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ON AUTOMORPHISM GROUPS OF CAYLEY GRAPHS

by

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Abstract

Let $X_{G,H}$ denote the Cayley graph of a finite group G with respect to a subset H. It is well-known that its automorphism group $A(X_{G,H})$ must contain the regular subgroup L_G corresponding to the set of left multiplications by elements of G. This paper is concerned with minimizing the index $[A(X_{G,H}):L_G]$ for given G, in particular when this index is always greater than 1. If G is abelian but not one of seven exceptional groups, then a Cayley graph of G exists for which this index is at most 2. Nearly complete results for the generalized dicyclic groups are also obtained.

1. Motivation

The symbol G will always denote a group assumed to be finite unless otherwise specified, e will denote its identity, and H will denote a subset of G subject to the conditions $e \notin H$ and $H = \{h^{-1} : h \in H\}$. The symbol X will always denote a simple graph assumed to be finite unless otherwise specified. The symbols $V(X)$, $E(X)$, $A(X)$, and $A_v(X)$ will denote, respectively, its vertex set, edge set, automorphism group, and the subgroup of $A(X)$ which stabilizes the vertex $v \in V(X)$. For any set S, $\mathbf{1}_S$ indicates the identity permutation on S. Let Z_n denote the cyclic group of order n.

To write $X = X_{G,H}$ means that $V(X) = G$ and $E(X) = \{[g, gh] : g \in G; h \in H\}$. This graph is called the *Cayley graph of G with respect to H*. Such a graph is connected if and only if H generates G. One readily observes that for each $g \in G$, the left multiplication $\lambda_g : G \to G$ given by $x \to gx$ belongs to $A(X_{G,H})$. Thus the set $L_G = \{\lambda_g : g \in G\}$ is a subgroup of $A(X_{G,H})$ for any H and, moreover, is a regular permutation group on $V(X_{G,H})$. Let us write

$$c(G, H) = [A(X_{G,H}) : L_G] = |A(X_{G,H})|/|G|,$$

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which equals, of course, $|A_g(X_{G,H})|$ for any $g \in G$. We define the *Cayley index* of G to be

$$c(G) = \min_H c(G, H).$$

A Cayley graph $X_{G,H}$ for which $c(G, H) = 1$ is called a *graphical regular representation* (*GRR*) of G. It has been shown by NOWITZ and WATKINS [11] that all non-abelian groups of order coprime to 6 have a GRR. (For an exposition of the state of the "GRR-problem" see [16].) More recently IMRICH [6], using results from [11] and some results of D. HETZEL [2], has shown that except for the non-abelian group of order 27 and exponent 3, every non-abelian group of odd order admits a GRR. All GRR's are connected, with the unique exception of the GRR of Z_2 consisting of precisely two isolated vertices. (See [13] Lemma 1.)

Let the function $\alpha : G \to G$ be defined by $x \to x^{-1}$. If G is abelian, $g \in G$, and $h \in H$, then the action of α on $E(X_{G,H})$ satisfies

$$\alpha[g, gh] = [g^{-1}, (gh)^{-1}] = [g^{-1}, g^{-1}h^{-1}].$$

Since H is closed with respect to inverses and $\alpha^2 = 1_G$, α maps edges into edges. Thus $\{1_G, \alpha\} \leq A_e(X_{G,H})$ for all H, and as SABIDUSSI [14] and CHAO [1] observed,

1.1. *If G is abelian but not an elementary abelian 2-group, then $c(G) \geq 2$.*

In [3] this was complemented as follows:

1.2. $c(Z_2^n) \geq 2$ *if and only if* $n = 2, 3$ *or* 4.

The proof in [3] consists of a nonexistence proof for $n = 2, 3, 4$ and an existence proof for $n \geq 5$. However, it should be noted that the construction given in [3] is wrong for $n = 5$ and that several authors, including R. FRUCHT and M. H. MCANDREW have found constructions for $n \geq 6$. The error in [3] has been pointed out and corrected by B. ALSPACH, P. HELL, D. HETZEL and CHONG-KEANG LIM. For the sake of completeness we include a GRR of Z_2^5 due to HETZEL:

Let $a_1, a_2, \ldots, a_5$ generate Z_2^5 and let H consist of these generators together with a_1a_2, a_1a_3, a_1a_5, a_2a_4, a_4a_5, $a_1a_2a_3$ and $a_2a_3a_4$. Then the Cayley graph of Z_2^5 with respect to H is a GRR of Z_2^5.

One of the two main results of this paper is the following:

THEOREM 1. *Let G be a finite abelian group. Then $c(G) \leq 2$ unless G is one of the following seven groups: Z_2^3, Z_2^4, $Z_4 \times Z_2$, $Z_4 \times Z_2^2$, Z_3^2, Z_3^3, and Z_4^2.*

As it is in most cases rather tedious to determine the Cayley index of the exceptional groups, we have investigated only the cases Z_2^3, $Z_4 \times Z_2$, and Z_3^2 (see 2.4 and 2.7 below). The Cayley indices of the other exceptional groups have been determined by D. HETZEL [2] with the aid of a computer.

A non-abelian group G is *generalized dicyclic* if it is generated by an abelian group L and an element $b \notin L$ such that (i) $b^2 \in L \setminus \{e\}$; (ii) $b^4 = e$; and (iii) $b^{-1}xb = x^{-1}$ for all $x \in L$. When G is generalized dicyclic, we define the function $\beta : G \to G$ given by

$$\beta(G) = \begin{cases} g, & \text{if } g \in L; \\ g^{-1}, & \text{if } g \in G \setminus L. \end{cases}$$

It has been shown in [9] and [15] that in addition to being a group-automorphism of every generalized dicyclic group G, β is also a graph-automorphism of $X_{G,H}$ for every H. Since $\beta^2 = 1_G$, we have

1.3. *If G is a generalized dicyclic group, then $c(G)$ is even.*

Letting Q denote the quaternion group, we state the second main result of this paper.

THEOREM 2. *Let G be a finite generalized dicyclic group generated by L and b as in the above definition.*

(a) *If G is not of the form $Q \times Z_2^m$ for some $m \geqq 0$ and L is not Z_4^2, $Z_4 \times Z_2$, or $Z_4 \times Z_2^2$, then* (i) $c(G) = 2$ *or* 4; (ii) *if* $|G| > 96$, *then* $c(G) = 2$.

(b) *If G is of the form $Q \times Z_2^m$, then $c(G) \leqq 16$.*

2. Preliminary results

For background on cartesian products of graphs, the reader is referred to [12]. The same reference contains the following result (Corollary 3.2), stated here for finite graphs.

