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ON A CONJECTURE OF FEJES TÓTH

by

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FEJES TÓTH [1] made the following conjecture. "If in a packing of translates of a square each square has at least six neighbours then the density of the packing is at least 11/15." Here a "square" is, for instance, the square $0 < x < 1$, $0 < y < 1$, a "packing" is a nonempty family of pairwise disjoint sets and, finally, two sets are said to be "neighbours" if their closures have a nonempty intersection. FEJES TÓTH has constructed a packing of density 11/15 which satisfies his requirements (see Fig. 1) and observed that every packing satisfying his requirements has density at least 2/3. It is not difficult to show [2] that in the investigation of this problem we can restrict ourselves to squares forming a grid, i.e., a set of squares joining along whole sides and filling the plane completely. For grids, HANANI improved the lower bound into 5/7 and restated the conjecture as follows. "In the planar square grid, color the squares blue and red, so that (a) there is at least one blue square and (b) each blue square has at least six blue neighbours. Then the density of the set of blue squares is at least 11/15." We shall prove this conjecture.

By the *order* of a square S we shall mean the number of the neighbours of S having the same color as S. By (b), there are no blue squares of order smaller than six. Moreover, there are no red squares of order greater than three. (This has been also observed by Hanani.) Let B_i (resp. R_i) be the set of all the blue (resp. red) squares of order i; let b_i (resp. r_i) be the density of B_i (resp. R_i). Then evidently

$$r_0 + r_1 + r_2 + r_3 + b_6 + b_7 + b_8 = 1.$$

Moreover, counting the red-blue connections we obtain

$$8r_0 + 7r_1 + 6r_2 + 5r_3 = 2b_6 + b_7.$$

Now, Hanani's bound follows since

$$\begin{aligned}7(b_6 + b_7 + b_8) &\geq -3r_0 - 2r_1 - r_2 + 7b_6 + 6b_7 + 5b_8 = \\ &= 5(r_0 + r_1 + r_2 + r_3 + b_6 + b_7 + b_8) - \\ &- (8r_0 + 7r_1 + 6r_2 + 5r_3 - 2b_6 - b_7) = 5.\end{aligned}$$

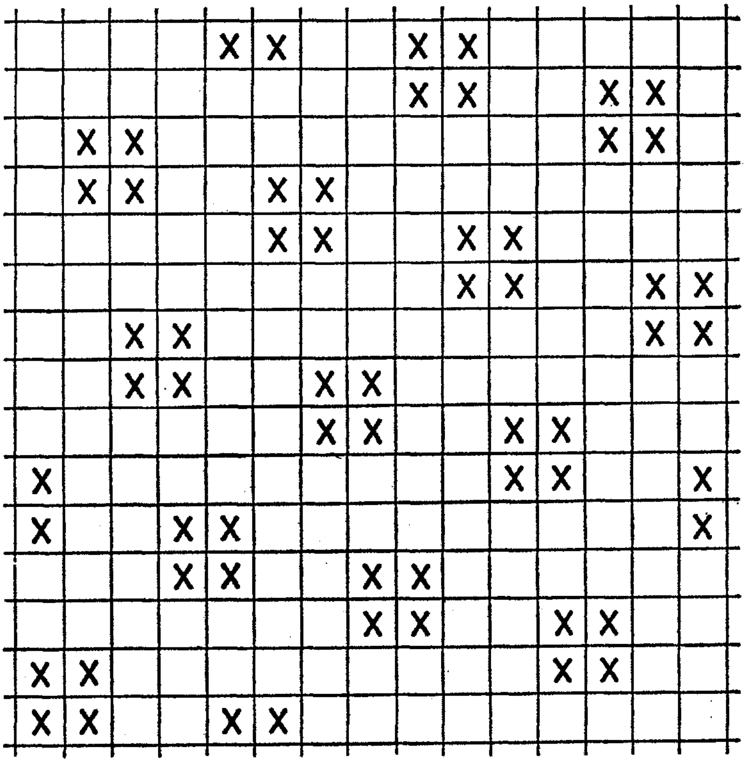

Fig. 1

To prove $b_6 + b_7 + b_8 \geq 11/15$, it will be enough to show that

$$(*) \qquad r_3 \leq 4r_0 + 2r_1 + 2b_7 + 4b_8,$$

for then

$$15(b_6 + b_7 + b_8) \geq -r_0 - r_1 - r_2 + 15b_6 + 15b_7 + 15b_8 =$$

$$= 11(r_0 + r_1 + r_2 + r_3 + b_6 + b_7 + b_8) -$$

$$- 2(8r_0 + 7r_1 + 6r_2 + 5r_3 - 2b_6 - b_7) +$$

$$+ (4r_0 + 2r_1 - r_3 + 2b_7 + 4b_8) \geq 11$$

as desired.

