BIT 30 (1990), 358–375

ON ERROR BEHAVIOUR OF PARTITIONED LINEARLY IMPLICIT RUNGE-KUTTA METHODS FOR STIFF AND DIFFERENTIAL ALGEBRAIC SYSTEMS

K. STREHMEL, R. WEINER and I. DANNEHL

Sektion Mathematik, Martin-Luther-Universität Halle-Wittenberg, Weinbergweg 17, DDR-4050 Halle, German Democratic Republic.

Abstract.

This paper studies partitioned linearly implicit Runge-Kutta methods as applied to approximate the smooth solution of a perturbed problem with stepsizes larger than the stiffness parameter ε. Conditions are supplied for construction of methods of arbitrary order. The local and global error are analyzed and the limiting case $\varepsilon \to 0$ considered yielding a partitioned linearly implicit Runge-Kutta method for differential-algebraic equations of index one. Finally, some numerical experiments demonstrate our theoretical results.

Subject Classifications: AMS(MOS) 65L05; CR:G1.7.

1. Introduction.

Consider the singularly perturbed initial value problem

$$\varepsilon z'(t) = g(t, y, z), \quad z(t_0) = z_0, \text{with } 0 < \varepsilon \ll 1 \tag{1.1a}$$

$$y'(t) = f(t, y, z), \quad y(t_0) = y_0 \tag{1.1b}$$

where $g\colon [t_0, t_e] \times \mathbb{R}^{n-N} \times \mathbb{R}^N \to \mathbb{R}^N$ and $f\colon [t_0, t_e] \times \mathbb{R}^{n-N} \times \mathbb{R}^N \to \mathbb{R}^{n-N}$ are sufficiently differentiable functions which may also depend smoothly on the small parameter ε. We suppose that the stiffness appears only in (1.1a). Furthermore, we assume that in a neighbourhood of the exact solution of (1.1) for a suitable inner product $\langle .,. \rangle$ the logarithmic norm of the Jacobian $\partial g/\partial z$ is strictly negative, i.e.

$$\left\langle \xi, \frac{\partial g(t, y, z)}{\partial z} \xi \right\rangle \leq \mu \|\xi\|^2 \quad \text{with } \mu \leq \mu_0 < 0, \quad \xi \in \mathbb{R}^N \tag{1.2}$$

where $\|x\|^2 := \langle x, x \rangle$. It is known (see [8]), that under the assumption (1.2) there exists a smooth solution of (1.1), i.e. all derivatives of $y(t), z(t)$ up to a sufficiently high

Received October 1988. Revised August 1989.

order are bounded independently of ε. All neighbouring solutions approach the smooth solution in a very small time. We suppose that the initial values y_0 and z_0 lie on the smooth solution. We will consider two cases:

1. The class F_ε of singularly perturbed systems ($0 < \varepsilon \ll 1$), that forms an important subclass of nonlinearly stiff systems.
2. The limiting case $\varepsilon \to 0$, i.e. the class F_0 of the reduced systems

$$\begin{aligned} 0 &= g(t, y, z), \qquad z(t_0) = z_0 \\ y'(t) &= f(t, y, z), \qquad y(t_0) = y_0 \end{aligned} \tag{1.3}$$

where the initial values are consistent, i.e. $g(t_0, y_0, z_0) = 0$.

Because of (1.2) this differential algebraic system is of index one. In (1.1) the stiff and nonstiff parts are separated, and we therefore use a partitioned linearly implicit Runge-Kutta method. This compound method consists of a customary explicit Runge-Kutta method for the nonstiff and of a linearly implicit Runge-Kutta method (see [12]) for the stiff subsystem. Therefore, at each integration step only systems of linear equations of dimension N must be solved.

The aim of this paper is to give convergence results for the class of partitioned linearly implicit Runge-Kutta methods when applied to the approximation of the smooth solution of (1.1) with stepsizes larger than the stiffness parameter ε. Because the Lipschitz constant of system (1.1) is of size $O(\varepsilon^{-1})$ the classical convergence theory for one-step methods (see e.g. [6]) is not applicable. We give a direct estimation of the local and global error and derive conditions so that the global error of a partitioned linearly implicit Runge-Kutta method satisfies $y_m - y(t_m) = O(h^q)$, $z_m - z(t_m) = O(h^q)$ for all ε with $\varepsilon \leq \text{const} \cdot h$. Here the constants in $O(\ldots)$ are independent of h, m and ε. These conditions enable us to construct methods of arbitrary order $q \geq 1$.

During the last years the numerical treatment of differential-algebraic systems has gained much interest (see [2]). According to [1] we consider the limiting case $\varepsilon \to 0$. This yields a partitioned linearly implicit Runge-Kutta method for differential-algebraic systems of index one. Order conditions are obtained from the conditions of the partitioned linearly implicit Runge-Kutta method for (1.1) putting $\varepsilon = 0$. Finally, some numerical examples demonstrate our theoretical results.