2.1. *If X_1 and X_2 are connected graphs which are relatively prime with respect to cartesian multiplication, then*

$$A(X_1 \times X_2) = A(X_1) \times A(X_2).$$

(Note: in the left-hand member of this expression the symbol $\times$ denotes cartesian multiplication of graphs, but in the right-hand member $\times$ denotes the direct product of permutation groups.)

If X is a graph, then its complement is denoted by X'. The complete graph on n vertices is denoted by K_n. We require the following result [5, Theorem 1]:

2.2. *If X is any finite or infinite graph, then either X or X' is prime with respect to cartesian multiplication unless X is one of the following six graphs:* $K_2 \times K_2$, $K_2 \times K_2'$, $K_2 \times K_2 \times K_2$, $K_4 \times K_2$, $K_3 \times K_3$, *and* $K_2 \times K_4^-$, *where* K_4^- *is obtained from* K_4 *by deletion of an edge.*

A Cayley graph $X_{G,H}$ with the property that $c(G,H) = c(G)$ is called a *most rigid representation* (*MRR*) of G. Clearly a GRR is always an MRR. Since $A(X) = A(X')$ for any graph X, every group admits a connected MRR. In addition, 2.2 implies that every group G admits an MRR which is relatively prime with respect to cartesian multiplication unless every MRR of G is one of the six "forbidden" graphs. In the interest of a smoother argument later on, let us first determine some MRR's for certain specific groups.

LEMMA 2.3. *For each $m \geq 3$, $c(Z_m) = 2$ and the m-circuit C_m is an MRR for Z_m.*

PROOF. Clearly C_m is a Cayley graph of Z_m. Moreover, $A(C_m)$ is the dihedral group D_m, and $|D_m| = 2|Z_m|$. The conclusion follows from 1.1.

LEMMA 2.4 $c(Z_3^2) = 8$, *and $K_3 \times K_3$ is an MRR of Z_3^2.*

PROOF. Let $X = X_{Z_3^2,H}$ be a connected MRR of Z_3^2. Since H is closed with respect to inverses and generates $Z_3 \times Z_3$, $|H| \geq 4$. If $|H| = 4$, then H has the form $\{a_1, a_1^{-1}, a_2, a_2^{-1}\}$, where $a_1^3 = a_2^3 = a_1 a_2 a_1^{-1} a_2^{-1} = e$. The automorphism φ of $Z_3 \times Z_3$ which interchanges a_1 and a_2 clearly belongs to $A_e(X)$ as does the automorphism ψ which fixes a_1 but interchanges a_2 with a_2^{-1}. Thus $A_e(X) = \langle\{\alpha, \varphi, \psi\}\rangle \cong D_4$, and $c(Z_3^2) \leq 8$. In this case, $X \simeq K_3 \times K_3 \simeq X'$. If $|H| > 4$, then $Z_3 \times Z_3 \setminus (H \cup \{e\})$ does not generate $Z_3 \times Z_3$ and so X' is not connected. Either X' consists of nine isolated vertices or $X' = K_3 \times K_3'$. Either way, $|A_e(X)| = |A_e(X')| > 8$.

LEMMA 2.5. $c(D_4) = 2$ *and an MRR of D_4 is shown in Figure* 1.

PROOF. Let X be the graph represented by Figure 1. Let e denote a vertex of that graph. We first show that $|A_e(X)| = 2$. Of the four edges incident

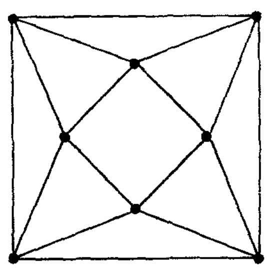

Fig. 1. MRR for D_4

with e, two of them lie on precisely one 3-circuit each, and two of them lie on precisely two 3-circuits each. One readily verifies that interchanging such a pair of edges lying on the same number of 3-circuits determines uniquely the only non-identity automorphism in $A_e(X)$.

Represent $D_4 = \langle a, b \mid a^4 = b^2 = (ba)^2 = e \rangle$. Let $H = \{a, a^{-1}, b, ba\}$. Then $X_{D_4,H} \simeq X$. Since D_4 has no GRR (see [15, Theorem 2]), $c(D_4) = 2$.

D. Hetzel has observed to us that the 8-circuit is also an MRR of D_4, obtained quite simply by taking $H = \{b, ba\}$. This method also affords MRR's of D_3 and D_5.

Lemma 2.6. *$c(Q) = 16$, and $(C_4 \times K_2')$ is an MRR for Q.*

Proof. The group Q admits the representation $Q = \langle a, b \mid a^2 = b^2 = (ba)^2 \rangle$. Let $X = X_{Q,H}$ be a connected MRR of Q. Then H contains at least two of the three pairs $a^{\pm 1}, b^{\pm 1}$ and $(ba)^{\pm 1}$. Since then $Q \setminus H$ does not generate Q, the graph X' cannot be connected. But since X' is vertex-transitive, it must be one of the following: K_8', $K_2 \times K_4'$, $K_4 \times K_2'$, or $C_4 \times K_2'$. Of these, $C_4 \times K_2'$ has the smallest permutation group, namely D_4 wreath Z_2, which has order 128. Hence the stabilizer of a vertex of $K_4 \times K_2'$ has order 16. It remains only to show that $(C_4 \times K_2')'$ is a Cayley graph of Q. This is immediate when one lets $H = \{a, a^{-1}, b, b^{-1}, a^2\}$.

Lemma 2.7. *If $G = Z_4 \times Z_2$ or Z_2^3, then $c(G) = 6$ and the 3-cube $K_2 \times K_2 \times K_2$ is an MRR of G.*

Proof. We represent $Z_4 \times Z_2 = \langle a_1, a_2 \mid a_1^4 = a_2^2 = a_1 a_2 a_1^{-1} a_2 = e \rangle$ and $Z_2^3 = \langle b_1, b_2, b_3 \mid b_i^2 = (b_i b_j)^2 = e;\ i, j = 1, 2, 3 \rangle$. The 3-cube is clearly a Cayley graph of $Z_4 \times Z_2$ with respect to $\{a_1, a_1^{-1}, a_2\}$ and a Cayley graph of Z_2^3 with respect to $\{b_1, b_2, b_3\}$. Its automorphism-group is known to have order 48, and any vertex-stabilizer is isomorphic to the symmetric group of a 3-set. Thus if $G = Z_4 \times Z_2$ or Z_2^3, then $c(G) \leq 6$. Let $X = X_{G,H}$ be a connected MRR of G. If $G = Z_2^3$, then clearly $|H| \geq 3$. If $G = Z_4 \times Z_2$, then H contains at least one pair of elements of order 4 together with some other element, and again $|H| \geq 3$. By the same argument as in the previous Lemma, X' must be connected, and so $G \setminus H$ also generates G. Hence $|H| \leq 4$. The proof that no other generating set H yields a Cayley graph with a smaller automorphism group is straightforward and is left to the reader.

