MAXIMUM LIKELIHOOD ESTIMATES AND LIKELIHOOD RATIO CRITERIA FOR LOCATION AND SCALE PARAMETERS OF THE MULTIVARIATE l_1-NORM SYMMETRIC DISTRIBUTIONS*

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Abstract

In this paper we introduce three families of multivariate and matrix l_1-norm symmetric distributions with location and scale parameters and discuss their maximum likelihood estimates and likelihood ratio criteria. It is shown that under certain condition sthey have the same form as those for independent exponential variates.

§ 1. Introduction

The exponential distribution is one of the most important distributions in statistics and has been studied extensively and thoroughly by many authors. There are a number of ways to extend exponential distribution to multivariate distribution[6]。 We have introduced and studied some families of multivariate symmetric distributions related to exponential distribution. We may call the multivariate l_1-norm symmetric distributions. In this paper we'll study related families with location and scale parameters.

Let $W(\mu, \sigma)$ denote the exponential distribution with p.d.f.

$$\sigma^{-1}e^{-(x-\mu)/\sigma}, \quad x>\mu. \tag{1.1}$$

It is well-known that the maximum likelihood estimates (MLE's) of μ and σ are [5]

$$\hat{\mu}=\min_i x_i$$

$$\hat{\sigma}=n^{-1}\sum_{i=1}^{n}(x_i-\hat{\mu}) \tag{1.2}$$

where $x_1, \cdots, x_n$ are independent random variables, each having p.d.f. (1.1). Furthermore, the hypothesis testing problems can be considered. For example Sukhatme (see [3] p. 133) has studied the problem of testing equality of μ_i when there are k independent samples, $x_{ij}\left(i=1, 2, \cdots, k; j=1, 2, \cdots, n_i; \sum_{i=1}^{k} n_i=N\right)$, with x_{ij} drawn from $W(\mu_i, \sigma)$. More generally, we consider the following problems.

Let $W=(W_{ij})$ be an $n\times p$ matrix, where $w_{ij}\sim W(\mu_j, a_j)$, $i=1, \cdots, n$, $j=1, \cdots,$

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p, are independent. Set

$$\Theta=\{(\mu, A), \mu=(\mu_1, \cdots, \mu_p), A=\operatorname{diag}(a_1,\cdots, a_p)>0\}$$
$$\Theta_1=\{(\mu, A)\in\Theta: a_1=\cdots=a_p\}$$
$$\Theta_2=\{(\mu, A)\in\Theta: \mu_1=\cdots=\mu_p\}$$
$$\Theta_3=\{(\mu, A)\in\Theta: \mu_1=\cdots=\mu_p, a_1=\cdots=a_p\}$$
$$\Theta_4=\{(\mu, A)\in\Theta: \mu=0\}$$
$$\Theta_5=\{(\mu, A)\in\Theta: \mu=0, a_1=\cdots=a_p\}.$$

We want to test H_j: $(\mu, A)\in\Theta_j$ versus K_j: $(\mu, A)\in\Theta-\Theta_j$, $j=1, \cdots, 5$, and H_6: $(\mu, A)\in\Theta_3$ versus K_6: $(\mu, A)\in\Theta_1$. It can be shown that the above likelihood ratio criteria (LRC) are

$$T_1(W)=\frac{\prod_{j=1}^{p}(\|w_j\|-n\min_j w_{ij})}{\left[\sum_{j=1}^{p}(\|w_j\|-\min_i w_{ij})\right]^p}, \tag{1.3}$$

$$T_2(W)=\prod_{j=1}^{p}\left(\frac{\|w_j\|-n\min_i w_{ij}}{\|w_j\|-n\min_{ij} w_{ij}}\right), \tag{1.4}$$

$$T_3(W)=\prod_{j=1}^{p}\left(\frac{\|w_j\|-n\min_i w_{ij}}{\|w\|-np\min_{ij} w_{ij}}\right), \tag{1.5}$$

$$T_4(W)=\prod_{j=1}^{p}\left(\frac{\|w_j\|-n\min_i w_{ij}}{\|w_j\|}\right), \tag{1.6}$$

$$T_5(W)=\prod_{j=1}^{p}\left(\frac{\|w_j\|-n\min_i w_{ij}}{\|W\|}\right), \tag{1.7}$$

$$T_6(W)=\frac{\sum_{j=1}^{p}n(\min_i w_{ij}-\min_{ij} w_{ij})/(p-1)}{\sum_{i=1}^{n}\sum_{j=1}^{p}(w_{ij}-\min_i w_{ij})/(np-p)}, \tag{1.8}$$

respectively, where $\|W\|=\sum_{i,j}w_{ij}$, $\|w_j\|=\sum_i w_{ij}$.