2. Partitioned linearly implicit Runge-Kutta methods for class F_ε.

For a general stiff initial value problem

$$\begin{aligned} &u'(t) = q(t, u(t)), \quad t \in [t_0, t_e] \quad q: [t_0, t_e] \times \mathbb{R}^n \to \mathbb{R}^n \\ &u(t_0) = u_0 \end{aligned}$$

the class of s-stage linearly implicit Runge-Kutta methods is defined by (see [12]):

$$u_{m+1}^{(1)} = u_m$$

(2.1)
$$u_{m+1}^{(i)} = R_0^{(i)}(c_i hT)u_m + h \sum_{j=1}^{i-1} A_{ij}(c_i hT)[q_j - Tu_{m+1}^{(j)}], \quad i = 2(1)s$$

$$u_{m+1} = R_0^{(s+1)}(hT)u_m + h \sum_{j=1}^{s} B_j(hT)[q_j - Tu_{m+1}^{(j)}]$$

with $q_j = q(t_m + c_j h, u_{m+1}^{(j)})$. The vector u_{m+1} approximates $u(t)$ at $t_{m+1} = t_m + h$, where h denotes the stepsize. $R_0^{(i)}(\xi)$ are rational approximations to $\exp(\xi)$ for $\xi \to 0$, $A_{ij}(\xi)$ and $B_j(\xi)$ are rational functions (coefficients of the method), c_i are real parameters ($c_1 = 0$) and T is an arbitrary (n, n)-matrix (usually an approximation to the Jacobian $\partial q/\partial u$ at (t_m, u_m)). Linearly implicit Runge-Kutta methods include the well known ROW- and W-methods and adaptive Runge-Kutta methods.

With the special choice of the matrix

(2.2)
$$T := \frac{1}{\varepsilon}\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$$

where T_1 is a (N, N)-matrix and T_2 a $(N, n - N)$-matrix (see [11]) we get from (2.1) the partitioned linearly implicit Runge-Kutta method

$$z_{m+1}^{(1)} = z_m, \qquad y_{m+1}^{(1)} = y_m$$

$$z_{m+1}^{(i)} = R_0^{(i)}\left(c_i \frac{h}{\varepsilon} T_1\right) z_m + c_i \frac{h}{\varepsilon} R_1^{(i)}\left(c_i \frac{h}{\varepsilon} T_1\right) T_2 y_m + \frac{h}{\varepsilon} \sum_{j=1}^{i-1} A_{ij}\left(c_i \frac{h}{\varepsilon} T_1\right) G_j$$
$$+ h \sum_{j=1}^{i-1} \left[A_{ij}\left(c_i \frac{h}{\varepsilon} T_1\right) - a_{ij} I\right] T_1^{-1} T_2 f_j$$

(2.3)
$$y_{m+1}^{(i)} = y_m + h \sum_{j=1}^{i-1} a_{ij} f_j$$

$$z_{m+1} = R_0^{(s+1)}\left(\frac{h}{\varepsilon} T_1\right) z_m + \frac{h}{\varepsilon} R_1^{(s+1)}\left(\frac{h}{\varepsilon} T_1\right) T_2 y_m + \frac{h}{\varepsilon} \sum_{j=1}^{s} B_j\left(\frac{h}{\varepsilon} T_1\right) G_j$$
$$+ h \sum_{j=1}^{s} \left[B_j\left(\frac{h}{\varepsilon} T_1\right) - b_j I\right] T_1^{-1} T_2 f_j$$

$$y_{m+1} = y_m + h \sum_{j=1}^{s} b_j f_j$$

with $a_{ij} = A_{ij}(0)$, $b_j = B_j(0)$ and

$$G_j = g(t_m + c_j h, y_{m+1}^{(j)}, z_{m+1}^{(j)}) - T_1 z_{m+1}^{(j)} - T_2 y_{m+1}^{(j)}, \quad f_j = f(t_m + c_j h, y_{m+1}^{(j)}, z_{m+1}^{(j)}),$$

I denotes the (N, N)-identity matrix and $R_1^{(i)}(\xi)$ is defined by

(2.4)
$$R_1^{(i)}(\xi) = (R_0^{(i)}(\xi) - 1)/\xi.$$

We symbolize the partitioned linearly implicit Runge-Kutta method by the tableau

$$\begin{array}{c|cccccc} c_2 & A_{21} & & & & & \\ c_3 & A_{31} & A_{32} & & & & \\ \vdots & \vdots & & \ddots & & & \\ c_s & A_{s1} & \cdot & \cdot & \cdot & A_{s,s-1} & \\ \hline & B_1 & \cdot & \cdot & \cdot & B_{s-1} & B_s \end{array}$$

We assume

$$T_1 = \frac{\partial g}{\partial z}(t_m, y_m, z_m) + O(h) \text{ for } h \to 0. \tag{2.5}$$

Note, that condition (1.2) guarantees the regularity of the matrix T_1 for sufficiently small h.

For the choice of the matrix T_2 we consider two cases:

(2.6)
(i) T_2 is an arbitrary $(N, n - N)$-matrix, especially $T_2 = 0$.
(ii) $T_2 = \dfrac{\partial g}{\partial y}(t_m, y_m, z_m) + O(h)$ for $h \to 0$.