Lemma 2.8. *Let G_1 and G_2 be groups having connected MRR's which are relatively prime to each other with respect to cartesian multiplication. Then*

$$c(G_1 \times G_2) \leq c(G_1)c(G_2).$$

PROOF. Let us choose connected, relatively prime MRR's X_1 and X_2 of G_1 and G_2, respectively. By 2.1, $A(X_1 \times X_2) = A(X_1) \times A(X_2)$. Hence

$$|A(X_1 \times X_2)| = |A(X_1)|\,|A(X_2)| = \\ = c(G_1)|G_1|\,c(G_2)|G_2| = \\ = c(G_1)c(G_2)|G_1 \times G_2|.$$

But $c(G_1 \times G_2) \leqq |A(X_1 \times X_2)|\,/\,|G_1 \times G_2|$.

PROPOSITION 2.9. *Let G be a group other than $Z_2^2, Z_2^3, Z_4, Z_4 \times Z_2$, or Z_3^2. Then G admits a connected MRR which is prime with respect to cartesian multiplication.*

PROOF. Let X be a connected MRR of the group G. We first verify that X is not one of the exceptional graphs listed in 2.2. Since $|G| \neq 4$, the graphs $K_2 \times K_2$ and $K_2 \times K_2'$ are precluded. If X is $K_2 \times K_2 \times K_2$ or its complement $K_4 \times K_2$, then $|G| = 8$ and $c(G) = 6$ by 2.7. Since G is neither Z_2^3 nor $Z_4 \times Z_2$, this is impossible by 2.5 and 2.6. If $X = K_3 \times K_3$, then $c(G) = 8$ by 2.4. But $c(Z_9) = 2$ by 2.3. Since $K_2 \times K_4^-$ is not vertex transitive, it is not a Cayley graph. Hence by 2.2, either X or X' is prime (with respect to cartesian multiplication).

If X is prime, we are done, so suppose that X' is prime but not connected. Since X' is vertex transitive, it is the union of some $n \geqq 2$ copies of some component Y. But then $X' \cong Y \times K_n'$. Since X' is prime, $Y \cong K_1$. Hence $X \cong K_n$. But K_n is prime with respect to cartesian multiplication.

COROLLARY 2.10. *Let G be an abelian group other than Z_4 or Z_2^m for some $m \geqq 1$. If $c(G) = 2$, then $c(G \times Z_2) = 2$.*

PROOF. By 1.1, $c(G \times Z_2) \geqq 2$. Since K_2 is a GRR of Z_2, the corollary will follow from 2.8 once it is established that G has a connected MRR X which is relatively prime to K_2. This is an immediate consequence of the hypothesis, 2.9, and 2.7.

If $K \subseteq G$, then $\langle K \rangle$ denotes the subgroup of G generated by K and $\varphi|_K$ denotes the restriction of φ to the subset K. Let $K^{-1} = \{k^{-1} : k \in K\}$. For any non-negative integer i, we define $K^0 = \{e\}$ and $K^{i+1} = KK^i$.

By [7, Corollary 1.2] the relation $\varphi(a) = b$ for all $\varphi \in A_e(X) \setminus \{1_G\}$ implies $\varphi(ca) = \varphi(c)\,\varphi(a)$ for all $c \in G$ and for all $\varphi \in A_e(X)$. If $\varphi|_K = 1|_K$ for all $\varphi \in A_e(X)$ we therefore have $e = \varphi(k^{-1}\,k) = \varphi(k^{-1})\,k$ and $\varphi(k^{-1}) = k^{-1}$ for $k \in K$. By the same result, $\varphi(ka) = ka$ and $\varphi(k^{-1}a) = k^{-1}a$ if $\varphi(a) = a$ for all $\varphi \in A_e(X)$. Proceeding by induction we therefore obtain $\varphi|_{K^i} = 1|_{K^i}$ for all integers i if $\varphi|_K = 1|_K$ for $\varphi \in A_e(X)$. Since $\langle K \rangle$ is the union of all K^i we have shown:

2.11. *Let X be a Cayley graph of a group G and let $K \subseteq G$. If $\varphi|_K = 1|_K$ for all $\varphi \in A_e(X)$, then $\varphi|_{\langle K \rangle} = 1_{\langle K \rangle}$ for all $\varphi \in A_e(X)$.*

For finite groups this has been formulated in [7, Corollary 1.4]. (See also [10, Proposition 2.3].)

The next result generalizes [7, Proposition 1.8] and is very important for the proofs of the Theorems.

PROPOSITION 2.12. *Let X be a Cayley graph of a finite or infinite abelian group G, and let $e \neq a \in K \subseteq G$. Suppose that for all $\varphi \in A_e(X)$:*

(i) $\varphi(a) \in \{a, a^{-1}\}$, *and* (ii) $\varphi(a) = a \Rightarrow \varphi|_K = 1_K$.

Then for all $\varphi \in A_e(X)$:

(iii) $\varphi(a) = a \Rightarrow \varphi|_{\langle K \rangle} = 1_{\langle K \rangle}$, *and* (iv) $\varphi(a) = a^{-1} \Rightarrow \varphi|_{\langle K \rangle} = \alpha|_{\langle K \rangle}$.

PROOF. Suppose the conditions of the theorem are satisfied and that $\varphi \in A_e$. We note first that $\varphi(a) = a^{-1}$ implies $\varphi|_K = \alpha|_K$. For, let $\varphi(a) = a^{-1}$. Then $\alpha\varphi \in A_e(X) \cap A_a(X)$ and $\alpha\varphi|_K = 1|_K$. Replacing K by $\langle K \rangle$ we see that (iv) is a consequence of (iii). It therefore suffices to prove (iii). Assume $\varphi(a) = a$.

For $k \in K$ we have $\lambda_{k^{-1}}\varphi\lambda_k \in A_e(X) \cap A_{k^{-1}}(X)$, whence $\lambda_{k^{-1}}\varphi\lambda_k|_K = 1|_K$ or $\alpha|_K$. In the second case $k^{-1} = k$ and $\psi(k) = k$ for all $\psi \in A_e$. Hence $\varphi(ak) = \varphi(a)\,\varphi(k) = k\,\varphi(a)$ by [7, Corollary 1.2]. Thus $\lambda_{k^{-1}}\varphi\lambda_k(a) = \varphi(a) = a$ and $\lambda_{k^{-1}}\varphi\lambda_k|_K = 1|_K$ by (ii). This implies $\varphi|_{kK} = 1|_{kK}$ for all $k \in K$, or equivalently, $\varphi|_{K^2} = 1|_{K^2}$. By induction we obtain $\varphi|_{K^i} = 1|_{K^i}$ for all positive integers i.

