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GENERALIZED RAMANUJAN'S SUM

by

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Introduction

Let V_t be the set of all ordered t-tuples of integers $X = \{x_i\}_{i=1}^{t}$, called integral t-vectors or simply t-vectors. Two t-vectors $X = \{x_i\}_{i=1}^{t}$ and $Y = \{y_i\}_{i=1}^{t}$ are said to be congruent modulo the positive integer r if $x_i = y_i \pmod r$ for $i = 1, 2, \ldots, t$. Any set of r^t t-vectors no two of which are congruent modulo r is called a complete residue system of t-vectors mor r. A t-vector $X = \{x_i\}_{i=1}^{t}$ is called k-prime to r if $((x_1, x_2, \ldots, x_t), r)_k = 1$; here by $(a, b, \ldots, e)_k$ we mean the largest kth power common divisor of $a, b, \ldots, e$ and $(a, b, \ldots, e)_1 = (a, b, \ldots, e)$ with the convention $(0, 0, \ldots, 0)_k = 0$. The set of all t-vectors in a complete residue system of t-vectors mod r which are k-prime to r is called a k-reduced residue system of t-vectors mod r.

With this terminology, RAMANUJAN's sum $C(n, r)$ is (see [12])

$$C(n, r) = \sum_{x} e(nx, r), \quad e(a, b) = \exp 2\pi ai/b; \tag{1.1}$$

and E. COHEN's generalized Ramanujan's sum (see [3]) is

$$C^{(k)}(n, r) = \sum_{x} e(nx, r^k) \tag{1.2}$$

where the sum in (1.1) is extended over a 1-reduced residue system of 1-vectors, i.e., a reduced residue system mod r, while the sum in (1.2) is extended over a k-reduced residue system of 1-vectors mod r^k. In [7], he obtained another generalization

$$C_k(n, r) = \sum_{X} e(n(x_1 + x_2 + \ldots + x_k), r), \tag{1.3}$$

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the sum now being extenedd over a 1-reduced residue system of k-vectors mod r. In [14] M. SUGUNAMMA further generalized (1.2) and (1.3), by combining them, as

$$C_k^{(s)}(n, r) = \sum_X e(n(x_1 + x_2 + \dots + x_k), r^s) \tag{1.4}$$

where the sum is extended over a s-reduced residue system of k-vectors mod r^s.

More recently, C. S. VENKATARAMAN and R. SIVARAMAKRISHNAN [15] obtained an entirely different extension of (1.1) based on a new generalization $\mu_u(r)$ of the Möbius function $\mu(r)$, defined as

$$\mu_u(r) = \begin{cases} 0 & \text{if } r \text{ is not square free} \\ e(w(r), 2u) & \text{if } r \text{ is square free,} \end{cases} \tag{1.5}$$

where $w(r)$ is the number of distict prime factors of r. Clearly, $\mu_1(r) = \mu(r)$ and their extended Ramanujan's sum is (with a slight change of symbolism)

$$C^{\mu_u}(n, r) = \sum_{d \mid (n,r)} \mu_u\left(\frac{r}{d}\right) d\,. \tag{1.6}$$

The purpose of this paper is to define and study a much more general Ramanujan's sum which we denote by $C_{f,t}^{k,\eta}(n, r)$; here $f = f(x)$ is a polynomial of positive degree with integer coefficients, $\eta = \eta(r)$ is a multiplicative function of r, and k and t are positive integres. This sum includes as special cases when $f(x) = x$ and special values of k and t and special choice of $\eta(r)$ all the generalizations of Ramanujan's sum mentioned before. As are the special cases, $C_{f,t}^{k,\eta}(n, r)$ is multiplicative in both variables n and r and also as a function of r. It is a k-even function of n (mod r) (see § 2) and the generalized Hölder identity holds (Theorem 3.1). Also it can be expressed as a trigonometric sum (Theorem 4.1). Specifically we extend all the results in [15] for $C^{\mu_u}(n, r)$ and the identities (3) through (13) and (16) of [9] involving $C^{(k)}(n, r)$ which identities are due to C. S. Venkataraman for $k = 1$, to $C_{f,t}^{k,\eta}(n, r)$. For the generalization of Ramanujan's sum to ordered structures we refer to the work of SCHEID [13] and MCDONALD [11].

§ 1. Preliminaries

We recall that an arithmetical function $a(r)$ is called multiplicative if $a(rs) = a(r)a(s)$ whenever $(r, s) = 1$, and is called completely multiplicative if $a(rs) = a(r)\,a(s)$ holds for all r and s. Let $N_f(r)$ denote the number of incongruent solutions (mod r) of

$$f(x) \equiv 0 \pmod{r}. \tag{1.7}$$

It is well known that $N_f(r)$ is a multiplicative function of r. We denote by $I(r)$ the function $I(r) = 1$ for all r. Given the integer coefficient polynomial $f = f(x)$ of positive degree, the multiplicative arithmetical function $\eta(r)$, and the positive integers k and t, let the functions $\mu_{f,t}^{k,\eta}(r)$ and $\varphi_{f,t}^{k,\eta}(r)$ be defined by

$$(1.8) \qquad \mu_{f,t}^{k,\eta}(r) = \mu(r)\,\eta(r)\,N_f^t(r^k)\,,$$

$$(1.9) \qquad \varphi_{f,t}^{k,\eta}(r) = r^{kt} \prod_{p \mid r} \left\{1 - \frac{N_f^t(p^k)\,\eta(p)}{p^{kt}}\right\} = \sum_{d \mid r} \mu_{f,t}^{k,\eta}(d) \left(\frac{r}{d}\right)^{kt},$$

where $N_f^t(r) = (N_f(r))^t$. In fact, $\varphi_{f,t}^{k,I}(r) = \Phi_{f,t}^{(k)}(r^k)$, where $\Phi_{f,t}^{(k)}(r)$ is the generalized totient function defined in [2] as the number of vectors in a complete residue system of t-vectors mod r which are k-prime to r with respect to the polynomial f, a vector $X = \{x_i\}_{i=1}^{t}$ being called k-prime to r with respect to the polynomial f if $((f(x_1), f(x_2), \ldots, f(x_t)), r)_k = 1$. Clearly $\mu_{f,t}^{k,\eta}(r)$ and $\varphi_{f,t}^{k,\eta}(r)$ are multiplicative functions of r. Let $M_{f,t}^{(k)}(r)$ and $M_{f,t}^{k,\eta}(r)$ be defined by

$$(1.10) \qquad M_{f,t}^{(k)}(r) \begin{cases} = 1, & \text{for} \quad r = 1, \\ = \prod_{p^\alpha \| r} N_f^{\alpha t}(p^k), & \text{for} \quad r > 1; \end{cases}$$