Let $X=(x_{ij})$ with $\{x_{ij}\}$ being i.i.d. $W(0, 1)$ variates. We can express W as follows:

$$W\stackrel{d}{=\!=}M+XA \tag{1.9}$$

where

$$\mathbf{1}_n=(1, \cdots, 1)', \mu=(\mu_1, \cdots, \mu_p)', M=\mathbf{1}_n\mu',$$
$$A=\operatorname{diag}(a_1, \cdots, a_p)$$

and " $\stackrel{d}{=}$ " denotes that the random variables on the two sides have the same distribution. This suggests that certain classes of distributions can be defined by replacing X with $Z\in F_{n\times p}(i)$, the matrix l_1-norm symmetric distributions[2].

In this paper, we'll lay stress on discussing MLE's and LRC of the parameters under the generalized model.

First of all, a brief review for the multivariate and matrix l_1-norm symmetric distributions is required.

$$R_+^{n\times p}=\{A=(a_{ij})_{n\times p}:\ a_{ij}\geqslant 0,\ i=1,\ \cdots,\ n,\ j=1,\ \cdots,\ p\}$$
$$R_+^n=R_+^{n\times 1}$$
$$B_n=\{b=(b_1,\ \cdots,\ b_n)':\ b\in R_+^n,\ \|b\|=1\}.$$

If y is uniformly distributed on B_n, we write $y\sim U_n$. If $Y=(y_1,\ \cdots,\ y_p)\in R_+^{n\times p}$, where y_j's are i.i.d. and $y_1\sim U_n$, we write $Y\sim U_{n\times p}(1)$. If Vec $Y=(y_1',\ \cdots,\ y_p')'\sim U_{np}$, we write $Y\sim U_{n\times p}(2)$. Let $L(z)$ denote the distribution of z. In [1] and [2], the families of multivariate and matrix l_1-norm symmetric distributions are defined as follows:

$$F_n=\{L(z):\ z=ru,\ \text{where } r\geqslant 0 \text{ is independent of } u\sim U_n.\} \tag{1.10}$$

$$F_{n\times p}(1)=\{L(Z):\ Z=UR,\ \text{where } U\sim U_{n\times p}(1),\ R=\operatorname{diag}(r_1,\ \cdots,\ r_p)\geqslant 0,\ U \text{ and } R \text{ are independent.}\} \tag{1.11}$$

$$F_{n\times p}(2)=\{L(Z):\ Z=Ur,\ \text{where } U\sim U_{n\times p}(2),\ r\geqslant 0,\ U \text{ and } r \text{ are independent}\}. \tag{1.12}$$

It is evedent that $F_{n\times 1}(2)=F_{n\times 1}(1)=F_n$ and it is shown that $F_{n\times p}(2)$ is a proper subset of $F_{n\times p}(1)\,(p>1)$. Moreover, there is a one-to-one correspondence between $F_{n\times p}(1)$ and the set $R_p=\{L((r_1,\ \cdots,\ r_p)):\ r_i\geqslant 0,\ i=1,\ \cdots,\ p\}$ and a one-to-one correspondence between $F_{n\times p}(2)$ and R_1. Specially, if $z\in R_+^{n\times p}$ is absolutely continuous, then (1) $Z\in F_{n\times p}(1)$ and $(\|z_1\|,\ \cdots,\ \|z_p\|)$ has p.d.f. $g(\cdot)$ iff Z has p.d.f. $f(\|z_1\|,\ \cdots,\ \|z_p\|)$. In this case, $f(x)=(\Gamma(n))^p g(x)\prod_{j=1}^{p} x_j^{-n+1}$; (2) $Z\in F_{n\times p}(2)$ and $\|Z\|$ has p.d.f. $g(\cdot)$ iff Z has p. d. f. $f(\|Z\|)$. In this case $f(x)=\Gamma(np)g(x)x^{-np+1}$.

Now we can, in the same way as (1.9), introduce parameters for the above three families. Let

$$S_n=\{(L(w):\ w=m+Az,\ z\in F_n,\ A=\operatorname{diag}(a_1,\ \cdots,a_p)>0\} \tag{1.13}$$

$$S_{n\times p}(i)=\{L(W):\ W=M+ZA,\ Z\in F_{n\times p}(i),\ A=\operatorname{diag}(a_1,\ \cdots,\ a_p)>0\},\quad i=1,\ 2. \tag{1.14}$$

Throughout the paper we assume that all random variables (vectors, matrices) considered have p.d.f.s. Hence we write $w=m+Az\in S_n(m,\ A,\ g)$ to mean that $L(w)\in S_n$ and $\|z\|$ has p. d. f. $g(\cdot)$. The notation $W=M+ZA\in S_{n\times p}(i,\ M,\ A,\ g)$ is adopted in the same way.

From papers [1] and [2], it is easy to obtain basic properties of the above three families, including distribution functions, probability density functions, characteristic functions, marginal distributions, conditional distributions and characterizations of exponential distributions. In this paper MLE's and LRC are obtained. We'll show that under certain conditions the MLE of M is independent of g for $W\in S_{n\times p}$ $(i,\ M,\ A,\ g)$, $i=1,\ 2$, the MLE of A is invariant except for a constant multiplier (depending on g) when $i=2$, and the LRC and their null distributions are independent of g for some tests.

