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REMARK ON A PAPER OF C. H. DOWKER

by

G. FEJES TÓTH and L. FEJES TÓTH (Budapest)

To Professor P. ERDŐS *on his sixtieth birthday*

Let a_n be the maximum of the areas of all n-gons inscribed in a convex domain d, and A_n the minimum of the areas of all n-gons circumscribed about d. DOWKER [1] proved the important facts — previously conjectured by KERSHNER — that the sequence $a_3, a_4, \ldots$ is concave and the sequence $A_3, A_4, \ldots$ is convex.

The method used by DOWKER in the proof of these theorems enabled him to prove two further theorems.

THEOREM A. *Among the $2n$-gons ($n = 2, 3, \ldots$) of maximal area inscribed in a centro-symmetric convex domain there is one which has central symmetry.*

THEOREM B. *Among the $2n$-gons ($n = 2, 3, \ldots$) of minimal area circumscribed about a centro-symmetric convex domain there is one which has central symmetry.*

The above theorems, which turned out to be fundamental in certain packing and covering problems [2], [3], [4], started further investigations. It has been observed that these theorems continue to hold in non-Euclidean geometries as well as in the case when we replace the word "area" by "perimeter" [5, 6]. A modified proof can be found in [7]. Further Dowker-type theorems are contained in [8].

The two last theorems of Dowker are far from being trivial. This is clearly shown by the fact, to be proved later, that analogous statements do not hold for bilateral symmetry instead of central symmetry. Therefore it is interesting to observe that the theorems under consideration can be generalized as follows.

THEOREM. *Among the polygons with kn sides ($k = 2, 3, \ldots; n = 1, 2, \ldots; kn > 2$) of maximal (minimal) area or perimeter inscribed in (circumscribed about) a convex domain of k-fold rotatory symmetry there is one which has k-fold rotatory symmetry.*

This theorem contains four statements. Because of the similarity of the proofs we restrict ourselves to inscribed polygons of maximal perimeter.

Let $p = AB \ldots A$ and $q = LM \ldots L$ be two m-gons both inscribed in a convex domain d. In the following investigation the case of coinciding vertices can be considered as the limiting case of distinct vertices. Therefore, for the sake of simplicity, we assume that no two vertices of p and q coincide.

Assigning in the plane a positive direction of rotation, with each side XY of a polygon a unique arc $\widehat{XY}$ of the boundary of d can be associated. Suppose that the arc $\widehat{AB}$ contains the arc $\widehat{LM}$. Then we say that the side LM of q is enclosed by the side AB of p. Replacing the sides AB and LM by AM and LB, the polygons p and q will unite in one double-polygon $s = AM \ldots LB \ldots A$. The name double-polygon refers to the fact that the arcs belonging to the sides of s cover the boundary of d twice. Since $AM + LB \geq AB + LM$, the total length Λ of the sides has not decreased by the above operation (Fig. 1).

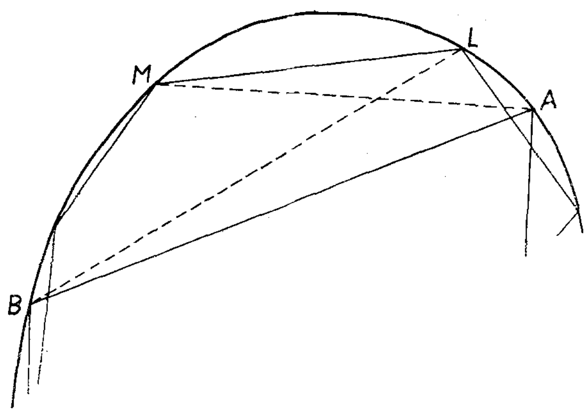

Fig. 1

Changing the notation of the vertices of s, we write $s = AB \ldots LM \ldots A$, and assume that among the sides of s there is one, say LM, which is enclosed by another side of s, say by AB. Replacing again AB and LM by AM and LB, the double-polygon s will decompose into two convex polygons $AM \ldots A$ and $LB \ldots L$ with a value of Λ not less than that of s.

Continuing this process the number of sides enclosed by another decreases in each step. Therefore we finally obtain either a pair of convex polygons or a double-polygon in which no side is enclosed by another. Consequently, the resulting figure arises by uniting the vertices of p and q, and joining cyclically every second vertex. Since the total number of vertices is an even number, namely $2m$, this figure consists of two m-gons. Since in no step of the above process has Λ decreased, the total perimeter of these m-gons is not less than that of p and q.