and

$$(1.11) \qquad M_{f,t}^{k,\eta}(r) = M_{f,t}^{(k)}(r)\,\eta(r)\,,$$

where in (1.10) the symbol $p^\alpha \| r$ means that p^α is the highest power of p dividing r. It is clear that $M_{f,t}^{(k)}(r)$ is a multiplicative function of r and so is $M_{f,t}^{k,\eta}(r)$ since $\eta(r)$ is. We need also the functions

$$(1.12) \qquad \eta_u(r) = \mu(r)\,\mu_u(r)$$

$$(1.13) \qquad a_{f,t}^{k,\eta}(n, r) = \sum_{d^k \mid (n, r^k)_k} \mu_{f,t}^{k,\eta}(d) \left(\frac{r}{d}\right)^{k(t-1)}$$

$$(1.14) \qquad A_{f,t}^{k,\eta}(r) = a_{f,t}^{k,\eta}(0, r) = \sum_{d \mid r} \mu_{f,t}^{k,\eta}(d) \left(\frac{r}{d}\right)^{k(t-1)}$$

It is well known that

$$(1.15) \qquad \begin{cases} \text{(i)} & C(n, r) = \sum_{d \mid (n,r)} \mu\left(\frac{r}{d}\right) d, \\ \text{(ii)} & C^{(k)}(n, r) = \sum_{d^k \mid (n, r^k)_k} \mu\left(\frac{r}{d}\right) d^k, \\ \text{(iii)} & C_k(n, r) = \sum_{d \mid (n,r)} \mu\left(\frac{r}{d}\right) d^k, \\ \text{(iv)} & C_k^{(s)}(n, r) = \sum_{d^s \mid (n, r^s)_s} \mu\left(\frac{r}{d}\right) d^{ks}. \end{cases}$$

We shall also need $\varphi_k(r)$ which is the number of integers in a k-reduced residue system mod r^k. It is well known that

$$\varphi_k(r) = r^k \sum_{d|r} \frac{\mu(d)}{d^k} = r^k \prod_{p|r} \left\{1 - \frac{1}{p^k}\right\}. \tag{1.16}$$

As usual $\sigma_k(r)$ and $\tau(r)$ denote respectively the sum of the kth powers of the divisors of r and the number of divisors of r. In the following the results referred to before the statement of a theorem are the special cases of the earlier extensions mentioned before of $C(n, r)$, of part of or the whole of that theorem.

§ 2.

We define the generalized Ramanujan's sum by

$$C_{f,t}^{k,\eta}(n, r) = \sum_{d^k | (n, r^k)_k} d^{kt} \mu_{f,t}^{k,\eta} \left(\frac{r}{d}\right). \tag{2.1}$$

Clearly, by (1.15), (1.6), (1.12), (1.8) and (1.9)

$$\begin{aligned} &\text{(i)} \quad C_{x,1}^{1,I}(n, r) = C(n, r), \\ &\text{(ii)} \quad C_{x,1}^{k,I}(n, r) = C^{(k)}(n, r), \\ &\text{(iii)} \quad C_{x,k}^{1,I}(n, r) = C_k(n, r), \\ &\text{(iv)} \quad C_{x,k}^{s,I}(n, r) = C_k^{(s)}(n, r), \end{aligned} \tag{2.2}$$

and

$$\text{(v)} \quad C_{x,1}^{1,\eta_u}(n, r) = C^{\mu_u}(n, r),$$

and as in the special cases $C_{f,t}^{k,\eta}(n, r)$ is a k-even function of n mod r [10]; i.e.,

$$C_{f,t}^{k,\eta}(n, r) = C_{f,t}^{k,\eta}((n, r^k)_k, r), \tag{2.3}$$

and

$$C_{f,t}^{k,\eta}(n, r) = \varphi_{f,t}^{k,\eta}(r), \quad \text{if } n \equiv 0 \pmod{r^k}, \tag{2.4}$$

$$C_{f,t}^{k,\eta}(1, r) = \mu_{f,t}^{k,\eta}(r). \tag{2.5}$$

We recall that an arithmetical function $S(n, r)$ of the variables n and r is called multiplicative in both n and r[1] if $(n_1, n_2) = (r_1, r_2) = (n_1, r_2) = (n_2, r_1) = 1$ implies that $S(n_1 n_2, r_1 r_2) = S(n_1, r_1) S(n_2, r_2)$, and that such a function is completely determined by the values $S(p^\alpha, p^\beta)$, p a prime and $\alpha \geq 0, \beta \geq 0$.

LEMMA 2.1. *If the arithmetical functions $g(r)$ and $h(r)$ are multiplicative, then*

$$S^{(k)}(n, r) = \sum_{d^k \mid (n, r^k)_k} g(d)\, h\left(\frac{r}{d}\right)$$

is

(i) *multiplicative in both n and r,*

(ii) *multiplicative as a function of r.*

PROOF. If $(n_1, n_2) = (r_1, r_2) = (n_1, r_2) = (n_2, r_1) = 1$, it is easily seen that

$$(n_1 n_2, r_1^k r_2^k)_k = (n_1, r_1^k)_k\, (n_2, r_2^k)_k, \quad ((n_1, r_1^k)_k, (n_2, r_2^k)_k) = 1;$$

and so,

$$S^{(k)}(n_1 n_2, r_1 r_2) = \sum_{d^k \mid (n_1 n_2, r_1^k r_2^k)_k} g(d)\, h\left(\frac{r_1 r_2}{d}\right) =$$

$$= \sum_{d_1^k \mid (n_1, r_1^k)_k;\ d_2^k \mid (n_2, r_2^k)_k} g(d_1)\, g(d_2)\, h\left(\frac{r_1}{d_1}\right) h\left(\frac{r_2}{d_2}\right) =$$

$$= \sum_{d_1^k \mid (n_1, r_1^k)_k} g(d_1)\, h\left(\frac{r_1}{d_1}\right) \sum_{d_2^k \mid (n_2, r_2^k)_k} g(d_2)\, h\left(\frac{r_2}{d_2}\right) = S^{(k)}(n_1, r_1)\, S^{(k)}(n_2, r_2),$$

giving (i).

If $(r_1, r_2) = 1$, $(n, r_1^k r_2^k)_k = (n, r_1^k)_k\, (n, r_2^k)_k$, $((n, r_1^k)_k, (n, r_2^k)_k) = 1$; using the fact that $S^{(k)}(n, r)$ is k-even mod r and (i) of this lemma,

$$S^{(k)}(n, r_1 r_2) = S^{(k)}((n_1, r_1^k)_k\, (n, r_2^k)_k, r_1 r_2) =$$

$$= S^{(k)}((n, r_1^k)_k, r_1)\, S^{(k)}((n, r_2^k)_k, r_2) = S^{(k)}(n, r_1)\, S^{(k)}(n, r_2).$$

Lemma 2.1 and (2.1) give

THEOREM 2.1 (Theorem 1, [3]; Theorem 3, [14]; (3.2), (3.4) of [15]).