In this paper, we use capital letters to express matricess, while corresponding small letters with subscripts stand for their row vectors, column vectors and elements, small lettersn for example $Z=(z_1,\cdots,\ z_p)=(z_{(1)},\ \cdots,\ z_{(n)})'=(z_{ij})_{n\times p}$ and $x=(x_1,\ \cdots,\ x_n)'_{n\times 1}$.

§ 2. MLE's and LRC of S_n and $S_{n\times p}(2)$

In this section we investigate MLE's for location and scale parameters of S_n, $S_{n\times p}(2)$ and give some LRC and their null distributions. The following lemmas can be proved similarly as those in [4].

Lemma 2.1. Suppose that $f(\boldsymbol{x})$, defined on R_+^p, is nonnegative and continuous such that

(1) $f(\boldsymbol{x})$ is decreasing for each x_j (sufficiently large);

(2) $$\int_{R_+^p} f(\boldsymbol{x}) \prod_{j=1}^p x_j^{n-1}\, d\boldsymbol{x} < \infty.$$

Let

$$u(\boldsymbol{x}) = f(\boldsymbol{x}) \prod_{j=1}^p x_j^n. \tag{2.1}$$

Then $u(\boldsymbol{x})$ has a maximum in $R_+^p - \{\boldsymbol{0}\}$.

Lemma 2.2. Suppose that $f(\|\boldsymbol{z}_1\|, \cdots, \|\boldsymbol{z}_p\|)$ is a density for $\boldsymbol{Z}\in R_+^{n\times p}$ such that $E\left(\prod_{j=1}^p z_{1j}\right) < \infty$ and $u(\boldsymbol{x}) = f(\boldsymbol{x}) \prod_{j=1}^p x_j^n$ is uniformly continuous in R_+^p. Then $u(\boldsymbol{x})$ has a maximum in $R_+^p - \{\boldsymbol{0}\}$.

Lemma 2.3. Suppose that the conditions of Lemmas 2.1 and 2.2 hold. For any fixed $y\in R_+^p - \{\boldsymbol{0}\}$, let

$$v(z) = u(\boldsymbol{y}z), \quad z\in R_+^1. \tag{2.2}$$

Then $v(z)$ has a maximum in $(0,\infty)$ as a function of z (note this maximum is depending on $\boldsymbol{y}$ when $\boldsymbol{y}$ varies and we denote it by y_0).

Corollary 1. Suppose that f has partial derivatives of order one for all its variables. If u defined by (2.1) has a maximum at $\boldsymbol{x}$, then it satisfies the equations

$$nf(\boldsymbol{x}) + x_j f'_{x_j}(\boldsymbol{x}) = 0, \quad j=1, \cdots, p. \tag{2.3}$$

If v defined by (2.2) has a maximum at y_0, then it satisfies the equation

$$\left[npf(\boldsymbol{x}) + \sum_{j=1}^p x_j f'_{x_j}(\boldsymbol{x})\right]\Big|_{x_j=y_j y_0} = 0. \tag{2.4}$$

For $\boldsymbol{W}\in S_{n\times p}(\mathring{\boldsymbol{i}}, \boldsymbol{M}, \boldsymbol{A}, g)$, Lemmas 2.1, 2.2 and 2.3 give some sufficient conditions for corresponding u and v, defined by (2.1) and (2.2) respectively, to have maxima. The following theorems give MLE's of $(\boldsymbol{M}, \boldsymbol{A})$ when u and v have maxima. Since the structure of $S_{n\times p}(2)$ is simpler than $S_{n\times p}(1)$, we first solve the problems of MLE and LRC of the former.

Theorem 2.1. Assume that Θ satisfies the condition that if $(\boldsymbol{m}, \boldsymbol{A})\in\Theta$ then $(\boldsymbol{m}, c\boldsymbol{A})\in\Theta$, $\forall c>0$. Suppose that $\boldsymbol{w}=\boldsymbol{m}+\boldsymbol{A}\boldsymbol{z}\in S_n(\boldsymbol{m}, \boldsymbol{A}, g)$ and $u(x)=f(x)x^n$ has a maximum at $x_0\in(0,\infty)$. Suppose also that MLE $(\tilde{\boldsymbol{m}}, \tilde{\boldsymbol{A}})$ exists when z_i's are i.i.d. $W(0, 1)$ variates. Let the likelihood function of $\boldsymbol{w}$ be

$$L(\boldsymbol{m}, \boldsymbol{A}) = f(\|\boldsymbol{A}^{-1}(\boldsymbol{w}-\boldsymbol{m})\|)\,|\boldsymbol{A}|^{-1}.$$