We have $\alpha\varphi\alpha \in A_e(X)$ and $\alpha\varphi\alpha(a^{-1}) = a^{-1}$. For $a = a^{-1}$ this is the same as $\alpha\varphi\alpha(a) = a$ and for $a \neq a^{-1}$ the relation $\alpha\varphi\alpha(a) = a$ follows by (i). Hence $\alpha\varphi\alpha|_{K^i} = 1|_{K^i}$ by what we have just shown, and therefore $\varphi|_{K^{-i}} = 1|_{K^{-i}}$ for all positive integers i. Now the observation that $\langle K \rangle$ is the union of all $K^{\pm i}$ completes the proof.

COROLLARY 2.13. *Assume the hypothesis of Proposition 2.12. If G is not an elementary abelian 2-group, and if $\langle K \rangle = G$, then $c(G) = 2$ and X is an MRR of G.*

If $X = X_{G,H}$, then X_e will denote the subgraph of X induced by the vertices adjacent to e, that is, the set H. It is clear that if $\varphi \in A_e(X)$, then its restriction to X_e belongs to $A(X_e)$. We shall see that, in particular if the set K of 2.11 is contained in H, consideration of the symmetries of X_e is very helpful in determining $A_e(X)$.

We shall require the following result which is a special case of [7, Theorem 3.1]:

2.14. *Let G be an abelian group of odd order and Cayley index* 2. *Suppose $X_{G,H}$ is an MRR of G and m is a positive integer. If*

(i) $$m = 1 \text{ and } |G| > 45,$$

or

(ii) $$m \geq 2 \text{ and } |G| > |H| + 11,$$

then

$$c(G \times Z_{2m+1}) = 2.$$

3. On the Cayley index of abelian groups

In the present section we presume G to be a finite abelian group. By the Fundamental Theorem of Abelian Groups, G may be expressed uniquely in the form $Z_{m_1} \times \ldots \times Z_{m_r}$ where $m_{i+1} | m_i$ for $i = 1, \ldots, r$. Henceforth G will be identified with the r-tuple $(m_1, \ldots, m_r)$, and we shall adopt as standard the representation

$$\langle a_1, \ldots, a_r : a_i^{m_i} = a_i a_j a_i^{-1} a_j^{-1} = e, \quad 1 \leq i \leq j \leq r\rangle$$

for the group $(m_1, \ldots, m_r)$. Further, we will denote the order of $g \in G$ by $o(g)$.

If $g_1, g_2 \in G$, we define

$$C(g_1, g_2) = \{g_1, g_1^{-1}, g_2, g_2^{-1}, g_1 g_2, g_1^{-1} g_2^{-1}\}.$$

$$C_2(g_1) = \{g_1^{\pm i} : i = 1, 2\}$$

$$C_3(g_1) = \{g_1^{\pm i} : i = 1, 2, 3\}.$$

The following includes a reformulation of some observations from [7, Section 4]. The proof is elementary and is omitted.

LEMMA 3.1. *Let $X = X_{G,H}$. Let $g_1, g_2 \in G$ and suppose $g_1^i = g_2^j$ only if $g_1^i = e$. Also suppose $o(g_1) \geq o(g_2) \geq 3$.*

a) *If $C(g_1, g_2) \subseteq H$, then $C(g_1, g_2)$ induces the following subgraph in X_e:*

(i) *the 6-circuit with vertices listed cyclically as g_1, $g_1 g_2$, g_2, g_1^{-1}, $g_1^{-1} g_2^{-1}$, g_2^{-1} if $o(g_2) > 3$;*

(ii) *the subgraph in* (i) *together with the edge $[g_2, g_2^{-1}]$ if $o(g_1) > o(g_2) = 3$;*

(iii) *the subgraph in* (ii) *together with edges $[g_1, g_1^{-1}]$ and $[g_1 g_2, g_1^{-1} g_2^{-1}]$ if $o(g_1) = o(g_2) = 3$.*

b) *If $C_2(g_1) \subseteq H$, then $C_2(g_1)$ induces the following subgraph in X_e:*

(i) *a 3-circuit if $o(g_1) = 4$;*

(ii) *a complete graph on its 4 vertices if $o(g_1) = 5$;*

(iii) *the 4-circuit $[g_1^{-2}, g_1^{-1}, g_1, g_2]$ if $o(g_1) = 6$;*

(iv) *the path $[g_2^{-2}, g_2^{-1}, g_1, g_2]$ if $o(g_1) \geq 7$.*

c) *If* $C_3(g_1) \subseteq H$, *then* $C_3(g_1)$ *induces the following subgraph* X_e:

(i) *the complete graph on its* 5 *vertices if* $o(g_1) = 6$;

(ii) *the complete graph on its* 6 *vertices if* $o(g_1) = 7$;

(iii) *a subgraph with at least* 9 *edges in which* g_1 *and* g_1^{-1} *have valence at least* 4 *if* $o(g_1) \geqq 8$.

The way has now been paved for the

PROOF of Theorem 1.

Let G be identified with $(m_1, \ldots, m_r)$. Due to Proposition 2.3, we may assume that $r \geqq 2$.

We first dispose of two special cases. First let $G = (4, 2, 2, 2)$, and consider the neighborhood graph X_e induced by

$$\{a_1^{\pm 1}, a_2, (a_1a_2)^{\pm 1}, a_1^2a_2, a_3, (a_1a_3)^{\pm 1}, a_2a_3, a_4, (a_1a_4)^{\pm 1}\}$$

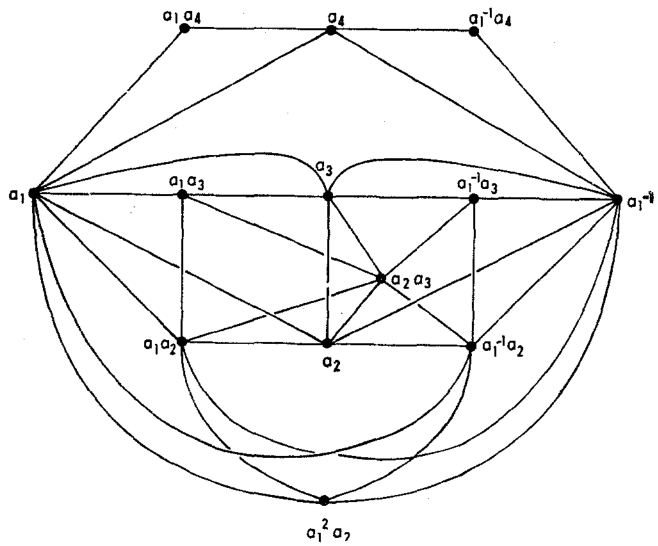

Fig. 2. MRR for (4, 2, 2, 2)

(see Figure 2). Let $\varphi \in A_e(X)$. Considering the restriction of φ to X_e, one sees that $\varphi(a_1) \in \{a_1, a_1^{-1}\}$ since these are the only 8-valent vertices of X_e. Suppose $\varphi(a_1) = a_1$. We note that $\varphi(a_2) = a_2$ since this is the only 6-valent vertex of X_e with no 4-valent neighbor. Similarly a_2a_3 is fixed by φ since it is the only 6-valent vertex of X_e with no 8-valent neighbor. Finally, φ fixes a_1a_4 since it is the only 2-valent neighbor of a_1. With $K = \{a_1, a_2, a_2a_3, a_1a_4\}$, we may conclude by Corollary 2.13 that $c(G) = 2$.