(i) $C_{f,t}^{k,\eta}(n, r)$ *is multiplicative in both n and r.*

(ii) $C_{f,t}^{k,\eta}(n, r)$ *is multiplicative as a function of r.*

THEOREM 2.2 (Theorem 3, [3]). *For the prime p*

$$C_{f,t}^{k,\eta}(p^\alpha, p^\beta) \begin{cases} = 1, \text{ if } \beta = 0, \\ = p^{\beta k t} - p^{(\beta-1)kt}\, \eta(p)\, N_f^t(p^k), & \text{if } \alpha \geq \beta k \geq k; \\ = -p^{(\beta-1)kt}\, \eta(p)\, N_f^t(p^k), & \text{if } 0 \leq (\beta - 1)k \leq \alpha < \beta k; \\ = 0, \text{ if } 0 \leq \alpha < (\beta - 1)k. \end{cases}$$

PROOF. Let $(p^\alpha, p^{\beta k})_k = p^{\gamma k}$ so that $0 \leq \gamma \leq \beta$. By (2.1) and (1.8),

$$C_{f,t}^{k,\eta}(p^\alpha, p^\beta) = \sum_{d \mid p^\gamma} d^{kt} \mu\left(\frac{p^\beta}{d}\right) \eta\left(\frac{p^\beta}{d}\right) N_f^t\left(\frac{p^{\beta k}}{d^k}\right). \tag{2.6}$$

If $\beta = 0$, $\gamma = 0$ and the r.h.s. of (2.6) is 1 while if $\beta \geq 1$, $\gamma = \beta$, $\beta - 1$, or $\leq \beta - 2$ according as $\alpha \geq \beta k$, $(\beta - 1)k \leq \alpha < \beta k$, or $\alpha < (\beta - 1)k$ and the r.h.s. of (2.6) is

$$p^{(\beta-1)kt} \mu(p)\, \eta(p)\, N_f^t(p^k) + p^{\beta kt} \mu(1)\, \eta(1)\, N_f^t(1),$$
$$p^{(\beta-1)kt} \mu(p)\, \eta(p)\, N_f^t(p^k),$$

or 0 according as $\gamma = \beta$, $\beta - 1$, or $\leq \beta - 2$ and Theorem 2.2 is clear.

THEOREM 2.3 (Theorem 3, [14]; (3.5) of [15]).

(i) *If* $(n_1, n_2) = 1$,

$$C_{f,t}^{k,\eta}(n_1, r)\, C_{f,t}^{k,\eta}(n_2, r) = C_{f,t}^{k,\eta}(n_1 n_2, r)\, C_{f,t}^{k,\eta}(1, r),$$

(ii) *If* $(r_1, r_2) = 1$,

$$C_{t,t}^{k,\eta}(n_1, r_1)\, C_{f,t}^{k,\eta}(n_2, r_2) = C_{f,t}^{k,\eta}(n_1 r_2^k + n_2 r_1^k, r_1 r_2).$$

PROOF. Let $r = \pi p^\beta$ be the canonical decomposition of r and let S_1 and S_2 denote respectively the set of all primes common to n_1 and r and n_2 and r and R the remaining prime factors of r; i.e. prime factors of r which are neither in S_1 nor in S_2. Since $(n_1, n_2) = 1$, S_1, S_2 and R are pairwise disjoint sets with union consisting of all prime factors of r. By (i) of Theorem 2.1,

$$C_{f,t}^{k,\eta}(n_1, r) = \{\prod_{\substack{p \in S_1 \\ p^\alpha \| n_1}} C_{f,t}^{k,\eta}(p^\alpha, p^\beta)\}\{\prod_{p \in S_2} C_{f,t}^{k,\eta}(1, p^\beta)\}\{\prod_{p \in R} C_{f,t}^{k,\eta}(1, p^\beta)\}$$

and similarly,

$$C_{f,t}^{k,\eta}(n_2, r) = \{\prod_{\substack{p \in S_2 \\ p^\alpha \| n_2}} C_{f,t}^{k,\eta}(p^\alpha, p^\beta)\}\{\prod_{p \in S_1} C_{f,t}^{k,\eta}(1, p^\beta)\}\{\prod_{p \in R} C_{f,t}^{k,\eta}(1, p^\beta)\}$$

and so

$$C_{f,t}^{k,\eta}(n_1, r)\, C_{f,t}^{k,\eta}(n_2, r) = \{\prod_{\substack{p \in S_1 \\ p^\alpha \| n_1}} C_{f,t}^{k,\eta}(p^\alpha, p^\beta) \prod_{\substack{p \in S_2 \\ p^\alpha \| n_2}} C_{f,t}^{k,\eta}(p^\alpha, p^\beta)\} \times$$
$$\times \{\prod_{p \in R} C_{f,t}^{k,\eta}(1, p^\beta)\}\{\prod_{p \in S_1 \cup S_2 \cup R} C_{f,t}^{k,\eta}(1\ p^\beta)\} = C_{f,t}^{k,\eta}(n_1 n_2, r)\, C_{f,t}^{k,\eta}(1, r),$$

giving (i) of Theorem 2.3.

If $(r_1, r_2) = 1$, then $(n_1 r_2^k + n_2 r_1^k, r_1^k)_k = (n_1, r_1^k)_k$ and $(n_1 r_2^k + n_2 r_1^k, r_2^k)_k = (n_2, r_2^k)_k$; hence by (ii) of Theorem 2.1 and (2.3)

$$C_{f,t}^{k,\eta}(n_1 r_2^k + n_2 r_1^k, r_1 r_2) = C_{f,t}^{k,\eta}(n_1, r_1)\, C_{f,t}^{k,\eta}(n_2, r_2),$$

giving (ii) of Theorem 2.3.