3. Behaviour of the local error on class $\mathbf{F}_\varepsilon$.

Assume that the stability functions $R_0^{(i)}(\xi)$, $i = 2(1)s + 1$, and the coefficients $A_{ij}(\xi)$ and $B_j(\xi)$ of method (2.3) fulfil the following conditions (see [12]):

(A1) The approximation order r_i of $R_0^{(i)}(\xi)$ to $\exp(\xi)$ for $\xi \to 0$ is sufficiently high, i.e. $r_i \geq p_i$ where p_i denotes the classical order of the ith stage.
(A2) $R_0^{(i)}(\xi)$ has no pole for Re $\xi \leq 0$ and $|R_0^{(i)}(\infty)| < \infty$.
(A3) $|A_{ij}(\xi)|$, $|B_j(\xi)|$ and $|\xi A_{ij}(\xi)|$, $|\xi B_j(\xi)|$ are uniformly bounded for Re $\xi \leq 0$.

Further, we introduce the index-sets

$$K_i := \{j\colon 1 \leq j \leq i - 1, A_{ij}(\xi) \neq 0\} \text{ for } i = 2(1)s, K_{s+1} := \{j\colon 1 \leq j \leq s, B_j(\xi) \neq 0\}$$

and define

$$R_{l+1}^{(i)}(\xi) = (l R_l^{(i)}(\xi) - 1)/\xi \text{ for } l = 1, 2, \ldots. \tag{3.1}$$

Next we give a lemma needed in the proofs of our results.

LEMMA 3.1. (*see* [3]). *Let* $\mu \in \mathbb{R}$ *and let* $R(\xi)$ *be a rational function without poles in* $\{\xi \in \mathbb{C}, Re\ \xi \leq \mu\}$ *and let the* (N,N)*-matrix* A *satisfy*

$$\langle Ax, x\rangle \leq \mu \|x\|^2 \textit{ for all } x \in \mathbb{R}^N.$$

Then $R(A)$ exists and

$$\|R(A)\| \leq \sup\{|R(\xi)|, \quad \xi \in \mathbb{C}, \quad Re\,\xi \leq \mu\}.$$

THEOREM 3.1. *Let T_2 be an arbitrary $(N, n-N)$-matrix. Assume that method (2.3) satisfies at the ith stage the conditions*

$$\sum_{j=1}^{i-1} A_{ij}(c_i\xi)\,c_j^l = c_i^{l+1}\,R_{l+1}^{(i)}(c_i\xi) \quad \text{for } l = 0(1)\bar{q}_{z,i} \tag{3.2a}$$

$$\sum_{j=1}^{i-1} a_{ij}c_j^{l-1} = l^{-1}c_i^l \quad \text{for } l = 1(1)\bar{q}_{y,i}. \tag{3.2b}$$

Then the following estimates for the local error components at the ith stage hold for all $h \in (0, h_0]$

$$z_{m+1}^{(i)} - z(t_m + c_ih) = O(h^{q_{z,i}+1}), \quad y_{m+1}^{(i)} - y(t_m + c_ih) = O(h^{q_{y,i}+1}) \tag{3.3}$$

where

$$q_{z,i} := \min\{q_{z,j} + 1, q_{y,j}, \bar{q}_{z,i}, \bar{q}_{y,i} \text{ for } j \in K_i\}$$

$$q_{y,i} := \min\{q_{z,j} + 1, q_{y,j} + 1, \bar{q}_{y,i} \text{ for } j \in K_i\}$$

with $q_{z,j} := q_{y,1} := \infty$ (because of $z_{m+1}^{(1)} = z(t_m)$, $y_{m+1}^{(1)} = y(t_m)$). The constants symbolized in the $O(\ldots)$ terms and the maximal stepsize h_0 which is to be sufficiently small are independent of ε.

PROOF. With the mean value theorem for vector functions we obtain

$$\begin{aligned} G_j &\equiv G_j - g(t_m + c_jh, y(t_m + c_jh), z(t_m + c_jh)) + \varepsilon z'(t_m + c_jh) \\ &= \varepsilon z'(t_m + c_jh) + [M_1(t_m + c_jh) - T_1]\cdot[z_{m+1}^{(j)} - z(t_m + c_jh)] \\ &\quad - T_1z(t_m + c_jh) + [M_2(t_m + c_jh) - T_2]\cdot[y_{m+1}^{(j)} - y(t_m + c_jh)] \\ &\quad - T_2y(t_m + c_jh) \end{aligned} \tag{3.4}$$

where

$$M_1(t_m + c_jh) = \int_0^1 g_z(t_m + c_jh, y(t_m + c_jh), z(t_m + c_jh) + \theta(z_{m+1}^{(j)} - z(t_m + c_jh)))\,d\theta$$

$$M_2(t_m + c_jh) = \int_0^1 g_y(t_m + c_jh, y(t_m + c_jh) + \theta(y_{m+1}^{(j)} - y(t_m + c_jh)), z_{m+1}^{(j)})\,d\theta.$$

With (2.5) we get $M_1(t_m + c_jh) - T_1 = O(h)$ and therefore

$$G_j = -T_1z(t_m + c_jh) - T_2y(t_m + c_jh) + \varepsilon z'(t_m + c_jh) + O(h^{\kappa_z+2}) + O(h^{\kappa_y+1})$$

with $\kappa_z = \min\{q_{z,j} \text{ for } j \in K_i\}$ and $\kappa_y = \min\{q_{y,j} \text{ for } j \in K_i\}$.