Next let $G = (4, 4, 2)$, and consider its neighborhood graph X_e induced by

$$C(a_1, a_2) \cup C_2(a_1) \cup \{a_3, a_2a_3, a_2^{-1}a_3\}$$

(see Figure 3). Since a_3 is the only vertex on two 3-circuits of X_e, it is fixed under $A(X_e)$. Its two 4-valent neighbors a_2, a_2^{-1} are thus either fixed or interchanged. Clearly, if a_2 is fixed, then so is a_1, and by Corollary 2.13, $c(G) = 2$.

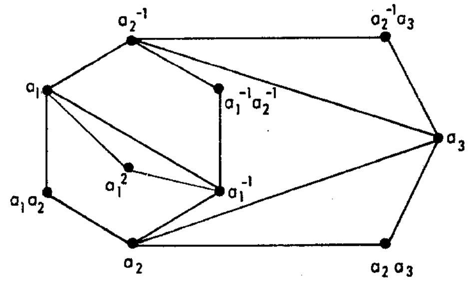

Fig. 3. MRR for (4, 4, 2)

In the light of these two examples and Corollary 2.10, we may safely assume that $m_r > 2$. The remaining argument will fall into five cases according to the values of m_1 and m_r, but the basic argument in each case will be essentially the same as in the two foregoing examples: a set H is proposed and the neighborhood graph X_e of $X_{G,H}$ is considered in order to show that the hypothesis of 2.13 is satisfied. The valence in X_e will be denoted by ϱ. We mention that $\varrho(h) = \varrho(h^{-1})$ for all $h \in H$. Indeed, h is incident with an edge of X_e for each relation $h = h_1h_2$ that holds for $h_1, h_2 \in H$. But then $h^{-1} = h_2^{-1}h_1^{-1}$. (This holds also when G is not abelian.)

Case 1: $m_1 \geqq 6$ and $m_r > 3$. We let

$$H = C_3(a_1) \cup \bigcup_{i=1}^{r-1} C(a_i, a_{i+1}) \cup C_2(a_r).$$

From Lemma 3.1 we compute:

$$\begin{aligned} \varrho(a_1) &\geqq \varrho(a_1^j) + 2, && j = \pm 2, \pm 3 \\ \varrho(a_1) &= 6 \text{ or } 7, \\ \varrho(a_i a_{i+1}) &= 2, && 1 \leqq i \leqq r-1 \\ \varrho(a_i) &= 4, && 2 \leqq i \leqq r-1 \\ \varrho(a_r) &= 4 \text{ or } 5, \\ \varrho(a_r^2) &\leqq 3. \end{aligned}$$

If $\varphi \in A(X_e)$, then $\varphi(a_1) \in \{a_1, a_1^{-1}\}$. Suppose $\varphi(a_1) = a_1$. Proceeding inductively, we note that if $\varphi(a_j) = a_j$ for $j \leqq i$, then $\varphi(a_{i+1}^{-1}) = a_{i+1}^{-1}$ since a_{i+1}^{-1} is the vertex of largest valence adjacent to a_i not already shown to be fixed by φ. By Proposition 2.12 we have $\varphi(a_{i+1}) = a_{i+1}$. Since $\langle a_1, \ldots, a_r\rangle = G$, the conclusion follows from Corollary 2.13.

Case 2: $m_1 \geqq 6$ and $m_r = 3$. Let us suppose that $m_1 \geqq \ldots \geqq m_q \geqq 6$ and $m_{q+1} = \ldots = m_r = 3$, where $1 \leqq q \leqq r-1$. (Recall that $m_{q+1} | m_q$.) We let

$$H = C_3(a_1) \cup \bigcup_{i=1}^{q} C(a_i, a_{i+1}) \cup \bigcup_{j=0}^{r-q-1} C(a_q a_{q+1} \ldots a_{q+j}, a_{q+j+1}).$$

From Lemma 3.1 we compute $\varrho(a_1^j)$ as in Case 1, but now

$$\begin{aligned} \varrho(a_i) &= 4 & 2 &\leqq i \leqq q \\ \varrho(a_i) &= 3 & q+1 &\leqq i \leqq r \\ \varrho(a_i a_{i+1}) &= 2 & 1 &\leqq i \leqq q-1 \\ \varrho(a_q a_{q+1} \ldots a_{q+j}) &= 4 & 1 &\leqq j \leqq r-q-1 \\ \varrho(a_q a_{q+1} \ldots a_r) &= 2. \end{aligned}$$

An induction argument proceeds as in Case 1 up to $i = q$. We then observe that if an element of $A(X_e)$ fixes a_q it also fixes $a_q a_{q+1}$, and continue by induction on j to obtain that $a_q a_{q+1} \ldots a_{q+j}$ is also fixed for $j = 1, \ldots, q-r$. Clearly G is generated by $\{a_1, \ldots, a_q, a_q a_{q+1}, \ldots, a_q a_{q+1} \ldots a_r\}$.

Case 3: $m_1 = 5$. Thus $m_1 = \ldots = m_r = 5$. The proof proceeds by induction on r. We begin with $r = 2$ and represent

$$Z_5^2 = \langle a, b : a^5 = b^5 = aba^{-1}b^{-1} = e\rangle.$$

Let H_2 consist of the six elements a, ba, ba^2, ba^3, ba^4, b^2a together with their inverses. Let $Y_2 = X_{Z_5^2, H_2}$ and let $\varphi \in A_e(X_2)$. Considering the restriction of φ to $(Y_2)_e$ (see Figure 4), one sees that $\varphi(ba^3) \in \{ba^3, b^4a^2\}$, since these are the

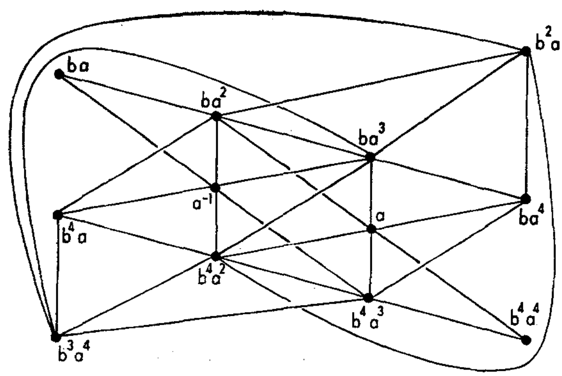

Fig. 4. MRR for (5, 5)

only 7-valent vertices of $(Y_2)_e$. Suppose $\varphi(ba^3) = ba^3$. Then $\varphi(ba^4) = ba^4$, since ba^4 is the only 4-valent neighbor of ba^3 in $(Y_2)_e$. With $K = \{ba^3, ba^4\}$, we conclude by Corollary 2.13 that $c(Z_5^2) = 2$ and Y_2 is an MRR of Z_5^2.