THEOREM 2.4 ((4) and (6) of [9]; Theorems 5.3 and 5.4 of [15]).

$$(2.7) \qquad \sum_{d\mid(n,r)} C_{f,t}^{k,\eta}\left(\left(\frac{n}{d}\right)^k, \frac{r}{d}\right) = \sum_{d\mid(n,r)} \mu_{f,t}^{k,\eta}\left(\frac{r}{d}\right) \sigma_{kt}(d);$$

$$(2.8) \qquad \sum_{d\mid n} C_{f,t}^{k,\eta}(d^k, r) = \sum_{d\mid(n,r)} \mu_{f,t}^{k,\eta}\left(\frac{r}{d}\right) \tau\left(\frac{n}{d}\right) d^{kt}.$$

PROOF. It is easy to see that both sides of (2.7) and (2.8) are multiplicative in both n and r and so we need only verify them when $n = p^\alpha$, $r = p^\beta$, p a prime, $\alpha \geq 0$, $\beta \geq 0$. We need to consider the cases $\alpha \geq \beta$, $\alpha = \beta - 1$, and $\alpha < \beta - 1$. If $\alpha \geq \beta$, the l.h.s. of (2.7) is, by Theorem 2.2,

$$\sum_{j=0}^{\beta} C_{f,t}^{k,\eta}(p^{(\alpha-j)k}, p^{\beta-j}) = \left\{\sum_{j=0}^{\beta-1} p^{(\beta-j)kt} - p^{(\beta-j-1)kt}\eta(p)\, N_f^t(p^k)\right\} + 1 =$$

$$= \sum_{j=0}^{\beta} p^{jkt} - \eta(p)\, N_f^t(p^k) \sum_{j=0}^{\beta-1} p^{jkt} = \sigma_{kt}(p^\beta) - \eta(p)\, N_f^t(p^k)\, \sigma_{kt}(p^{\beta-1}),$$

and the r.h.s. of (2.7) is

$$\sum_{j=0}^{\beta} \mu_{f,t}^{k,\eta}(p^{\beta-j})\, \sigma_{kt}(p^j) = \sigma_{kt}(p^\beta) - \sigma_{kt}(p^{\beta-1})\, \eta(p)\, N_f^t(p^k)$$

which is the same as the l.h.s. of (2.7); the verification when $\alpha = \beta - 1$ and $\alpha < \beta - 1$ is done similarly and (2.7) follows.

Similarly, when $n = p^\alpha$, $r = p^\beta$, the l.h.s. of (2.8) is $\sum_{j=0}^{\alpha} C_{f,t}^{k,\eta}(p^{jk}, p^\beta)$ and this, by Theorem 2.2, is easily seen to be

$$(\alpha - \beta + 1)\, p^{\beta kt} - (\alpha - \beta + 2)\, \eta(p)\, N_f^t(p^k)\, p^{(\beta-1)kt}, \quad -p^{(\beta-1)kt}\eta(p)\, N_f^t(p^k)$$

or 0 according as $\alpha \geq \beta$, $\alpha + 1 = \beta$, or $\alpha + 1 < \beta$, and the r.h.s. of (2.8) is

$$\sum_{j=0}^{\min\{\alpha,\beta\}} \mu_{f,t}^{k,\eta}(p^{\beta-j})\, \tau(p^{\beta-j})\, p^{jkt}$$

which is

$$\tau(p^{\alpha-\beta})\, p^{\beta kt} - \eta(p)\, N_f^t(p^k)\, \tau(p^{\alpha-\beta+1})\, p^{(\beta-1)kt}, \quad -p^{(\beta-1)kt}\eta(p)\, N_f^t(p^k)$$

or 0 according as $\alpha \geq \beta$, $\alpha + 1 = \beta$ or $\alpha + 1 < \beta$ and (2.8) is clear.

THEOREM 2.5 ((5) of [9]). *If $\eta(r)$ is completely multiplicative, then*

$$\sum_{d\mid r} \mu_{f,t}^{k,\eta}(d)\, M_{f,t}^{k,\eta}\left(\frac{r}{d}\right) = \begin{cases} 1, & \text{if } r = 1 \\ 0, & \text{if } r > 1. \end{cases} \tag{2.9}$$

$$\sum_{d\mid r} \sum_{e\mid n} C_{f,t}^{k,\eta}(e^k, d)\, M_{f,t}^{k,\eta}\left(\frac{r}{d}\right) = \begin{cases} \tau\left(\frac{n}{r}\right) r^{kt}, & \text{if } r \mid n; \\ 0, & \text{if } r \nmid n. \end{cases} \tag{2.10}$$

PROOF. We need only to verify (2.9) when r is a prime power p^α, since both sides are multiplicative functions of r. If $\alpha = 0$ both sides are 1 and if $\alpha > 0$, by (1.8), (1.10), and (1.11) we have

$$\sum_{d\mid p^\alpha} \mu_{f,t}^{k,\eta}(d)\, M_{f,t}^{k,\eta}\left(\frac{p^\alpha}{d}\right) = N_f^{\alpha t}(p^k)\,\eta(p^\alpha) - N_f^t(p^k)\,\eta(p)\,N_f^{(\alpha-1)t}(p^k)\,\eta(p^{\alpha-1}) = 0$$

and (2.9) follows.

Now, by (2.8) the l.h.s. of (2.10) is

$$\sum_{d\mid r} M_{f,t}^{k,\eta}\left(\frac{r}{d}\right) \sum_{e\mid (n,d)} \mu_{f,t}^{k,\eta}\left(\frac{d}{e}\right) \tau\left(\frac{n}{e}\right) e^{kt} = \sum_{e\mid (n,r)} \tau\left(\frac{n}{e}\right) e^{kt} \sum_{D\delta = r/e} \mu_{f,t}^{k,\eta}(\delta)\, M_{f,t}^{k,\eta}(D)$$

and this is by (2.9) $\tau\left(\frac{n}{r}\right) r^{kt}$ or 0 according as $r|n$ or $r \nmid n$ and (2.10) follows.

THEOREM 2.6 ((3) of [9]; (2.11) of [14]). *If $\eta(r)$ is completely multiplicative,*

$$\sum_{d\mid r} C_{f,t}^{k,\eta}\left(d^k, \frac{r}{d}\right) M_{f,t}^{k,\eta}(d) = \begin{cases} (\sqrt{r})^{kt} M_{f,t}^{k,\eta}(\sqrt{r}), & \text{if } r \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