To prove (*), we first observe that the red squares of order three come in two by two quadruples. Let Q denote the set of these quadruples; set $q = r_3/4$. It will suffice to construct a bipartite graph G together with disjoint subsets S, T of R_1 such that

(i) the bipartition of G consists of Q in one part and $R_0 \cup S \cup T \cup \cup B_7 \cup B_8$ in the other,

(ii) each element of $S \cup B_7 \cup B_8$ has degree at most one in G, each element of $R_0 \cup T$ has degree at most two in G,

(iii) if some A with $A \in Q$ is adjacent to ϱ_0 elements of R_0, to σ elements of S, to τ elements of T, to β_7 elements of B_7 and to β_8 elements of B_8 then

$$\frac{1}{2}\varrho_0 + \frac{1}{2}\sigma + \frac{1}{4}\tau + \frac{1}{2}\beta_7 + \beta_8 \geq 1,$$

(iv) if some A with $A \in Q$ is adjacent to w then w comes from the 8×8 square centered at A.

Suppose for the moment that we have constructed G. Assign to each edge of G the weight

1 if the edge has an endpoint in B_8,

1/2 if the edge has an endpoint in $R_0 \cup S \cup B_8$,

1/4 if the edge has an endpoint in T.

Then, for every A with $A \in Q$, the sum of the weights of edges incident with A is at least one. The corresponding sum for each element of $R_0 \cup B_8$ is at most one and, for each element of $R_1 \cup B_7$, the sum is at most 1/2. Counting now the weights of the edges of G, we obtain

$$q \leq r_0 + \frac{1}{2}r_1 + \frac{1}{2}b_7 + b_8$$

and (*) follows.

It remains to construct G.

Step 1. If a quadruple is a part of one of the five configurations in Figure 2 (or its image under a rotation), join it (by an edge of G) to the element of B_8 indicated by the arrow. (Do this whenever applicable, then go on to the next step.)

Step 2. Call a quadruple *available* if it is adjacent to no element of B_8. If an available quadruple is a part of one of the three configurations in Figure 3 (or its image under a rotation), join it to the element of B_7 indicated by the star.

Step 3. If an available quadruple is a part of the configuration in Figure 4 (or its image under a rotation), join it to the element of R_0 indicated by the star.