Now let $r \geqq 2$, suppose that $c(Z_5^r) = 2$, and let $Y_r = X_{Z_5^r, H_r}$ be an MRR of Z_5^r. Without loss of generality it may be assumed that $|H_r| < 5^r/2$. Since $5^r > 5^r/2 + 11$ for $r \geqq 2$, we apply 2.14 to conclude that $c(Z_5^{r+1}) = 2$.

We remark that HETZEL [2] has independently determined all MRR's of Z_5^2. Using other techniques, IMRICH [7, Theorem 4.5] showed that $c(Z_5^n) = 2$ for $n \geqq 4$.

Case 4: $m_1 = 4$ and $r \geqq 3$. Under our assumptions, $m_1 = \ldots = m_r = 4$. We let

$$H = C_2(a_1) \cup C_2(a_1 a_2) \cup \bigcup_{i=1}^{r-1} C(a_i, a_{i+1}) \cup C_2(a_r) \cup \{a_1^2 a_r^2\}.$$

Again, using Lemma 3.1, we compute:

$$\varrho(a_i) = 4 \qquad\qquad 1 \leqq i \leqq r$$

$$\varrho(a_i a_{i+1}) = \begin{cases} 4, & i = 1 \\ 2, & 2 \leqq i \leqq r-1 \end{cases}$$

$$\varrho(a_1^2) = \varrho(a_r^2) = 4$$

$$\varrho(a_1^2 a_2^2) = \varrho(a_1^2 a_r^2) = 2.$$

The vertices a_1 and a_1^{-1} are the only 4-valent vertices all four of whose neighbors are each 4-valent. Hence $\varphi(a_1) \in \{a_1, a_1^{-1}\}$ for all $\varphi \in A(X_e)$. Suppose $\varphi(a_1) = a_1$. The only neighbor of a_1 lying on no 3-circuit is a_2^{-1}. We conclude that $\varphi|_{\langle K \rangle} = 1_{\langle K \rangle}$ where $K = \{a_1, a_2\}$. We now suppose that $\varphi(a_j) = a_j$ for all $j \leqq i$, where $i \geqq 2$, and continue inductively as in Case 1.

Case 5: $m_1 = 3$ and $r \geqq 4$. Thus $m_1 = \ldots = m_r = 3$. Because of 2.14 (i), it suffices to demonstrate an MRR for Z_3^4. To simplify notation, let $\{a,b,c,d\}$ be a generating set for $G_1 = Z_3^4$ instead of the standard $\{a_1, a_2, a_3, a_4\}$, and let $G = \langle a, b, c \rangle$. Define

$$H = G \setminus \{a, c, a^2, c^2, e\},$$

$$H_1 = G \cup \{cd, d, ad, abd, abc^2d\}^{\pm 1}$$

and set $X = X_{G,H}$, $Y = X_{G_1,H_1}$. By [4, Corollary 1] the cosets of G are blocks of $A(X_{G_1,H_1})$, i.e., every element of $A_e(Y)$ stabilizes G and maps dG into itself or interchanges dG with d^2G.

We note that any element of $A_e(X)$ fixes $\{a, a^2, c, c^2\}$ setwise and, moreover, either stabilizes the subsets $\{a, a^2\}$ and $\{c, c^2\}$ or interchanges them. The neighbors of a in $dG \cap H_1$ and $d^2G \cap H_1$ are ad and d^2, respectively, while a^2 has the neighbors d and a^2d^2 in $dG \cap H_1$ and $d^2G \cap H_1$, respectively. The neighbors of c lying in $dG \cap H_1$ are abd and cd; those in $d^2G \cap H_1$ are $a^2b^2d^2c$ and d^2. Finally, c^2 has the neighbors $abdc^2$ and d in $dG \cap H_1$ and $a^2b^2d^2$, c^2d^2 in $d^2G \cap H_1$.

As c has more neighbors in $H_1 \setminus G$ than a does, the elements of $A_e(Y)$ fix $\{a, a^2\}$ and $\{c, c^2\}$ setwise. Since d and d^2 are the only elements of $H_1 \setminus G$ with two neighbors in $\{a, a^2, c, c^2\}$, it is clear that every $\varphi \in A_e(Y)$ stabilizes $\{d, d^2\}$. Suppose $\varphi(d) = d$. Since d has the neighbors a^2, c^2 in $G \setminus H$, which cannot be interchanged, φ fixes each of a, a^2, c, c^2. As c^2 has only the neighbor abc^2d besides d in $dG \cap H_1$, the element abc^2d is also fixed. Now an application of 2.13 shows that Y is an MRR of G_1.

Remark. An MRR of Z_3^5 can be obtained from the one given in [7] for Z_3^6 by an application of [17, Lemma 2.7]. If X is the just-mentioned MRR for Z_3^6 we only have to set $N = \langle a_1 a_3^{-1} a_5^{-1} a_6 \rangle$ in order to obtain an MRR X/N of Z_3^5. This method does not work for Z_3^4.

4. On the Cayley index of generalized dicyclic groups

In [8] we defined a graph X to belong to the class $\mathcal{S}_{n,q}$ ($n \geqq 2$; $q \geqq 1$) if it is isovalent and if the set $V(X)$ admits a partition $\{V_1, V_2, \ldots, V_p\}$ with $2 \leqq p \leqq n$ such that every vertex in V_i is adjacent to at most q vertices in V_j for $j \neq i$. It was shown [8, Corollary 1B] that

4.1. *If* $X \in \mathcal{S}_{n,q}$ *and* $|V(X)| > qn(n+2)$, *then* $X' \notin \mathcal{S}_{n,q}$.

Proof of Theorem 2. Let G be a generalized dicyclic group generated by the abelian group L and an element b as in the definition in § 1 above, and let β denote the automorphism of G also given in § 1. Clearly, the subgroup L must have even order. Since G is non-abelian, L is not an elementary abelian 2-group.