PROOF. The multiplicativity of $C_{f,t}^{k,\eta}(n, r)$ in both n and r and that of $M_{f,t}^{k,\eta}(r)$ as a function of r imply the multiplicativity of the l.h.s. of (2.11) and clearly the r.h.s. of (2.11) is multiplicative. We need to verify (2.11) only when r is a prime power p^α. If $\alpha = 0$, both sides are 1. Let $\alpha > 0$. Then, by Theorem 2.2, (1.10), and (1.11), if $\alpha = 2u + 1$, $u \geq 0$, the l.h.s. of (2.11) is

$$C_{f,t}^{k,\eta}(p^{uk}, p^{u+1})\, M_{f,t}^{k,\eta}(p^u) + \sum_{j=u+1}^{\alpha-1} C_{f,t}^{k,\eta}(p^{jk}, p^{\alpha-j})\, M_{f,t}^{k,\eta}(p^j) + M_{f,t}^{k,\eta}(p^\alpha) =$$

$$= -\, p^{ukt}\eta(p^{u+1})\, N_f^{t(u+1)}(p^k) +$$

$$+ \sum_{j=u+1}^{\alpha-1} \{p^{(\alpha-j)kt}\eta(p^j)\, N_f^{jt}(p^k) - p^{(\alpha-j-1)kt}\eta(p^{j+1})\, N_f^{t(j+1)}(p^k)\} + \eta(p^\alpha)\, N_f^{\alpha t}(p^k) = 0,$$

while if $\alpha = 2u$, $u > 0$, it is

$$\sum_{j=u}^{\alpha-1} C_{f,t}^{k,\eta}(p^{jk}, p^{\alpha-j}) M_{f,t}^{k,\eta}(p^j) + M_{f,t}^{k,\eta}(p^\alpha) =$$

$$= \sum_{j=u}^{\alpha-1} \{p^{(\alpha-j)kt}\eta(p^j) N_f^{jt}(p^k) - p^{(\alpha-j-1)kt}\eta(p^{j+1}) N_f^{t(j+1)}(p^k)\} +$$

$$+ \eta(p^\alpha) N_f^{\alpha t}(p^k) = p^{ukt}\eta(p^u) N_f^{ut}(p^k),$$

and (2.11) follows.

THEOREM 2.7 ((11) and (12) of [9]; Theorems 5.8 and 5.9 of [15]).

(a) *If* $r^k | n$,

(i) $$\sum_{a=1}^{r^k} C_{f,t}^{k,\eta}(na, r) = r^k \varphi_{f,t}^{k,\eta}(r),$$

(ii) $$\sum_{(a, r^k)_k=1} C_{f,t}^{k,\eta}(na, r) = \varphi_k(r)\, \varphi_{f,t}^{k,\eta}(r).$$

(b) (i) $$\sum_{\substack{1 \le a \le r^k \\ (a, r^k)_k = g^k}} C_{f,t}^{k,\eta}(a, r) = C_{f,t}^{k,\eta}(g^k, r)\, \varphi_k\left(\frac{r}{g}\right)$$

(ii) $$\sum_{\substack{1 \le a \le r^k \\ (a, r^k)_k = g^k}} C_{f,t}^{k,\eta}(a, r)\, a = C_{f,t}^{k,\eta}(g^k, r)\, \frac{r^k}{2}\, \varphi_k\left(\frac{r}{g}\right), \quad r > g.$$

REMARKS.

(i) A glance at the suggestion of the profs of (11) and (12) in [9] might tend one to think that (12) in [9] is true without the condition $r^k | n$. That this is not the case is seen by taking $n = 3$, $r = 3$, $k = 2$, since in this case, the l.h.s. of (12) is 10 and the r.h.s. is -8.

(ii) The g in Theorem 5.8 of [15] can be any divisor of r and that of Theorem 5.9 of [15] can be any proper divisor of r.

PROOF. (i) and (ii) of (a) follow from (2.4) and the definition of $\varphi_k(r)$.

Since, $1 \le a \le r^k$, $(a, r^k)_k = g^k$ if and only if $1 \le \frac{a}{g^k} \le \frac{r^k}{g^k}$ and $\left(\frac{a}{g^k}, \frac{r^k}{g^k}\right)_k = 1$, for a given divisor g of r there are $\varphi_k\left(\frac{r}{g}\right)$ numbers $1 \le a \le r^k$, and $(a, r^k)_k = g^k$. Hence, by (2.3), the l.h.s. of (i) of (b) is

$$\sum_{\substack{1 \le a \le r^k \\ (a, r^k)_k = g^k}} C_{f,t}^{k,\eta}(g^k, r) = C_{f,t}^{k,\eta}(g^k, r)\, \varphi_k\left(\frac{r}{g}\right),$$

giving (i) of (b).

Similarly,

$$\sum_{\substack{1 \le a \le r^k \\ (a, r^k)_k = 1}} a = \frac{r^k}{2} \varphi_k(r), \quad r > 1.$$

This is well known for $k = 1$ and essentially the same proof works for $k > 1$. Hence, the l.h.s. of (ii) of (b) is

$$C_{f,t}^{k,\eta}(g^k, r)\, g^k \sum_{\substack{1 \le a \le r^k \\ (a/g^k, r^k/g^k)_k = 1}} \frac{a}{g^k} = \text{the r.h.s. of (ii) of (b).}$$

§ 3.

The following lemma is due to ANDERSON and APOSTOL for $k = 1$ (Theorem 2, [1], and to MCCARTHY for $k > 1$, (Theorem 5, [10]).

LEMMA 3.1 *If $g(r)$ is completely multiplicative, $h(r)$ multiplicative, $g(p) \neq 0$, $h(p) \neq g(p)$ for all primes p,*

$$u(n, r) = \sum_{d^k \mid (n, r^k)_k} g(d)\, h\left(\frac{r}{d}\right) \mu\left(\frac{r}{d}\right),$$

and $U(r) = u(0, r)$, then

$$u(n, r) = \frac{U(r)\, \mu(m)\, h(m)}{U(m)},$$

where $m^k = \dfrac{r^k}{(n, r^k)_k}$.

Taking $g(r) = r^{kt}$, $h(r) = \eta(r)\, N_f^t(r^k)$ in Lemma 3.1, we have, by (2.1), (1.8), and (1.9)

THEOREM 3.1 (Theorem 1, [5]; Theorem 2, [7] with Theorem 5, [6]; Theorem 2, [14] and Theorem 5.1, [15]). *If $\eta(p)\, N_f^t(p^k) \neq p^{kt}$ for all primes, p, then*

$$C_{f,t}^{k,\eta}(n, r) = \frac{\varphi_{f,t}^{k,\eta}(r)\, \mu_{f,t}^{k,\eta}(m)}{\varphi_{f,t}^{k,\eta}(m)}, \quad m^k = \frac{r^k}{(n, r^k)_k}.$$

THEOREM 3.2 (Corollary 2.1, [4]; Theorem 5.5, [15]). *If $\eta(p)\, N_f^t(p^k) \neq p^{kt}$ for all primes p, then*

$$\sum_{a=1}^{r^k} C_{f,t}^{k,\eta}(a, r) = r^k A_{f,t}^{k,\eta}(r). \tag{3.1}$$

PROOF. The numbers Xd^k run through the numbers 1 through r^k as d runs through the divisors of r and for each d, X runs through the numbers

