly one point of $K_k^{M+1}, \ldots, K_k^{M+k}$; and, in general, w_j to exactly one point of $K_k^{(j-1)M+1}, \ldots$ $\ldots, K_k^{(j-1)M+k}$ for $j = 1, \ldots, N$. It is now easily seen that no new path joining any w_i and w_j is of length $< n + 1$. It is clear that $A_n = 1$ and $V_n = t + 1$ in this new graph for any path of length $\leq n$ joining u and v must lie entirely within the original $G(n, t)$ part of this new graph. It is equally clear that the new graph $G(n, t, k)$ is k-connected.

4. A different type of Mengerian result

In this section we take a different approach. Recall that $V_n(u, v) \geq A_n(u, v)$ and moreover, strict inequality can occur. One's intuition may indicate that even in this case, if the subscript on A_n is allowed to increase to some new value n' one can always obtain $V_n \leq A_{n'}$. The next theorem says that such a conjecture is not only appealing, but true.

THEOREM 4. *Let n and h be positive integers. Then there is a constant $f(n, h)$ such that if $V_n(u, v) \geq h$,* then $A_{f(n,h)}(u, v) \geq h$.

In the proof we need the following result.

THEOREM 5 (BOLLOBÁS [2], KATONA [6], JAEGER—PAYAN [5]). *Given any family of r-sets which needs at least t points to cover, then there exists a subfamily with $\leq \binom{r+t-1}{r}$ elements which still needs t points to cover.*

REMARK. It is trivial to see that instead of "r-sets" one can say "sets of size at most r".

PROOF of Theorem 4. Consider sets of interior points of u-v paths of length $\leq n$. By the assumption we need $\geq h$ points to cover the members of this family. By the preceding theorem and the remark following it we can select $\binom{n+h-2}{n-1}$ paths of length $\leq n$ such that we still need h points to cover these paths. So let G_1 be the union of these paths and apply Menger's theorem to G_1 to see that there are $\geq h$ openly disjoint u-v paths. So how long can a longest path in G_1 be? We have $\binom{n+h-2}{n-1}$ paths of length $\leq n$.

So $G_1 - u - v$ has $\leq (n-1)\binom{n+h-2}{n-1}$ points. Now among all sets of $\geq h$ openly disjoint u-v paths in G_1, the longest path one could find would be of length $(n-1)\binom{n+h-2}{n-1} - (h-1) + 1$. (This of course happens when one has $h-1$ paths of length 2 and a single long path of the above length.)

Thus set $f(n, h) = (n-1)\binom{n+h-2}{n-1} - h + 2$ and we have $A_{f(n,h)}(u, v) \geq h$.

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