Then the MLE of $(\boldsymbol{m}, \boldsymbol{A})$ is

$$(\hat{\boldsymbol{m}}, \hat{\boldsymbol{A}}) = \left(\tilde{\boldsymbol{m}}, \frac{n}{x_0}\tilde{\boldsymbol{A}}\right) \tag{2.5}$$

and the maximum of the likelihood function is

$$L(\hat{\boldsymbol{m}}, \hat{\boldsymbol{A}}) = f(x_0)\,|\hat{\boldsymbol{A}}|^{-1}. \tag{2.6}$$

Proof. The proof is similar to that in [4]. Let

$$\boldsymbol{B} = |\boldsymbol{A}|^{-1/n}\boldsymbol{A},\ x = \|\boldsymbol{A}^{-1}(\boldsymbol{w}-\boldsymbol{m})\| = |\boldsymbol{A}|^{-1/n}\|\boldsymbol{B}^{-1}(\boldsymbol{w}-\boldsymbol{m})\|.$$

Then

$$L(\boldsymbol{m}, \boldsymbol{A}) = f(x)x^n\|\boldsymbol{B}^{-1}(\boldsymbol{w}-\boldsymbol{m})\|^{-n}. \tag{2.7}$$

If z_{ij}'s are i.i.d. $W(0, 1)$ variates, $f(x) = e^{-x}$. Hence the maximum of (2.7) is attained at

$$x = n,\ \boldsymbol{m} = \tilde{\boldsymbol{m}},\ \boldsymbol{B} = \tilde{\boldsymbol{B}} = |\tilde{\boldsymbol{A}}|^{-1/n}\tilde{\boldsymbol{A}}.$$

In general the maximum of (2.7) is attained at

$$\hat{x} = x_0,\ \hat{\boldsymbol{m}} = \tilde{\boldsymbol{m}},\ \hat{\boldsymbol{B}} = \tilde{\boldsymbol{B}}.$$

Therefore

$$\hat{\boldsymbol{A}} = |\hat{\boldsymbol{A}}|^{1/n}\hat{B} = (\|\hat{\boldsymbol{B}}^{-1}(\boldsymbol{w}-\tilde{\boldsymbol{m}})\|/\hat{x})\tilde{\boldsymbol{B}} = (|\tilde{\boldsymbol{A}}|^{1/n}n/x_0)\tilde{\boldsymbol{B}} = (n/x_0)\tilde{\boldsymbol{A}}.$$

Substitution of these values into (2.7) yields (2.6).

For $\boldsymbol{W} \in S_{n\times p}(2, \boldsymbol{M}, \boldsymbol{A}, g)$, if $\boldsymbol{M} = \mathbf{1}_n\boldsymbol{\mu}'$, then $\boldsymbol{w}_{(i)} \in S_p(\boldsymbol{\mu}, \boldsymbol{A}, g)$, so $\boldsymbol{W}$ can be regarded as a matrix of samples, each row of which is from $S_p(\boldsymbol{\mu}, \boldsymbol{A}, g)$, but need not be independent. In case there are two rows being independent, all are easily shown to be independent with $w_{ij} \sim W(\mu_j, a_j\sigma)$ for some $\sigma > 0$[1].

Theorem 2. 2. Let Θ and $\{\Theta_i\}$ be defined as in Section 1. Suppose that $\boldsymbol{W} = \boldsymbol{M} + \boldsymbol{Z}\boldsymbol{A} \in S_{n\times p}(2, \boldsymbol{M}, \boldsymbol{A}, g)$ with $\boldsymbol{M} = \mathbf{1}_n\boldsymbol{\mu}'$ and $n > 1$. Suppose also that $u(x) = f(x)x^{np}$ has a maximum at $x_0 \in (0, \infty)$ where f is the p. d. f. of $\boldsymbol{Z}$. Let the likelihood function of $\boldsymbol{W}$ be

$$L(\boldsymbol{\mu}, \boldsymbol{A}) = f(\|(\boldsymbol{W}-\boldsymbol{M})\boldsymbol{A}^{-1}\|)\,(a_1\cdots a_p)^{-n},\ (\boldsymbol{W}-\boldsymbol{M})\boldsymbol{A}^{-} \in R_+^{n\times p}.$$

Then

(1) for $(\boldsymbol{\mu}, \boldsymbol{A}) \in \Theta$, Θ_2 and Θ_4, the MLE's of $\boldsymbol{\mu}$ and $\boldsymbol{A}$ are $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \cdots, \hat{\mu}_p)$ and $\hat{\boldsymbol{A}} = \mathrm{diag}(\hat{a}_1, \cdots, \hat{a}_p)$, where

$$\hat{\mu}_j = \begin{cases} \min\limits_i w_{ij}, & (\boldsymbol{\mu}, \boldsymbol{A}) \in \Theta \\ \min\limits_{ik} w_{ik}, & (\boldsymbol{\mu}, \boldsymbol{A}) \in \Theta_2 \\ 0, & (\boldsymbol{\mu}, \boldsymbol{A}) \in \Theta_4 \end{cases} \tag{2.8}$$