$\leq \frac{r^k}{d^k}$ and k-prime to $\frac{r^k}{d^k}$. Hence by (2.3), Theorem 3.1, (1.9), (1.16) and (1.14), the l.h.s. of (3.1) is

$$\sum_{d\mid r} \sum_{\left(X, \frac{r^k}{d^k}\right)_k = 1} C_{f,t}^{k,\eta}(Xd^k, r) = \sum_{d\mid r} \sum_{\left(X, \frac{r^k}{d^k}\right)_k = 1} C_{f,t}^{k,\eta}(d^k, r) = \varphi_{f,t}^{k,\eta}(r) \sum_{d\mid r} \frac{\mu_{f,t}^{k,\eta}(d)\,\varphi_k(d)}{\varphi_{f,t}^{k,\eta}(d)} =$$

$$= r^{kt} \prod_{p\mid r} \left\{1 - \frac{\eta(p)\,N_f^t(p^k)}{p^{kt}}\right\} \prod_{p\mid r} \left\{1 - \frac{\eta(p)\,N_f^t(p^k)\,\varphi_k(p)}{\varphi_{f,t}^{k,\eta}(p)}\right\} =$$

$$= r^{kt} \prod_{p\mid r} \left\{\frac{p^{kt} - \eta(p)\,N_f^t(p^k)}{p^{kt}}\right\} \prod_{p\mid r} \left\{\frac{p^{kt} - \eta(p)\,N_f^t(p^k)\,p^k}{p^{kt} - \eta(p)\,N_f^t(p^k)}\right\} =$$

$$= r^{kt} \prod_{p\mid r} \left\{1 - \frac{\eta(p)\,N_f^t(p^k)}{p^{k(t-1)}}\right\} = r^{kt} \sum_{d\mid r} \frac{\mu_{f,t}^{k,\eta}(d)}{d^{k(t-1)}} = r^k A_{f,t}^{k,\eta}(r)\,.$$

THEOREM 3.3 ((7), (8), (9) of [9]; Theorem 5.2, [15]). *If* $\eta(p)N_f^t(p^k) \neq p^k$ *for all primes*, p, *then*

$$C_{f,t}^{k,\eta}(n, r)\,\varphi_{f,t}^{k,\eta}\left(\frac{r}{d}\right) = C_{f,t}^{k,\eta}\left(\frac{n}{d^k}, \frac{r}{d}\right)\varphi_{f,t}^{k,\eta}(r), \quad d^k \mid (n, r^k)_k; \tag{3.2}$$

$$C_{f,t}^{k,\eta}(n, r)\,\tau\left((n, r^k)^{\frac{1}{k}}\right) = \varphi_{f,t}^{k,\eta}(r) \sum_{d^k \mid (n, r^k)_k} \frac{C_{f,t}^{k,\eta}\left(\frac{n}{d^k}, \frac{r}{d}\right)}{\varphi_{f,t}^{k,\eta}\left(\frac{r}{d}\right)}; \tag{3.3}$$

$$C_{f,t}^{k,\eta}(n, r) \sum_{d^k \mid (n, r^k)_k} \varphi_{f,t}^{k,\eta}\left(\frac{r}{d}\right) = \varphi_{f,t}^{k,\eta}(r) \sum_{d^k \mid (n, r^k)_k} C_{f,t}^{k,\eta}\left(\frac{n}{d^k}, \frac{r}{d}\right). \tag{3.4}$$

PROOF. We need only to prove (3.2) since the other two identities directly follow from it. Let $(n, r^k)_k = D^k$. Then for every $d \mid D$,

$$\left(\frac{n}{d^k}, \frac{r^k}{d^k}\right)_k = \left(\frac{D}{d}\right)^k,$$

and so

$$\frac{r^k/d^k}{(n/d^k, r^k/d^k)_k} = \frac{r^k}{D^k} = \frac{r^k}{(n, r^k)_k}.$$

Hence, by Theorem 3.1,

$$\frac{C_{f,t}^{k,\eta}(n, r)}{\varphi_{f,t}^{k,\eta}(r)} = \frac{\mu_{f,t}^{k,\eta}\left(\frac{r}{D}\right)}{\varphi_{f,t}^{k,\eta}\left(\frac{r}{D}\right)} = \frac{C_{f,t}^{k,\eta}\left(\frac{n}{d^k}, \frac{r}{d}\right)}{\varphi_{f,t}^{k,\eta}\left(\frac{r}{d}\right)}.$$

and (3.2) is clear.

THEOREM 3.4 ((2.8), [6]; (2.2.3), [15]).

$$\varphi_{f,t}^{k,\eta}(mn)\,\varphi_{f,t}^{k,\eta}((m,n)) = \varphi_{f,t}^{k,\eta}(m)\,\varphi_{f,t}^{k,\eta}(n)\,(m,n)^{kt}; \tag{3.5}$$

$$\varphi_{f,t}^{k,\eta}(m)\,\varphi_{f,t}^{k,\eta}(n) = \varphi_{f,t}^{k,\eta}((m,n))\,\varphi_{f,t}^{k,\eta}([m,n]), \tag{3.6}$$

where in (3.6) $[m, n]$ *stands for the L.C.M. of* m *and* n.

PROOF. We have by (1.9), denoting

$$1 - \frac{\eta(p)\,N_f^t(p^k)}{p^{kt}}$$

by $T(p)$,

$$\varphi_{f,t}^{k,\eta}(mn)\,\varphi_{f,t}^{k,\eta}((m,n)) = m^{kt}n^{kt}(m,n)^{kt}\Big\{\prod_{p\mid mn}T(p)\Big\}\Big\{\prod_{p\mid(m,n)}T(p)\Big\}, \tag{3.7}$$

$$\Big\{\prod_{p\mid mn}T(p)\Big\}\Big\{\prod_{p\mid(m,n)}T(p)\Big\} = \Big\{\prod_{\substack{p\mid m\\ p\mid n}}T(p)\Big\}\Big\{\prod_{\substack{p\mid m\\ p\nmid n}}T(p)\Big\}\Big\{\prod_{\substack{p\mid n\\ p\nmid m}}T(p)\Big\}\Big\{\prod_{\substack{p\mid m\\ p\mid n}}T(p)\Big\} =$$

$$= \Big\{\prod_{p\mid m}T(p)\Big\}\Big\{\prod_{p\mid n}T(p)\Big\}, \tag{3.8}$$

and (3.5) is clear from (3.7) and (3.8). Now, by (3.5).

$$\varphi_{f,t}^{k,\eta}(m)\,\varphi_{f,t}^{k,\eta}(n)\,(m,n)^{kt} = \varphi_{f,t}^{k,\eta}(mn)\,\varphi_{f\,t}^{k\,\eta}((m,n)) =$$

$$= \varphi_{f,t}^{k,\eta}((m,n)\,[m,n])\,\varphi_{f,t}^{k,\eta}(((m,n),[m,n])) = \varphi_{f,t}^{k,\eta}(m,n)\,\varphi_{f,t}^{k,\eta}([m,n])\,(m,n)^{kt},$$

and (3.6) is clear.