Analogously we obtain

$$f_j = y'(t_m + c_j h) + O(h^{\kappa_z + 1}) + O(h^{\kappa_y + 1}). \tag{3.5}$$

Because of the regularity of the matrix T_1 for all $h \in (0, h_0]$ we get with lemma 3.1, with (2.4), (3.4), (3.5) and a straightforward Taylor expansion from (2.3)

$$\begin{aligned} z_{m+1}^{(i)} - z(t_m + c_i h) &= \left[c_i \frac{h}{\varepsilon} T_1 R_1^{(i)}(c_i \frac{h}{\varepsilon} T_1) - \frac{h}{\varepsilon} T_1 \sum_{j=1}^{i-1} A_{ij}(c_i \frac{h}{\varepsilon} T_1) \right] [z(t_m) \\ &+ T_1^{-1} T_2 y(t_m)] + \sum_{l=1}^{\bar{q}_{z,i}} \left[l \sum_{j=1}^{i=1} A_{ij} c_j^{l-1} - \frac{h}{\varepsilon} T_1 \sum_{j=1}^{i-1} A_{ij} c_j^l - c_i^l \right] \frac{h^l}{l!} z^{(l)}(t_m) \\ &+ \sum_{l=1}^{\bar{q}_{y,i}} \left[l \sum_{j=1}^{i-1} A_{ij} c_j^{l-1} - \frac{h}{\varepsilon} T_1 \sum_{j=1}^{i-1} A_{ij} c_j^l - l \sum_{j=1}^{i-1} a_{ij} c_j^{l-1} \right] T_1^{-1} T_2 \frac{h^l}{l!} y^{(l)}(t_m) \\ &+ O(h^{q_{z,i}+1}). \end{aligned}$$

With (3.1) and (3.2) it follows

$$z_{m+1}^{(i)} - z(t_m + c_i h) = O(h^{q_{z,i}+1}) \text{ for } h \in (0, h_0).$$

For the local error of the nonstiff components we obtain

$$\begin{aligned} y_{m+1}^{(i)} - y(t_m + c_i h) = y(t_m) + h \sum_{j=1}^{i-1} a_{ij} y'(t_m + c_j h) - y(t_m + c_i h) \\ + O(h^{\kappa_z+2}) + O(h^{\kappa_y+2}). \end{aligned}$$

By Taylor expansion and (3.2b) the statement follows. ■

For case (ii) we can improve the estimates for $q_{z,i}$ of theorem 3.1.

THEOREM 3.2. *Assume that the partitioned system* (1.1) *is autonomous. Let T_2 be given by* (2.6) *and let* (3.2) *be fulfilled. Then* (3.3) *holds with*

$$q_{z,i} = \min_{j \in k_i} \{q_{z,j} + 1, q_{y,j} + 1, \bar{q}^*_{z,i}, \bar{q}_{y,i}\}; \qquad \bar{q}^*_{z,i} = \max(1, \bar{q}_{z,i}).$$

PROOF. With (3.1) we obtain from (2.3)

$$z_{m+1}^{(2)} = z(t_m) + c_2 h R_1^{(2)}\left(c_2 \frac{h}{\varepsilon} T_1\right) z'(t_m) + c_2^2 \frac{h^2}{\varepsilon} R_2^{(2)}\left(c_2 \frac{h}{\varepsilon} T_1\right) T_2 y'(t_m).$$

Because of

$$T_1 z'(t_m) + T_2 y'(t_m) = \varepsilon z''(t_m) + O(h)$$

and (3.1) we have

$$z_{m+1}^{(2)} - z(t_m + c_2 h) = O(h^2).$$

For $\bar{q}_{z,i} = 0$, $i > 2$, we get analogously $q_{z,i} = 1$. With (2.6) we get

$$M_2(t_m + c_2 h) - T_2 = O(h)$$

and therefore

$$G_j = -T_1 z(t_m + c_j h) - T_2 y(t_m + c_j h) + \varepsilon z'(t_m + c_j h) + O(h^{\kappa_z + 2}) + O(h^{\kappa_y + 2}).$$