and

$$\hat{a}_j = (p/x_0)(\|\boldsymbol{w}_j\| - n\hat{\mu}_j) \tag{2.9}$$

and the maximum of the likelihood function is

$$L(\hat{\mu}, \hat{A}) = f\Big(\sum_{j=1}^{p}(\|\boldsymbol{w}_j\| - n\hat{\mu}_j)/\hat{a}_j\Big)\prod_{j=1}^{p}\hat{a}_j^{-n} = f(x_0)\prod_{j=1}^{p}\hat{a}_j^{-n} \tag{2.10}$$

with corresponding $\hat{\mu}_j$ and $\hat{a}_j$;

(2) for $(\boldsymbol{\mu}, \boldsymbol{A}) \in \Theta_1$, Θ_3 and Θ_5, the MLE's of $\boldsymbol{\mu}$ and $\boldsymbol{A}$ are $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \cdots, \hat{\mu}_p)$ and $\tilde{\boldsymbol{A}} = \mathrm{diag}(\tilde{a}, \cdots, \tilde{a})$, where

$$\hat{\mu}_j = \begin{cases} \min\limits_i w_{ij}, & (\boldsymbol{\mu}, \boldsymbol{A}) \in \Theta_1 \\ \min\limits_{i,k} w_{ik}, & (\boldsymbol{\mu}, \boldsymbol{A}) \in \Theta_3 \\ 0, & (\boldsymbol{\mu}, \boldsymbol{A}) \in \Theta_5 \end{cases} \tag{2.11}$$

and

$$\tilde{a}=\sum_{j=1}^{p}(\|w_j\|-n\hat{\mu}_j)/x_0 \tag{2.12}$$

and the maximum of the likelihood function is

$$L(\hat{\mu},\tilde{A})=f\left(\sum_{j=1}^{p}(\|w_j\|-n\hat{\mu}_j)/\tilde{a}\right)\tilde{a}^{-np}=f(x_0)\tilde{a}^{-np} \tag{2.13}$$

with corresponding $\hat{\mu}_j$ and $\tilde{a}$.

Proof. Since

$$\text{vec } W'=\text{Vec } M'+(I\otimes A)\text{ Vec } Z'\in S_{np}(\text{Vec } M', I\otimes A, g),$$

the result follows from Theorem 2. 1.

Let

$$F^{+}_{n\times p}(1)=\{L(Z)\in F_{n\times p}(1): P(\|z_j\|=0)=0,\ j=1,\cdots,p\},$$
$$F^{+}_{n\times p}(2)=\{L(Z)\in F_{n\times p}(2): P(\|Z\|=0)=0\}.$$

We call a statistic $t(Z)$ invariant in $F^{+}_{n\times p}(i)$ if $t(Z)\overset{d}{=}t(W)$ for any $Z, W\in F^{+}_{n\times p}(i)$, $i=1, 2$. The following lemma is from [2].

Lemma 2.4. A statistic $t(Z)$ is invariant in $F^{+}_{n\times p}(2)$ iff $t(aZ)\overset{d}{=}t(Z)$ for any constant $a>0$ and $Z\in F^{+}_{n\times p}(2)$.

Theorem 2.3. Suppose that the conditions of Theorem 2.2 hold. Let $X=(x_{ij})$, where x_{ij}'s are i. i. d. and $x_{11}\sim W(0,1)$. Then the LRC for H_j versus K_j is equivalent to $T_j(W)$, $j=1,\cdots,6$ (see (1.3)—(1.8)).

Proof. The results are from Theorem 2.2 directly.

Theorem 2.4. Suppose that the conditions of Theorem 2.3 hold. Then under H_j, $T_j(W)\overset{d}{=}T_j(X)$, $j=1,\cdots,6$.

Proof. Under H_j, $T_j(W)\overset{d}{=}T_j(Z)$. Then Lemma 2.4 leads to the desired conclusion.

Theorem 2.5. Suppose that the conditions of Theorem 2.3 hold. Then

$$T_1(X)\overset{d}{=}y_1\cdots y_p \tag{2.14}$$

$$T_4(X)\overset{d}{=}v_1\cdots v_p \tag{2.15}$$

$$T_5(X)\overset{d}{=}\prod_{j=1}^{p}(b_jv_j) \tag{2.16}$$

where $y=(y_1,\cdots,y_p)'\sim U_p$; $v_1,\cdots,v_p$ are i.i.d. with $v_1\sim B(1,n-1)$; $b=(b_1,\cdots,b_p)\sim D_p(n,\cdots,n)$ (Dirichlet distribution); v and b are independent. And

$$T_6(X)\sim F(2(p-1), 2p(n-1)). \tag{2.17}$$

Proof. Form [1] we have $x_j/\|x_j\|\sim U_n$, so

$$P(\min_i x_{ij}/\|x_j\|>a)=P(x_{1j}/\|x_j\|>a,\cdots,x_{nj}/\|x_j\|>a)$$
$$=(1-na)^{n-1},\ na\le 1.$$