THEOREM 3.5 ((16), [9]; Theorem 5.6, [15]). *If* $\eta(p)\,N_f^t(p^k) \neq p^{kt}$ *for all primes* p,

$$C_{f,t}^{k,\eta}(n,r) = \frac{\mu_{f,t}^{k,\eta}\left(\frac{r}{D}\right)\varphi_{f,t}^{k,\eta}(D)\left(D,\frac{r}{D}\right)^{kt}}{\varphi_{f,t}^{k,\eta}\left(\left(D,\frac{r}{D}\right)\right)} =$$

$$= \mu_{f,t}^{k,\eta}\left(\frac{r}{D}\right)\varphi_{f,t}^{k,\eta}(D)\,\frac{s^{kt}}{\varphi_{f,t}^{k,\eta}(s)}, \quad D^k = (n.\,r^k)_k; \tag{3.9}$$

where s *is the product of all the distinct prime factors common to* D *and* $\dfrac{r}{D}$.

$$C_{f,t}^{k,\eta}(n^k,r)\,C_{f,t}^{k,\eta}(r^k,n) = \varphi_{f,t}^{k,\eta}((n,r))\,C_{f,t}^{k,\eta}((n,r)^k,[n,r]). \tag{3.10}$$

PROOF. By Theorems 3.1 and 3.4,

$$C_{f,t}^{k,\eta}(n, r) = \frac{\varphi_{f,t}^{k,\eta}(r)\,\mu_{f,t}^{k,\eta}\left(\frac{r}{D}\right)}{\varphi_{f,t}^{k,\eta}\left(\frac{r}{D}\right)} = \frac{\mu_{f,t}^{k,\eta}\left(\frac{r}{D}\right)\varphi_{f,t}^{k,\eta}(D)\left(D, \frac{r}{D}\right)^{kt}}{\varphi_{f,t}^{k,\eta}\left(\left(D, \frac{r}{D}\right)\right)}$$

and (3.9) is clear in virtue of (1.9). Now, by (2.3), Theorem 3.1, and (3.6),

$$C_{f,t}^{k,\eta}(n^k, r)\, C_{f,t}^{k,\eta}(r^k, n) = C_{f,t}^{k,\eta}((n, r)^k, r)\, C_{f,t}^{k,\eta}((n, r)^k, n) =$$

$$= \frac{\varphi_{f,t}^{k,\eta}(r)\,\mu_{f,t}^{k,\eta}\left(\frac{r}{(n, r)}\right)}{\varphi_{f,t}^{k,\eta}\left(\frac{r}{(n, r)}\right)}\;\frac{\varphi_{f,t}^{k,\eta}(n)\,\mu_{f,t}^{k,\eta}\left(\frac{n}{(n, r)}\right)}{\varphi_{f,t}^{k,\eta}\left(\frac{n}{(n, r)}\right)} = \frac{\varphi_{f,t}^{k,\eta}(r)\,\varphi_{f,t}^{k,\eta}(n)\,\mu_{f,t}^{k,\eta}\left(\frac{rn}{(n, r)^2}\right)}{\varphi_{f,t}^{k,\eta}\left(\frac{rn}{(n, r)^2}\right)} =$$

$$= \frac{\varphi_{f,t}^{k,\eta}((n, r))\,\varphi_{f,t}^{k,\eta}([n, r])\,\mu_{f,t}^{k,\eta}\left(\frac{[n, r]}{(n, r)}\right)}{\varphi_{f,t}^{k,\eta}\left(\frac{[n, r]}{(n, r)}\right)} = \varphi_{f,t}^{k,\eta}(n, r)\, C_{f,t}^{k,\eta}((n, r)^k, [n. r]),$$

giving (3.10).

THEOREM 3.6 (Theorem 1, [6]).

(a) *If* $\eta(r)$ *is completely multiplicative, then*

$$\sum_{d\delta=r} \varphi_{f,t}^{k,\eta}(d)\, M_{f,t}^{k,\eta}(\delta) = r^{kt}$$

$$\text{(b)}\quad \sum_{\substack{d\delta=r\\(d,\delta)=1}} \mu_{f,t}^{k,\eta}(d)\, M_{f,t}^{k,\eta}(\delta) = \begin{cases} M_{f,t}^{k,\eta}(r), & \textit{if } r = 1 \textit{ or every prime factor of } r \textit{ is repeated.}\\ 0, & \textit{otherwise.}\end{cases}$$

PROOF. We prove (a), the proof of other part being similar. It is enough to verify (a) when r is a prime power p^α. If $\alpha > 0$, the l.h.s. is, by (1.9) and (1.11)

$$= M_{f,t}^{k,\eta}(p^\alpha) + \sum_{j=1}^{\alpha} \varphi_{f,t}^{k,\eta}(p^j)\, M_{f,t}^{k,\eta}(p^{\alpha-j}) =$$

$$= \eta(p^\alpha)\, N_f^{\alpha t}(p^k) + (p^{kt} - \eta(p)\, N_f^t(p^k)) \sum_{j=1}^{\alpha} p^{(j-1)kt}(\eta(p)\, N_f^t(p^k))^{\alpha-j} =$$

$$= \eta(p^\alpha)\, N_f^{\alpha t}(p^k) + p^{\alpha kt} - (\eta(p)\, N_f^t(p^k))^\alpha = p^{\alpha kt};$$

and if $\alpha = 0$, both sides are 1 and the result follows.

THEOREM 3.7 ((10), (13), [9]).

$$(3.11) \qquad \sum_{\substack{d\delta=n \\ (d,\delta)=1}} C_{f,t}^{k,\eta}(\delta^k, d)\, M_{f,t}^{k,\eta}(\delta) = \begin{cases} M_{f,t}^{k,\eta}(n), & \text{if } n=1 \text{ or every prime factor of } n \text{ is repeated}, \\ 0, & \text{otherwise}; \end{cases}$$

and if $\eta(r)$ is completely multiplicative,

$$(3.12) \qquad \sum_{d\mid D} C_{f,t}^{k,\eta}(d^k, d)\, M_{f,t}^{k,\eta}\left(\frac{D}{d}\right) = D^{kt}, \quad D^k = (n, r^k)_k .$$

PROOF (3.11) follows from (2.3) and (b) of Theorem 3.6 and (3.12) follows from (2.4) and (a) of Theorem 3.6.