Now, the proof is analogous to the proof of theorem 3.1. ■

COROLLARY 3.1. *Let method* (2.3) *satisfy*

$$\sum_{j=1}^{s} B_j(\xi) c_j^l = R_{l+1}^{(s+1)}(\xi), \quad l = 0(1)\bar{q}_z \tag{3.6a}$$

$$\sum_{j=1}^{s} b_j c_j^{l-1} = l^{-1}, \quad l = 1(1)\bar{q}_y. \tag{3.6b}$$

Then it holds for all $h \in (0, h_0]$

$$z_{m+1} - z(t_m + h) = O(h^{q_z + 1}), \quad y_{m+1} - y(t_m + h) = O(h^{q_y + 1}) \tag{3.7}$$

where

$$q_y = \min\{q_{z,j} + 1, q_{y,j} + 1, \bar{q}_y \text{ for } j \in K_{s+1}\}$$

and

a) $q_z = \min\{q_{z,j} + 1, q_{y,j}, \bar{q}_z, \bar{q}_y \text{ for } j \in K_{s+1}\}$ *for* T_2 *arbitrary,*

b) $q_z = \min\{q_{z,j} + 1, q_{y,j} + 1, \bar{q}_z^*, \bar{q}_y \text{ for } j \in K_{s+1}\}$, $\bar{q}_z^* = \max(1, \bar{q}_z)$ *for autonomous systems and* T_2 *given by* (2.6).

REMARK 3.1. Theorem 3.2 and corollary 3.1b) with $\bar{q}_z^* = \bar{q}_z$ hold for nonautonomous systems (1.1), too. However, for the second stage we have only $q_{z,2} = 0, q_{y,2} = 1$ and for $s = 1$, $q_z = 0$, $q_y = 1$, respectively.

It is well known that for a nonstiff system the conditions (3.6b), the so called simplifying condition $B(\bar{q}_y)$, are necessary to obtain

$$y_{m+1} - y(t_m + h) = O(h^{\bar{q}_y + 1}) \text{ for } h \in (0, h_0].$$

LEMMA 3.2. *For a partitioned linearly implicit Runge-Kutta method the conditions* (3.6) *are necessary to obtain* (3.7) *with*

$$q_z = \bar{q}_z, \quad q_y = \bar{q}_y.$$

PROOF. We consider the model problem

$$\varepsilon z'(t) = -z(t) + d(t) + \varepsilon d'(t), \quad z(0) = d(0), \quad 0 < \varepsilon \ll 1$$
$$y'(t) = f(t)$$

with smooth functions $d(t)$, $f(t)$ and with exact solution $z(t) = d(t)$. With $T_1 = -1$, $T_2 = 0$ and $y_m = y(t_m)$, $z_m = d(t_m)$ we have from (2.3)

$$z_{m+1} = \left[1 - \frac{h}{\varepsilon} R_1^{(s+1)}\left(-\frac{h}{\varepsilon}\right) + \frac{h}{\varepsilon}\sum_{j=1}^{s} B_j\left(-\frac{h}{\varepsilon}\right)\right] d(t_m)$$

$$+ \sum_{l=1}^{\infty}\left[l \sum_{j=1}^{s} B_j\left(-\frac{h}{\varepsilon}\right) c_j^{\,l-1} + \frac{h}{\varepsilon}\sum_{j=1}^{s} B_j\left(-\frac{h}{\varepsilon}\right) c_j^{\,l}\right] \frac{h^l}{l!} d^{(l)}(t_m)$$

$$y_{m+1} = y_m + \sum_{l=1}^{\infty}\sum_{j=1}^{s} b_j c_j^{\,l-1} \frac{h^l}{(l-1)!} y^{(l)}(t_m).$$

A comparison with the exact solution yields with (3.1) the conditions (3.6). ■

4. Construction of methods with higher local error order on F_ε

Theorem 3.1, 3.2 and corollary 3.1 allow us to construct s-stage partitioned linearly implicit Runge-Kutta methods with high local error order q_z and q_y on class F_ε. We consider some examples.

1. The partitioned linearly implicit Euler method,

$$\begin{array}{c|c} 0 & \\ \hline & R_1^{(2)} \end{array}$$

fulfils condition (3.6a) only for $l = 0$ and condition (3.6b) only for $l = 1$. Therefore, from corollary 3.1 we get the local error order

a) for case (i): $q_z = 0$ and $q_y = 1$,

b) for autonomous systems and case (ii): $q_z = 1$ and $q_y = 1$.

2. Two-stage formulas.

From the conditions

$$A_{21}(c_2\xi) = c_2 R_1^{(2)}(c_2\xi), \quad B_1(\xi) + B_2(\xi) = R_1^{(3)}(\xi), \quad c_2 B_2(\xi) = R_2^{(3)}(\xi)$$

we obtain the class of two-stage methods

$$\begin{array}{c|cc} c_2 & c_2 R_1^{(2)} & \\ \hline & R_1^{(3)} - c_2^{-1} R_2^{(3)} & c_2^{-1} R_2^{(3)} \end{array} \tag{4.1}$$

Corollary 3.1 yields the error order

a) for case (i): $q_z = 1$ and $q_y = 1$

b) for autonomous systems and case (ii): $q_z = 1$ and $q_y = 2$.