Thus $\min_i x_{ij}/\|x_j\|$ has a p.d.f. $n(n-1)(1-na)^{n-2}$, $0\le a\le\frac{1}{n}$. Let $v_j=1-n\min_i x_{ij}/\|x_j\|$. Then $v_j\sim B(1,n-1)$. Since $\|x_j\|\sim\Gamma(n;1)$ and is independent of $x_j/\|x_j\|$, $\|x_j\|$

and v_j are independent. Thus $\|\boldsymbol{x}_j\| v_j \sim W(0,1)$. In view of the independence of $\|\boldsymbol{x}_j\| v_j$, $j=1, \cdots, p$, we have

$$(\|\boldsymbol{x}_1\| v_1, \cdots, \|\boldsymbol{x}_p\| v_p)' \Big/ \sum_{j=1}^{p} \|\boldsymbol{x}_j\| v_j \overset{\mathrm{d}}{=} \boldsymbol{y} \sim U_p$$

and then

$$T_1(\boldsymbol{X}) \overset{\mathrm{d}}{=\!=} y_1 \cdots y_p.$$

(2.15) can be shown similarly. We now prove (2.16). Let $\boldsymbol{b} = (\|\boldsymbol{x}_1\|/\|\boldsymbol{x}\|, \cdots, \|\boldsymbol{x}_p\|/\|\boldsymbol{x}\|)$. Then $\boldsymbol{b} \sim D_p(n, \cdots, n)$ and $\boldsymbol{x}_1/\|\boldsymbol{x}_1\|, \cdots, \boldsymbol{x}_p/\|\boldsymbol{x}_p\|$ and $\boldsymbol{b}$ are independent[1]. Since $1 - n \min_i x_{ij}/\|\boldsymbol{x}_j\| = v_j \sim B(1, n-1)$ as above,

$$T_5(\boldsymbol{X}) = \prod_{j=1}^{p} (\|\boldsymbol{x}_j\|/\|\boldsymbol{x}\|)(1 - n \min_i x_{ij}/\|\boldsymbol{x}_j\|) \overset{\mathrm{d}}{=} \prod_{j=1}^{p} (b_j v_j).$$

For the last assertion (2.17), refer to [3] p. 133.

By making the transformation $x_1 = v_1$, $x_2 = v_1 v_2$, $\cdots$, $x_p = v_1 \cdots v_p$ we find that the p. d. f. of $x = v_1 \cdots v_p$ in Theorem 2.5 is

$$\frac{(-1)^{p-1}(n-1)^p}{(p-1)!} x^{n-2} (\log x)^{p-1}, \quad 0 < x < 1. \tag{2.18}$$

Finally, we give some examples to illustrate the lemmas and theorems.

Example 2.1. Let $\boldsymbol{X}$ be as in Theorem 2.3. Then $\boldsymbol{X} \in F_{n\times p}(2, g)$ $(= S_{n\times p}(2, \boldsymbol{0}, \boldsymbol{I}, g))$ with $f(x) = e^{-x}$. We have

$$x_0 = np, \ u(x_0) = e^{-np}(np)^{np},$$

$$\hat{a}_j = (1/n)(\|\boldsymbol{w}_j\| - n\hat{\mu}_j), \ \tilde{a} = \sum_{j=1}^{p} (\|\boldsymbol{w}_j\| - n\hat{\mu}_j)/(np).$$

Let $\boldsymbol{W} \in S_{n\times p}(2, \boldsymbol{M}, \boldsymbol{A}, g)$ in the following examples.

Example 2.2. Let g be the p.d.f. of $\Gamma(k, 1)$ (Gamma distribution). We have

$$x_0 = k, \ u(x_0) = \frac{\Gamma(np) k^k}{\Gamma(k)} e^{-k}.$$

Specially, if $k = np$, i.d. $\boldsymbol{Z} \overset{\mathrm{d}}{=} \boldsymbol{X}$ (see Example 2.1), then $x_0 = np$.

Example 2.3. Let g be the p.d.f. of $F(k, q)$ (F-distribution). We have

$$x_0 = 1, \ u(x_0) = \frac{\Gamma(np) k^{k/2} q^{q/2}}{B(k/2, q/2)(k+q)^{(k+q)/2}}.$$

Example 2.4. Let g be the p.d.f. of $B(k, q)$, $k > 0$, $q > 1$. We have

$$x_0 = \frac{k}{k+q-1}, \ u(x_0) = \frac{\Gamma(np)\Gamma(k+q) k^k (q-1)^{q-1}}{\Gamma(k)\Gamma(q)(k+q-1)^{k+q-1}}.$$

§ 3. MLE's and LRC of $S_{n\times p}(1)$

In this section we come to the same problems as in the last section for $S_{n\times p}(1)$. The following theorems are derived from lemmas in Section 2. Note that in proving Theorems 2.1 and 2.2 we use Lemmas 2.1 and 2.2 with $p=1$ and do not need Lemma 2.3, while we need Lemma 2.3 for (3.2) below.