From (1.8) and (1.14), we get

THEOREM 3.8 (Theorem 5.7, [15]).

$$\sum_{\substack{d\mid r \\ (d^k, n)_k=1}} C_{f,1}^{k,\eta}(n, d) = A_{f,1}^{k,\eta}(R_1)$$

where R_1 is the largest divisor of r such that $(R_1^k, n)_k = 1$.

§ 4.

The representation of $C_{f,t}^{k,\eta}(n, r)$ as a trigonometric sum depends on the following lemma which is a generalization of a theorem (Theorem 4, [1]) of Anderson and Apostol.

LEMMA 4.1 *For any arithmetical functions $g(r)$ and $h(r)$ the function*

$$(4.1) \qquad S^{(k)}(n, r) = \sum_{d^k \mid (n, r^k)_k} g(d)\, h\left(\frac{r}{d}\right)$$

can be represented as

$$(4.2) \qquad S^{(k)}(n, r) = \sum_{m \pmod{r^k}} \alpha(m, r)\, e(nm, r^k),$$

where the sum in (4.2) is extended over a complete residue system mod r^k, and

$$(4.3) \qquad \alpha(m, r) = \frac{1}{r^k} \sum_{d^k \mid (m, r^k)_k} d^k h(d)\, g\left(\frac{r}{d}\right).$$

Further, if $g(r)$ is completely multiplicative,

$$(4.4) \qquad \alpha(m, r) = \frac{1}{r^k}\, g\left(\frac{r}{D}\right) \sum_{d\mid D} d^k h(d)\, g\left(\frac{D}{d}\right),$$

where $(m, r^k)_k = D^k$.

PROOF. Since $m \equiv m_1 \pmod{r^k}$ implies $(m, r^k)_k = (m_1, r^k)_k$, it is clear that the sun on the r.h.s. of (4.2) is independent of the residue system mod r^k. Now,

$$\sum \alpha(m, r)\, e(nm, r^k) = \sum_{m=1}^{r^k} \frac{1}{r^k} \sum_{d^k \mid (m, r^k)_k} d^k h(d)\, g\left(\frac{r}{d}\right) e(nm, r^k) =$$

$$= \frac{1}{r^k} \sum_{d \mid r} d^k h(d)\, g\left(\frac{r}{d}\right) \sum_{m=1,\, d^k \mid m}^{r^k} e(nm, r^k) =$$

$$= \frac{1}{r^k} \sum_{d \mid r} d^k h(d)\, g\left(\frac{r}{d}\right) \sum_{j=1}^{r^k/d^k} e(nj, r^k/d^k);$$

since the inner sum above is r^k/d^k or 0 according as n is or is not divisible by r^k/d^k, the above sum is

$$= \frac{1}{r^k} \sum_{\substack{d \mid r \\ \frac{r^k}{d^k} \mid n}} d^k h(d)\, g\left(\frac{r}{d}\right) \frac{r^k}{d^k} = \sum_{d^k \mid (n, r^k)_k} g(d)\, h\left(\frac{r}{d}\right) = S^{(k)}(n, r).$$

That (4.3) can be expressed as (4.4) in case $g(r)$ is completely multiplicative is obvious.

Taking $g(r) = r^{kt}$, $h(r) = \mu_{f,t}^{k,\eta}(r)$ in Lemma 4.1, we get from (2.1) and (1.13)

THEOREM 4.1 ((3), [1]; Theorem 4.1, [15]).

$$C_{f,t}^{k,\eta}(n, r) = \sum_{m \pmod{r^k}} a_{f,t}^{k,\eta}(m, r)\, e(nm, r^k)$$

where $a_{f,t}^{k,\eta}(n, r)$ *is given by* (1.13).

Let us write $r_k(m) = ((m, r^k)_k)^{1/k}$, so that $r_k(m) = 1$ if and only if $(m, r^k)_k = 1$. Clearly, since $\mu_{x,t}^{k,\eta}(r)$ does not depend on t, we have from (1.13), (1.8), and (1.9)

$$a_{f,1}^{k,\eta}(m, r) = \sum_{d \mid r_k(m)} \mu_{f,1}^{k,\eta}(d); \tag{4.5}$$

$$a_{x,t}^{k,\eta}(m, r) = \left(\frac{r}{r_k(m)}\right)^{k(t-1)} \varphi_{x,t-1}^{k,\eta}(r_k(m)), \quad t > 1. \tag{4.6}$$

Hence, ;we have

COROLLARY 4.1.1

$$C_{f,1}^{k,\eta}(n, r) = \sum_{m \pmod{r^k}} \Big(\sum_{d \mid r_k(m)} \mu_{f,1}^{k,\eta}(d)\Big)\, e(nm, r^k); \tag{4.7}$$

$$C_{x,t}^{k,\eta}(n, r) = r^{k(t-1)} \sum_{m \pmod{r^k}} \frac{\varphi_{x,t-1}^{k,\eta}((r_k(m))}{(r_k(m))^{k(t-1)}}\, e(nm, r^k), \quad t > 1. \tag{4.8}$$

In particular, since $\mu_{\chi,t}^{k,I}(r) = \mu(r)$,

$$\mu_{\chi,t}^{k,\eta_u}(r) = \mu^2(r)\,\mu_u(r) = \mu_u(r)$$

(see (1.5)), $\varphi_{\chi,t}^{k,I}(r) = \varphi_{kt}(r)$, $\sum_{d|r}\mu(d) = 1$ or 0 according as $r = 1$ or $r > 1$, and

$$\sum_{d|r}\mu_u(d) = \big(1 + e(1, 2u)\big)^{w(r)},$$

we have

COROLLARY 4.1.2.

$$C(n, r) = \sum_{\substack{m \pmod r \\ (m,r)=1}} e(nm, r); \tag{4.9}$$

$$C^{(k)}(n, r) = \sum_{\substack{m \pmod{r^k} \\ (m,r^k)_k=1}} e(nm, r^k); \tag{4.10}$$

$$C_k(n, r) = r^{k-1} \sum_{m \pmod r} \frac{\varphi_{k-1}((m, r))}{(m, r)^{k-1}}\, e(nm, r). \quad k > 1; \tag{4.11}$$

$$C_k^{(s)}(n, r) = r^{s(k-1)} \sum_{m \pmod{r^s}} \frac{\varphi_{s(k-1)}(r_s(m))}{(r_s(m))^{s(k-1)}}\, e(nm, r^s), \quad k > 1; \tag{4.12}$$

$$C^{\mu_u}(n, r) = \sum_{m \pmod r} \big(1 + e(1, 2u)\big)^{w((m,r))} e(nm, r). \tag{4.13}$$

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