3. Four-stage formulas.

Let at the second stage ($i = 2$) the condition (3.2a) be fulfilled for $l = 0$, at the third stage ($i = 3$) for $l = 0, 1$ and at the fourth stage ($i = 4$) for $l = 0, 1, 2$. Furthermore, let the condition (3.6a) be fulfilled for $l = 0, 1, 2$ and let $B_2 = 0$ hold. Then we obtain the following family of 4-stage partitioned linearly implicit Runge-Kutta methods

(4.2)

$$\begin{array}{c|cccc}
c_2 & c_2 R_1^{(2)} & & & \\
c_3 & c_3 R_1^{(3)} - c_3^2 c_2^{-1} R_2^{(3)} & c_3^2 c_2^{-1} R_2^{(3)} & & \\
c_4 & A_{41} & A_{42} & A_{43} & \\
\hline
 & R_1^{(5)} - \dfrac{(c_3 + c_4) R_2^{(5)} - R_3^{(5)}}{c_3 c_4} & 0 & \dfrac{c_4 R_2^{(5)} - R_3^{(5)}}{c_3(c_4 - c_3)} & \dfrac{R_3^{(5)} - c_3 R_2^{(5)}}{c_4(c_4 - c_3)}
\end{array}$$

with

$$A_{41} = c_4 R_1^{(4)} - A_{42} - A_{43},\ A_{42} = \frac{c_4^2}{c_2} \frac{c_3 R_2^{(4)} - c_4 R_3^{(4)}}{c_3 - c_2},\ A_{43} = \frac{c_4^2}{c_3} \frac{c_4 R_3^{(4)} - c_2 R_2^{(4)}}{c_3 - c_2}.$$

From theorem 3.1 and 3.2 we get for $z_{m+1}^{(i)}$ and $y_{m+1}^{(i)}$, $i = 2, 3, 4$,

a) for case (i):

i	$q_{z,i}$	$q_{y,i}$
2	0	1
3	1	1
4	1	1

b) for autonomous systems and case (ii):

i	$q_{z,i}$	$q_{y,i}$
2	1	1
3	1	2
4	2	2

Now, we obtain from corollary 3.1

a) for case (i): $q_z = 1$ and $q_y = 2$;
b) for autonomous systems and case (ii): $q_z = 2$ and $q_y = 2$ if $c_4 \neq \frac{2}{3}$. For $c_4 = \frac{2}{3}$, i.e. $b_3 = 0$, we get $q_z = 2$ and $q_y = 3$.

5. Analysis of the global error on F_ε.

We rewrite the partitioned method (2.3) in the form

$$\begin{aligned}
z_{m+1}^{(1)} &= z_m, & y_{m+1}^{(1)} &= y_m \\
z_{m+1}^{(i)} &= \Psi^{(i)}(t_m, y_m, z_m; h), & y_{m+1}^{(i)} &= y_m + h\Phi^{(i)}(t_m, y_m, z_m; h) \qquad (5.1)\\
z_{m+1} &= \Psi(t_m, y_m, z_m; h), & y_{m+1} &= y_m + h\Phi(t_m, y_m, z_m; h).
\end{aligned}$$

Obviously, Φ satisfies a Lipschitz condition with respect to y and z where the Lipschitz constants are independent of ε.

Next, we give a lemma on differentiation of rational expressions of operator valued functions (for the proof see Hundsdorfer [7]).

LEMMA 5.1. *Let* $p(\xi) = p_0 + p_1 \xi + \ldots + p_k \xi^k$, $q(\xi) = q_0 + q_1 \xi + \ldots + q_k \xi^k$,

$\xi \in \mathbb{C}$, $p_j, q_j \in \mathbb{R}$ $(0 \leq j \leq k)$ and $q_k \neq 0$. Let D be an open set in $\mathbb{R}^N$ and assume that T: $\mathbb{R}^N \to \mathbb{R}^{N \times N}$ is continuously (Gateaux-) differentiable on D, and $q(T(x))$ is invertible for all $x \in D$. Further, let R: $\mathbb{R}^N \to \mathbb{R}^{N \times N}$ be defined by $R(x) = q^{-1}(T(x))p(T(x))$, $x \in \mathbb{R}^N$. Then the function R is continuously (Gateaux-) differentiable on D and for all $x \in D$, $v \in \mathbb{R}^N$ we have

$$\frac{dR(x)}{dx}v = \sum_{l=1}^{k} \phi_l(T(x)) \frac{dT(x)}{dx} v \Psi_l(T(x)).$$

Here ϕ_1, Ψ_1 are rational functions, which can be written with denominator q and which satisfy $\phi_l(\infty) = \Psi_l(\infty) = 0$ for $l = 1(1)k$.

For the proof of the following lemma see [4].

LEMMA 5.2. *Let $\{u_m\}$, $\{v_m\}$ be two sequences of non-negative numbers satisfying*

$$\begin{pmatrix} u_{m+1} \\ v_{m+1} \end{pmatrix} \leq \begin{pmatrix} 1 + O(h) & O(h) \\ O(1) & \alpha + O(h) \end{pmatrix} \begin{pmatrix} u_m \\ v_m \end{pmatrix} + M \begin{pmatrix} h \\ 1 \end{pmatrix}$$

with $0 < \alpha < 1$, $M \geq 0$. Then it holds for $h \leq h_0$ and $t_0 + mh \leq t_e$

$$u_m \leq C(u_0 + hv_0 + M), \quad v_m \leq C(u_0 + (h + \alpha^m) v_0 + M).$$

Now, we prove a lemma whixh we need in the proof of theorem 5.1.