Now we suppose that d has k-fold rotatory symmetry. Let p_0 be a convex polygon with $m = kn$ sides inscribed in d. We shall construct an m-gon with k-fold rotatory symmetry inscribed in d whose perimeter is not less than that of p_0.

The rotations through $2\pi/k, \ldots, (k-1)2\pi/k$ carrying d into itself transform p_0 into $p_1, \ldots, p_{k-1}$. Let t be the total number of those sides of $p_0, p_1, \ldots, p_{k-1}$ which are enclosed by one or more sides of these polygons, counting each side with the multiplicity of the sides which enclose it.

Suppose that $t > 0$. Then there are two m-gons such that one of them has a side enclosed by a side of the other. We replace these two polygons by the two m-gons obtained as a result of the above process. Obviously, in each step of this process t decreases. If for the two new and the $k-2$ original m-gons we still have $t > 0$, then we repeat the above operation. Continuing this process, we finally obtain k convex m-gons with $t = 0$. It is easily seen that these m-gons arise by uniting the km vertices of $p_0, p_1, \ldots, p_{k-1}$ and joining cyclically every kth vertex. Since the set of the km vertices has k-fold rotatory symmetry, the same holds for the k new m-gons. Since, furthermore, the total perimeter of the k new m-gons is not less than that of $p_0, p_1, \ldots, p_{k-1}$, there is among the new m-gons one whose perimeter is not less than that of p_0.

This completes the proof.

We still show that the theorems of Dowker concerning centro-symmetric domains stop to hold for bilateral symmetry.

Among the quadrangles of maximal area inscribed in an ellipse e there are exactly two quadrangles with bilateral symmetry: a rhombus and a rectangle. Let P be a point on the boundary of e other than the vertices of these quadrangles. Let p be the quadrangle of maximal area inscribed in e having P as a vertex. Let Q be a point near to P outside e, P', p' and Q' the images of P, p and Q reflected in one of the axes of e, and h the convex hull of e, Q and Q' (Fig. 2). Let s be the set of the quadrangles of maximal area inscribed in h. For $Q \to P$ s converges to the set consisting of p and p'. Therefore, if Q is

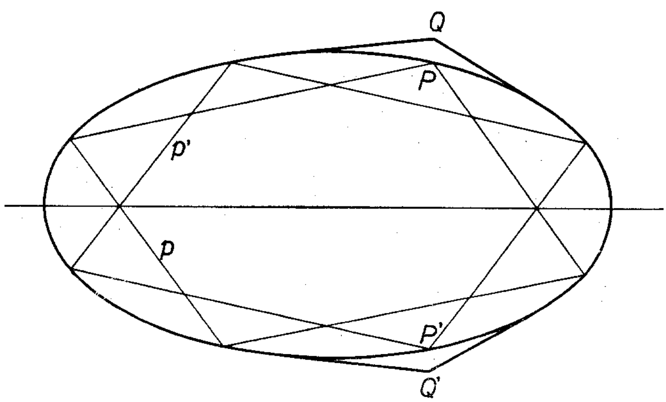

Fig. 2

sufficiently near to P, no quadrangle of maximal area inscribed in h has bilateral symmetry.

A similar counterexample can be given for circumscribed polygons.

We still phrase two further theorems. Let U and V be two convex domains. The area (perimeter) deviation of U and V is defined to be the difference between the areas (perimeters) of the union and the intersection of U and V.

Among the polygons with kn sides ($k = 2, 3, \ldots; n = 1, 2, \ldots; kn > 2$) having minimal area (perimeter) deviation from a convex domain of k-fold rotatory symmetry there is one which has k-fold rotatory symmetry.

The proof of the theorem concerning the area deviation rests on rather intricate considerations used by EGGLESTON [8]. The theorem concerning the perimeter deviation is an immediate consequence of our first theorem and the fact observed by EGGLESTON [8] that the n-gon having minimal perimeter deviation from a convex domain is inscribed in the domain.

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EÖTVÖS LORÁND TUDOMÁNYEGYETEM
VALÓSZÍNŰSÉGSZÁMÍTÁS TANSZÉK
H-1088 BUDAPEST
MÚZEUM KRT. 6–8.

MTA MATEMATIKAI KUTATÓ INTÉZETE
H-1053 BUDAPEST
REÁLTANODA U. 13–15.
HUNGARY