To prove (a) let it be assumed that L is neither (4,4) nor (4,2) nor (4,2,2). By Theorem 1, L admits an MRR $W = X_{L,J}$ such that $A_e(W) = \{1_L, \alpha\}$. Let $H = J \cup \{b, b^{-1}\}$ and let $X = X_{G,H}$. It will be shown that $A_e(X)$ can be embedded in Z_2^2. We first prove

$$\varphi[L] = L \quad \text{for all} \quad \varphi \in A_e(X). \tag{4.1}$$

Clearly, (4.1) is equivalent to the condition that $\varphi[bL] = bL$ for all $\varphi \in A_e(X)$.

If $|G| > 32$, then $|L| > 16$, and by 4.1, either W or W' is not in $\mathcal{S}_{2,2}$ and (4.1) follows immediately.

If $|G| \leq 32$, then $|L| \leq 16$. Under our assumptions, L is the cyclic group Z_{2m} $(3 \leq m \leq 8)$ or L is (6,2) or (8,2). If $L = \langle a \rangle$, let $J = \{a, a^{-1}\}$ whence $W = C_{2m}$ (cf. Lemma 2.3). We compute that the edges $[e, a^{\pm 1}]$ each lie on precisely two 4-circuits. (For the edge $[e, a]$ these are defined by $\{a, e, ba^{-1}b\}$ and $\{e, a, b^{-1}a^{-1}, b^{-1}\}$.) The edges $[e, b^{\pm 1}]$ each lie on precisely three 4-circuits. (For the edge $[e, b]$ these are defined by $\{e, a, ba^{-1}, b\}$, $\{e, a^{-1}, ba, b\}$ and $\{e, b^{-1}, b^2, b\}$.)

If L is (6,2) or (8,2), then b^2 is one of a_1^m, a_2, and $a_1^m a_2$ where $o(a_1) = 2m$. If $b^2 = a_1^m$, then $G = \langle a_1, b \rangle \times \langle a_2 \rangle$. By Theorem 1, $c(\langle a_1, b \rangle) = 2$, and so $c(G) = 2$ by 2.9 and 2.8. Now suppose $b^2 = a_2$. (The alternative $b^2 = a_1^m a_2$ is equivalent under the automorphism $a_1 \to a_1$, $a_2 \to a_1^m a_2$ of the group L.) Let $J = \{a_1, a_1^{-1}, a_2\}$. Using 2.3 and 2.1, the reader can readily verify that $X_{L,J}$ is an MRR of L. This time, however, let W be the complement of $X_{L,J}$, and note that every edge with both vertices in L or both vertices in bL lies on a 3-circuit. No edge with one vertex in L and one vertex in bL has this property; for if $[g, gb]$ were such an edge, then the other two edges on the 3-circuit would have to be $[gb, gb^2]$ and $[g, gb^2]$ or $[gb^{-1}, g]$ and $[gb^{-1}, gb]$. But $[g, gb^2]$, $[gb^{-1}, gb] \notin E(X)$ since $b^2 = a_2 \notin H$. Since in every instance W is a connected graph, (4.1) follows.

Now let $\varphi \in A_e(X)$. By (4.1), either $\varphi|_L = 1_L$ or $\varphi|_L = \alpha$.

First suppose $\varphi|_L = 1_L$. Since $\varphi(b) \in \{b, b^{-1}\}$, we begin by supposing $\varphi(b) = b$, i.e., $\varphi|_{bL}$ has a fixed-point, and so $\varphi|_{bL} = 1_{bL}$ or $\varphi|_{bL} = \lambda_b \alpha \lambda_{b^{-1}}$. In the former case $\varphi = 1_G$; in the latter case $\varphi(bx) = bx^{-1}$ for all $x \in L$. By our assumptions on L, there exists an $x \in L$ such that $x^2 \neq e, b^2$. In particular, if $L = Z_4 \times Z_2^m$ for some $m \geq 3$, then $b^2 \neq a_1^2$, or else G would be isomorphic to $Q \times Z_2^m$. Hence $x = a_1$. If $\varphi(bx) = bx^{-1}$, then the neighbors in L of bx, namely bxb and x^{-1}, must coincide with the neighbors in L of bx^{-1}, namely $bx^{-1}b$ and x. But clearly $x \neq x^{-1}$, while $x = bxb$ implies $x^2 = (bx)^2 = b^2$, which is a contradiction. Now suppose that $\varphi(b) = b^{-1}$. Then $(\beta\varphi)|_L = 1_L$ and $\beta\varphi(b) = b$. It follows by what we have just shown that $\beta\varphi = 1_G$. Since $\beta^2 = 1_G$, we have $\varphi = \beta$.

On the other hand, suppose $\varphi|_L = \alpha$. Again $\varphi(b) \in \{b, b^{-1}\}$, and we suppose first that $\varphi(b) = b$. Since $A_b(X \setminus W)$ is isomorphic as a permutation group to $\{1_L, \alpha\}$, there are at most two possibilities γ_1 and γ_2 for φ. Clearly, $\gamma_1^2 = \gamma_2^2 = 1_G$, $(\gamma_2\gamma_1)|_L = 1_L$ and $\gamma_2\gamma_1(b) = b$. Hence as before, $\gamma_2\gamma_1 = 1_G$. Under the given assumptions, there exists at most one such automorphism, which, if it exists, we denote by γ. Finally, suppose $\varphi(b) = b^{-1}$. Then $(\beta\varphi)|_L = \alpha$ and $\beta\varphi(b) = b$. Hence $\beta\varphi = \gamma$ and so $\varphi = \beta\gamma$. Thus $\varphi \in A_e(X)$ if and only if $\gamma \in A_e(X)$. We have shown that $A_e(X) \leq \{1_G, \beta, \gamma, \beta\gamma\} \cong Z_2^2$, completing the proof of (i) in part (a).

To prove (ii), suppose $|G| > 96$, in which case $|L| > 48$. Since, moreover, G is not of the form $Q \times Z_2^m$, one verifies by exhaustion that L must contain some two distinct elements s_1 and s_2 subject to all the following conditons for $i, j \in \{1, 2\}$ and $i \neq j$:

$$s_i^2 \neq e, \tag{4.2}$$

$$s_i s_j \neq e, \tag{4.3}$$

$$s_i^2 \neq b^2, \tag{4.4}$$

$$s_i \neq s_j^2, \tag{4.5}$$

$$s_i s_j \neq b^2, \tag{4.6}$$

$$s_i \neq b^2 s_j^2. \tag{4.7}$$

Let W be as before, but redefine

$$H = J \cup \{b, b^{-1}, bs_1, b^{-1}s_1, bs_2, b^{-1}s_2\} \tag{4.8}$$

and let $X = X_{G,H}$. Arbitrarily choose $\varphi \in A_e(X)$. We apply 4.1 again with $n = 2$ and $q = 6$. Since $|L| > 48$, we may assume that $W \notin \mathcal{S}_{2,6}$. Thus $\varphi[L] = L$ and $\varphi[bL] = bL$. If $\varphi(b) \notin \{b, b^{-1}\}$, then we may assume without loss of generality that $\varphi(b) = bs_1$. There are precisely two possibilities for the action of φ on bL, namely $\lambda_{s_1^{-1}}$ and $\lambda_{bs_1} \alpha \lambda_{b^{-1}}$; that is, either

$$\varphi(bx) = bxs_1 \quad \text{for all} \quad x \in L \tag{4.9}$$

or

$$\varphi(bx) = bx^{-1}s_1 \quad \text{for all} \quad x \in L, \tag{4.10}$$

respectively.