LEMMA 5.3, *Let the linearly implicit Runge-Kutta method (2.3) be strongly A-stable (i.e. A-stable and $|R_0^{(s+1)}(\infty)| < 1$) and let (3.2a) and (3.6a) be satisfied for at least $l = 0$. Let T_1 be given by $T_1 = g_z(t, y, z) + hA$ where A is a constant (N, N)-matrix and T_2 be continuously (Gateaux-) differentiable. Then it holds for $\varepsilon \leq C^*h$, $h \leq h_0$*

$$\|\Psi(t, y, z; h) - \Psi(t, y, \tilde{z}; h)\| \leq (\alpha + O(h))\|z - \tilde{z}\| \text{ with } \alpha < 1$$

for all $(t, y, z), (t, y, \tilde{z}) \in G_\gamma := \{(t, y, z) : t_0 \leq t \leq t_e, \quad \|y - y(t)\| \leq \gamma, \quad \|z - z(t)\| \leq \gamma\}$ where $(y(t), z(t))$ is the exact solution of (1.1).

PROOF. We have

$$\Psi(t, y, z; h) - \Psi(t, y, \tilde{z}; h) = \int_0^1 \Psi_z(t, y, \tilde{z} + \theta(z + \tilde{z}); h)(z - \tilde{z})d\theta.$$

From (2.3) and (5.1) we obtain

$$(5.2) \qquad \Psi_z(t, y(t), z(t); h)v = Q_1(t, y(t), z(t); h) + (h/\varepsilon) Q_2(t, y(t), z(t); h) + h Q_3(t, y(t), z(t); h)$$

where

$$Q_1 := R_0^{(s+1)}\left(\frac{h}{\varepsilon}T_1\right)v + \frac{\partial R_0^{(s+1)}\left(\frac{h}{\varepsilon}T_1\right)}{\partial z} vz(t)$$

$$Q_2 = \frac{\partial\left(R_1^{(s+1)}\left(\frac{h}{\varepsilon}T_1\right)T_2\right)}{\partial z} vy(t) + \sum_{j=1}^{s} \frac{\partial B_j\left(\frac{h}{\varepsilon}T_1\right)}{\partial z} vg_j - \sum_{j=1}^{s} \frac{\partial\left(B_j\left(\frac{h}{\varepsilon}T_1\right)T_1\right)}{\partial z} vz^{(j)}$$

$$- \sum_{j=1}^{s} \frac{\partial\left(B_j\left(\frac{h}{\varepsilon}T_1\right)T_2\right)}{\partial z} vy^{(j)}$$

$$+ \sum_{j=1}^{s} B_j\left(\frac{h}{\varepsilon}T_1\right)[g_z(t + c_jh, y^{(j)}, z^{(j)})\Psi_z^{(j)} - T_1\Psi_z^{(j)} - hT_2\Phi_z^{(j)}]\, v$$

$$Q_3 = \sum_{j=1}^{s} \frac{\partial\left(B_j\left(\frac{h}{\varepsilon}T_1\right)T_1^{-1}T_2\right)}{\partial z} vf_j - \sum_{j=1}^{s} b_j \frac{\partial(T_1^{-1}T_2)}{\partial z} vf_j$$

$$+ \sum_{j=1}^{s}\left[B_j\left(\frac{h}{\varepsilon}T_1\right) - b_j\right]T_1^{-1}\,T_2\frac{\partial f_j}{\partial z}v$$

with $v := z(t) - \tilde{z}$. By our assumptions we have

$$z^{(j)} = z(t) + O(h), \qquad y^{(j)} = y(t) + O(h) \qquad \text{for } h \to 0.$$

Therefore we get

$$g_j = \varepsilon z'(t) + O(h), \qquad f_j = y'(t) + O(h) \qquad \text{for } h \to 0.$$

Condition (3.6a) for $l = 0$ yields

$$\sum_{j=1}^{s} \frac{\partial\left(B_j\left(\frac{h}{\varepsilon}T_1\right)T_2\right)}{\partial z} = \frac{\partial\left(R_1^{(s+1)}\left(\frac{h}{\varepsilon}T_1\right)T_2\right)}{\partial z}$$

$$\sum_{j=1}^{s} \frac{\partial\left(B_j\left(\frac{h}{\varepsilon}T_1\right)\frac{h}{\varepsilon}T_1\right)}{\partial z} = \frac{\partial R_0^{(s+1)}\left(\frac{h}{\varepsilon}T_1\right)}{\partial z}.$$

Obviously, $\Phi_z^{(j)}(t, y(t), z(t); h)$, $j = 1(1)s$, are bounded and by induction we can show with lemma 5.1 that $\Psi_z^{(j)}(t, y(t)\, z(t); h)$ are bounded, too. Now, we get with lemma 3.1 and lemma 5.1 from (5.2) the estimates

$$\|\Psi_z(t, y(t), z(t); h)v\| \le (r + O(h))\,\|v\| \text{ for } \varepsilon \le C^*h, \quad h \le h_0 \tag{5.3}$$

where

$$r = \sup\{|R_0^{(s+1)}(\xi)| < 1 : \operatorname{Re}\xi \le \xi^*\}, \quad \xi^* = \mu_0/c^* < 0.$$