Theorem 3.1. Let Θ and $\{\Theta_j\}$ be defined as in section 1. Suppose that $\boldsymbol{W} = \boldsymbol{M}$

$+ZA\in S_{n\times p}(1, M, A, g)$ with $M=1_n\mu'$, $n>1$. Suppose also that $u(x)=f(x)\prod_{j=1}^{p}x_j^n$ has a maximum at some $c\in R_+^p-\{0\}$, where $f(\|z_1\|, \cdots, \|z_p\|)$ is the p.d.f. of Z and that for any fixed $y\in R_+^p-\{0\}$, $v(x)=u(yx)$, as a function of $x\in R_+^1$, has a maximum at $y_0\in(0, \infty)$. Let the likelihood function of W be

$$L(\mu, A)=f((\|w_1\|-n\mu_1)/a_1, \cdots, (\|w_p\|-n\mu_p)/a_p)(a_1\cdots a_p)^{-n},$$
$$(W-M)A^{-1}\in R_+^{n\times p}.$$

Then

(1) for $(\mu, A)\in\Theta$, Θ_2 and Θ_4, the MLE's of μ and A are $\hat{\mu}$, given in (2.8), and $\hat{A}=\mathrm{diag}(\hat{a}_1, \cdots, \hat{a}_p)$, where

$$\hat{a}_j=(\|w_j\|-n\hat{\mu}_j)/c_j \tag{3.1}$$

and the maximum of the likelihood function is

$$L(\hat{\mu}, \hat{A})=f(c)\prod_{j=1}^{p}c_j^n(\|w_j\|-n\hat{\mu}_j)^{-n}=u(c)\prod_{j=1}^{p}(\|w_j\|-n\hat{\mu}_j)^{-n}$$

with corresponding $\hat{\mu}_j$ and $\hat{a}_j$;

(2) for $(\mu, A)\in\Theta_1$, Θ_3 and Θ_5, the MLE's of μ and A are $\hat{\mu}$, given in (2.11), and $A=\mathrm{diag}(\tilde{a}, \cdots, \tilde{a})$, where

$$\tilde{a}=y_0^{-1} \tag{3.2}$$

when $y_j=\|w_j\|-n\hat{\mu}_j$ and the maximum of the likelihood function is

$$L(\hat{\mu}, \tilde{A})=f((\|w_1\|-n\hat{\mu}_1)y_0, \cdots, (\|w_p\|-n\hat{\mu}_p)y_0)y_0^{np}$$
$$=v(y_0)\prod_{j=1}^{p}(\|w_j\|-n\hat{\mu}_j)^{-n}$$

with corresponding $\hat{\mu}_j$ and $\tilde{a}$.

Theorem 3.2. Suppose that the conditions of Theorem 3.1 hold. Let X be as in Theorem 2.3. Then the LRC for H_j versus K_j, $j=1, \cdots, 6$, are (or are equivalent to) the following:

(1) $$\Lambda_1(W)=v(y_0)/u(c) \tag{3.3}$$

where $y_j=\|w_j\|-n\min w_{ij}$ (see Lemma 2.3 for y_0's definition);

(2) $$T_2(W)=\prod_{j=1}^{p}((\|w_j\|-n\min_i w_{ij})/(\|w_j\|-n\min_{ij} w_{ij})); \tag{3.4}$$

(3) $$\Lambda_3(W)=(v(y_0)/u(c))\prod_{j=1}^{p}((\|w_j\|-n\min_i w_{ij})/y_j)^n, \tag{3.5}$$

where $y_j=\|w_j\|-n\min_{ij} w_{ij}$;

(4) $$T_4(W)=\prod_{j=1}^{p}((\|w_j\|-n\min_i w_{ij})/\|w_j\|); \tag{3.6}$$

(5) $$\Lambda_5(W)=(v(y_0)/u(c))\prod_{j=1}^{p}((\|w_j\|-n\min_i w_{ij})/y_j)^n, \tag{3.7}$$

where $y_j=\|w_j\|$;

(6) $$\Lambda_6(W)=(u(xx_0)/u(yy_0))\prod_{j=1}^{p}(y_j/x_j)^n, \tag{3.8}$$

where $x_j=\|w_j\|-n\min_{ij} w_{ij}$, $y_j=\|w_j\|-n\min_i w_{ij}$.

Lemma 3.1. A statistic $t(\boldsymbol{Z})$ is invariant in $F^{+}_{n\times p}(1)$ iff $t(\boldsymbol{ZA}) \stackrel{d}{=} t(\boldsymbol{Z})$ for any constant matrix $\boldsymbol{A}=\mathrm{diag}(a_1, \cdots, a_p)>\boldsymbol{0}$ and $\boldsymbol{Z}\in F^{+}_{n\times p}(1)$ (see [2]).