If (4.9) holds, then $\varphi(bs_1) = bs_1^2$ must be one of the six neighbors of e in the right-hand member of (4.8). By (4.2), $bs_1^2 \neq b$ or $(bs_1)^{\pm 1}$, and by (4.4) $bs_1^2 \neq b^{-1}$. Further, by (4.5) $bs_1^2 \neq bs_2$ and by (4.7), $bs_1^2 \neq b^{-1}s_2$. On the other hand, if (4.10) were to hold, we would apply the same argument to $\varphi(bs_2) = bs_2^{-1}s_1$. Since $s_1 \neq s_2$, this image is not b, and by (4.3), it is not b^{-1}. By (4.2) we can eliminate $(bs_1)^{\pm 1}$, and by (4.5), $bs_2^{-1}s_1 \neq bs_2$. Finally, $bs_2^{-1}s_1 \neq b^{-1}s_2$ by (4.7). Hence $\varphi(b) \in \{b, b^{-1}\}$.

If $\varphi(b) = b$ and if $\varphi \neq 1_G$, then as before $\varphi(bx) = bx^{-1}$ for all $x \in L$. We now consider the image $\varphi(bs_1) = bs_1^{-1}$ which, since $\varphi^2 = 1_G$, cannot equal b. By (4.2) it is neither b^{-1} nor bs_1. By (4.4), $bs_1^{-1} \neq b^{-1}s_1$, and by (4.2) one may eliminate bs_2. Finally, by (4.6), $bs_1^{-1} \neq b^{-1}s_2$.

We have thus shown that either $\varphi = 1_G$ or $\varphi(b) = b^{-1}$. (The reader should be certain to note that this time it was not necessary to consider two cases depending upon the behavior of φ on L.) To conclude, suppose $\varphi(b) = b^{-1}$. Then $\beta\varphi(b) = b$, and by what we have just shown, $\beta\varphi = 1_G$. Hence $\varphi = \beta$, and $A_e(X) = \{1_G, \beta\} \cong Z_2$.

To prove (b), we let $G = Q \times Z_2^n$ and proceed by induction on n. When $n = 0$, $c(G) = 16$ by Lemma 2.6. If $c(Q \times Z_2^n) \leq 16$, then by 2.8 and 2.9, $c(Q \times Z_2^{n+1}) \leq c(Q \times Z_2^n)\, c(Z_2) \leq 16$.

REFERENCES

[1] C. Y. CHAO, On a theorem of Sabidussi, *Proc. Amer. Math. Soc.* **15** (1964), 291—292 *MR* **28** # 2538

[2] D. HETZEL, private communication.

[3] W. IMRICH, Graphs with transitive abelian automorphism groups, *Combinatorial Theory and its Applications* (Colloq. Math. Soc. J. Bolyai 4), Proc. Colloq. Balatonfüred, August 24—29, 1969, edited by P. Erdős, A. Rényi and Vera T. Sós, North-Holland Publishing Co., Amsterdam—London, 1970, 651—656. (Vol. I: 390 pp.; Vol. II: IV + pp. 395—799; Vol. III: III + pp. 803—1201) *Zbl* **206**, 262; (*MR* **45** # 4981)

[4] W. IMRICH, On graphs with regular groups, *J. Combinatorial Theory Ser. B* **19** (1975), 174—180. *Zbl* **282.** 05107 (**305.** 05112)

[5] W. IMRICH, On products of graphs and regular groups, *Israel J. Math.* **11** (1972), 258—264. *MR* **47** # 3234

[6] W. IMRICH, Graphical regular representations of groups of odd order, *Combinatorics* (Colloq. Math. Soc. J. Bolyai **18**), Proc. Colloq. Keszthely, 1976, North-Holland Publishing Co., Amsterdam—London (to appear).

[7] W. IMRICH, On graphical regular representations of groups, *Infinite and Finite Sets* (Colloq. Math. Soc. J. Bolyai **10**), Proc. Colloq. Keszthely, June 25—July 1, 1973, edited by A. Hajnal, R. Rado and Vera T. Sós, North-Holland Publishing Co., Amsterdam—London, 1975, 905—925. (Vol. I: 604 pp; Vol. II: pp. 605—1080; Vol. III: pp. 1081—1555) (*MR* **50** # 12526).

[8] W. IMRICH and M. E. WATKINS, On graphical regular representations of cyclic extensions of groups, *Pacific J. Math.* **55** (1974), 461—477. *Zbl.* **279.** 05114 (**298.** 05128)

[9] L. A. NOWITZ, On the non-existence of graphs with transitive generalized dicyclic groups, *J. Combinatorial Theory* 4 (1968), 49—51. *MR* **36** # 1359

[10] L. A. NOWITZ and M. E. WATKINS, Graphical regular representations of non-abelian groups, I, *Canad. J. Math.* **24** (1972), 993—1008. *MR* **47** # 8345

[11] L. A. NOWITZ and M. E. WATKINS, Graphical regular representations of non-abelian groups, II. *Canad. J. Math.* **24** (1972), 1009—1018. *MR* **47** # 8345

[12] G. SABIDUSSI, Graph multiplication, *Math. Z.* **72** (1959/60), 446—457. *MR* **35** # 80

[13] G. SABIDUSSI, On a class of fixed-point-free graphs, *Proc. Amer. Math. Soc.* **9** (1958), 800—804. *MR* **20** # 3548

[14] G. SABIDUSSI, Vertex-transitive graphs, *Monatsh. Math.* **68** (1964), 426—438. *MR* **31** # 91

[15] M. E. WATKINS, On the action of non-Abelian groups on graphs, *J. Combinatorial Theory Ser. B* **11** (1971), 95—104. *MR* **43** # 6136

[16] M. E. WATKINS, The state of the GRR problem, *Recent Advances in Graph Theory* (Proc. Second Czechoslovak Sympos. Prague, June 1974), edited by Miroslav Fiedler, Academia, Prague, 1975, 517—522 (see *MR* **51** # 217).

[17] M. E. WATKINS, Graphical regular representations of free products of groups, *J. Combinatorial Theory* **21** (1976), 47—56.

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