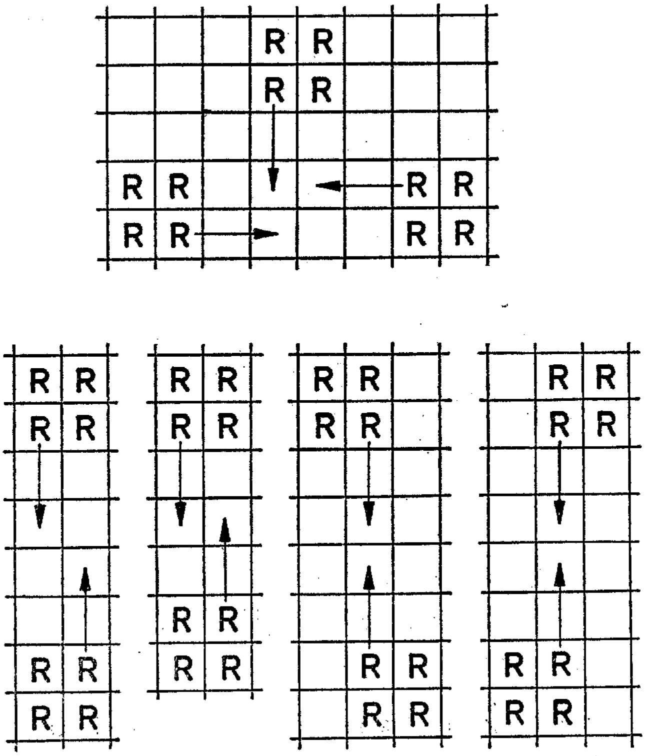

Fig. 2

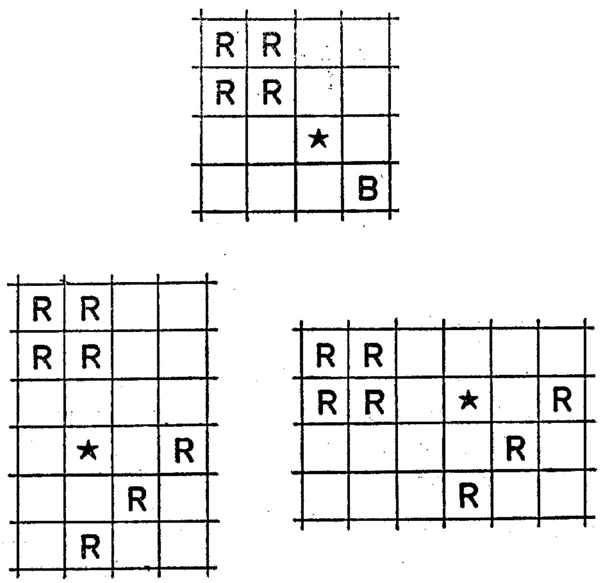

Fig. 3

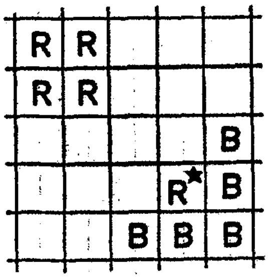

Fig. 4

Step 4. If an available quadruple is a part of one of the three configurations in Figure 5 (or its image under a rotation), join it to the element of R_1 indicated by the star and put this element into S.

Step 5. If an available quadruple is a part of one of the three configurations in Figure 6 (or its image under a rotation), join it to the two elements of R_1 indicated by the stars and put these two elements into T.

It is a little messy, but manageable, to show that the resulting graph has all of the required properties. The proof is finished.

Note that (*) actually implies that every optimal coloring has $r_0 = r_1 = r_2 = 0$. Under this assumption, an argument similar to the above (but much simpler) yields $r_3 \leq 2b_7 + b_8$. Then

$$15(b_6 + b_7 + b_8) - 3b_8 = 15b_6 + 15b_7 + 12b_8 =$$
$$= 11(r_3 + b_6 + b_7 + b_8) - 2(5r_3 - 2b_6 - b_7) - (r_3 - 2b_7 - b_8) \geq 11 .$$

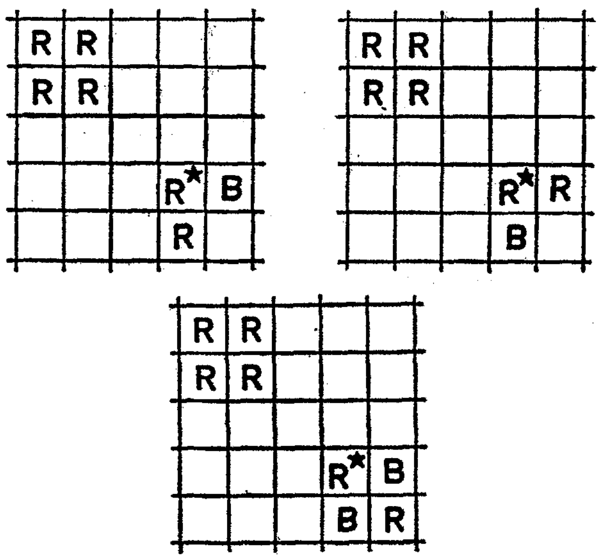

Fig. 5

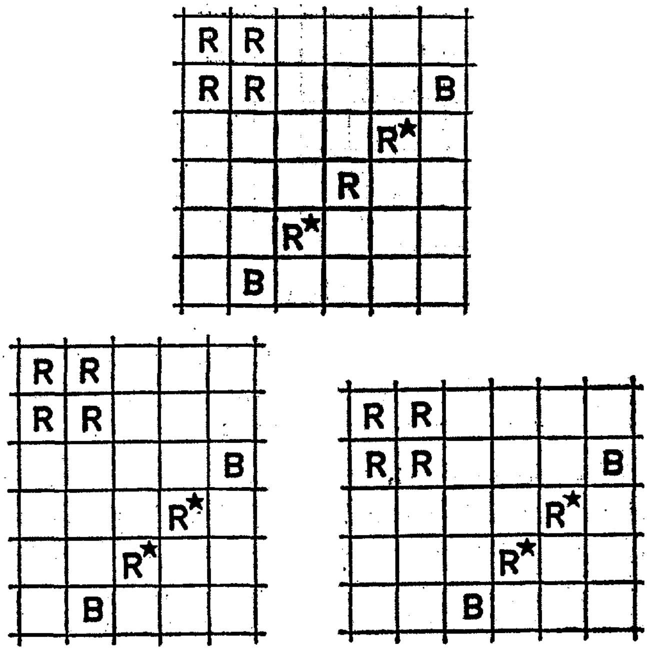

Fig. 6

Hence every optimal coloring satisfies $r_0 = r_1 = r_2 = b_8 = 0$, $r_3 = 4/15$, $b_7 = 2/15$ and $b_6 = 9/15$. Now, it follows that Fejes Tóth's coloring is unique in a sense.

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REFERENCES

[1] L. Fejes Tóth, Five-neighbour packing of convex discs, *Period. Math. Hungar.* 4 (1973), 221—229. *MR* 49 # 9745

[2] L. Fejes Tóth and N. Sauer, Thinnest packing of cubes with a given number of neighbours (to appear).

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