Theorem 3.3. Suppose that the conditions of Theorem 3.2 hold. Then under H_j, $T_j(\boldsymbol{W}) \stackrel{d}{=} T_j(\boldsymbol{X})$, $j=2, 4$.

We see that the expressions and null distributions of T_2 and T_4 are independent of g, while those of Λ_i, $i=1, 3, 5, 6$, are depending on g in general. Moreover, we can obtain the null distribution of T_4 by applying Theorem 2.5. Here are some illustrative examples.

Example 3.1. Let $\boldsymbol{X}$ be as in Example 2.1. $\boldsymbol{X}$ can be considered in $F_{n\times p}(1)$. Then

$$f(\boldsymbol{x})=\exp(-\|\boldsymbol{x}\|),\ c_j=n,\ u(\boldsymbol{c})=e^{-np}n^{np}.$$

The same $\hat{a}_j$ is obtained by using Theorem 3.1. For any fixed $\boldsymbol{y}$ (see Theorem 3.1),

$$y_0=np/\|\boldsymbol{y}\|,\ v(y_0)=e^{-np}(np)^{np}\prod_{j=1}^{p} y_j^n\|\boldsymbol{y}\|^{-np}$$

and

$$\tilde{a}=\|\boldsymbol{y}\|/(np)=\sum_j(\|\boldsymbol{w}_j\|-n\hat{\mu}_j)/(np).$$

Let $\boldsymbol{W}\in S_{n\times p}(1, \boldsymbol{M}, \boldsymbol{A}, g)$ in the following examples.

Example 3.2. Let g be the p.d.f. of $D_{p+1}(k_1, \cdots, k_p; k_{p+1})$ with $k_q>0$, $q=1, \cdots, p$, $k_{p+1}>1$. Then

$$c_j=k_j\Big/\Big(\sum_{q=1}^{p+1}k_q-1\Big),\ y_0=\Big(\sum_{q=1}^{p}k_q\Big/\sum_{q=1}^{p+1}\Big(k_q-1\Big)\Big)\|\boldsymbol{y}\|^{-1},$$

$$\hat{a}_j=\Big(\Big(\sum_{q=1}^{p+1}k_q-1\Big)\Big/k_j\Big)(\|\boldsymbol{w}_j\|-n\hat{\mu}_j),$$

$$\tilde{a}=\Big(\Big(\sum_{q=1}^{p+1}k_q-1\Big)\Big/\sum_{q=1}^{p}k_q\Big)\sum_{j=1}^{p}(\|\boldsymbol{w}_j\|-n\hat{\mu}_j),$$

$$\Lambda_1=K\prod_{j=1}^{p}\left(\frac{\|\boldsymbol{w}_j\|-n\min_i w_{ij}}{\sum_{q=1}^{p}(\|\boldsymbol{w}_q\|-n\min_i w_{iq})}\right)^{k_j},$$

$$\Lambda_3=K\prod_{j=1}^{p}\frac{(\|\boldsymbol{w}_j\|-n\min_i w_{ij})^n}{(\|\boldsymbol{w}_j\|-n\min_{ij} w_{ij})^{n-k_j}}\Big(\sum_{q=1}^{p}(\|\boldsymbol{w}_q\|-n\min_{ij} w_{ij})\Big)^{-\sum_{q=1}^{p}k_q}$$

and

$$\Lambda_5=K\prod_{j=1}^{p}\frac{(\|\boldsymbol{w}_j\|-n\min_i w_{ij})^n}{\|\boldsymbol{w}_j\|^{n-k_j}}\Big(\sum_{q=1}^{p}\|\boldsymbol{w}_q\|\Big)^{-\sum_{q=1}^{p}k_q},$$

where $K=\Big(\sum_{q=1}^{p}k_q\Big)^{\sum_{q=1}^{p}k_q}\Big/\prod_{q=1}^{p}k_q$.

Example 3.3. Let $p=2$ and g be the joint p.d.f. of the two independent $F(1, 1)$ variates. Then

$$\boldsymbol{c}=(1, 1),\ u(\boldsymbol{c})=\Big(\frac{\Gamma(n)}{B(1/2, 1/2)}\Big)^2\Big/4,\ y_0=(y_1y_2)^{-1/2},$$

$$v(y_0)=\left(\frac{\Gamma(n)}{B(1/2,\ 1/2)}\right)^2\frac{(y_1y_2)^{1/2}}{y_1+y_2+2(y_1y_2)^{1/2}},$$

$$\hat{a}_j=\|\boldsymbol{w}_j\|-n\hat{\mu}_j,\ \tilde{a}=\prod_{j=1}^{2}(\|\boldsymbol{w}_j\|-n\hat{\mu}_j)^{1/2}.$$

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