Wih the assumption (1.2) it follows from (5.3) that there exists a γ so that for all $(t, y, z) \in G_\gamma$ it holds

$$\|\Psi_z(t, y, z; h)v\| \leq (\alpha + O(h))\|v\| \text{ with } \alpha < 1. \qquad \blacksquare$$

LEMMA 5.4. *Under the assumptions of lemma 5.3 the increment function $\Psi(t, y, z; h)$ satisfies a Lipschitz condition with respect to y*

$$\|\Psi(t, y, z; h) - \Psi(t, \tilde{y}, z; h)\| \leq L\|y - \tilde{y}\|, \quad (t, y, z), (t, \tilde{y}, z) \in G_\gamma$$

where the Lipschitz constant L is independent of ε.

PROOF. Analogous to the proof of lemma 5.3. ■

THEOREM 5.1. *Let method* (2.3) *be strongly A-stable and let* (3.7) *hold for the local error with $q_z \geq q - 1$ and $q_y = q$, $q \geq 1$. Then for $\varepsilon \leq C^*h$, $h \leq h_0$, the following estimates for the global error hold*

$$z_m - z(t_m) = O(h^q), \quad y_m - y(t_m) = O(h^q) \tag{5.4}$$

where the constants symbolized in the $O(\ldots)$ terms and h_0 are independent of ε.

PROOF. We consider the modified method

$$z^*_{m+1} = \Psi(t_m, \bar{y}_m, \bar{z}_m; h), \qquad y^*_{m+1} = \bar{y}_m + h\Phi(t_m, \bar{y}_m, \bar{z}_m; h)$$

with

$$\bar{z}_m = \begin{cases} z^*_m \text{ for } z^*_m \in G_\gamma, \; z^*_0 = z_0 \\ z(t_m) + \gamma(z^*_m - z(t_m))/\|z^*_m - z(t_m)\| \text{ for } z^*_m \notin G_\gamma \end{cases}$$

$\bar{y}_m$ analogous.

Let z^*_{m+1}, y^*_{m+1} be a numerical solution obtained with $T_1 = g_z(t_m, \bar{y}_m, \bar{z}_m) + hA$ (A constant (N, N)-matrix), and let $\tilde{z}_{m+1}$, $\tilde{y}_{m+1}$ be a numerical solution obtained with $\tilde{T}_1 = g_z(t_m, \tilde{y}_m, \tilde{z}_m) + hA$ starting from $\tilde{z}_m = z(t_m)$, $\tilde{y}_m = y(t_m)$. We have

$$z^*_{m+1} - z(t_m + h) = z^*_{m+1} - \tilde{z}_{m+1} + \tilde{z}_{m+1} - z(t_m + h)$$

$$y^*_{m+1} - y(t_m + h) = y^*_{m+1} - \tilde{y}_{m+1} + \tilde{y}_{m+1} - y(t_m + h).$$

With the Lipschitz conditions of Φ and Ψ and lemma 5.3 we get

$$\|z^*_{m+1} - \tilde{z}_{m+1}\| \leq (\alpha + O(h))\,\|\bar{z}_m - \tilde{z}_m\| + \text{const.}\,\|\bar{y}_m - \tilde{y}_m\|$$

and by definition of $\bar{z}_m$, $\bar{y}_m$

$$\|z^*_{m+1} - \tilde{z}_{m+1}\| \leq (\alpha + O(h))\,\|z^*_m - z(t_m)\| + \text{const.}\,\|y^*_m - y(t_m)\|.$$

Analogously we get

$$\|y_{m+1} - \tilde{y}_{m+1}\| \leq \text{const.}\,h\,\|z^*_m - \tilde{z}_m\| + (1 + O(h))\,\|y^*_m - \tilde{y}_m\|.$$

Because of $\tilde{T}_1 = g_z(t_m, \tilde{y}(t_m), \tilde{z}(t_m)) + O(h)$ it holds

$$\|z^*_{m+1} - z(t_m + h)\| \leq (\alpha + O(h)) \|z^*_m - z(t_m)\| + \text{const.} \|y^*_m - y(t_m)\| + O(h^q)$$

$$\|y^*_{m+1} - y(t_m + h)\| \leq \text{const.}\, h \|z^*_m - z(t_m)\| + (1 + O(h)) \|y^*_m - y(t_m)\| + O(h^{q+1}).$$

With lemma 5.2 and $z_0 = z(t_0)$, $y_0 = y(t_0)$ it follows

$$\|z^*_m - z(t_m)\| \leq \text{const.}\, h^q, \qquad \|y^*_m - y(t_m)\| \leq \text{const.}\, h^q \tag{5.5}$$

for all $t_0 \leq t_m \leq t_e$. From (5.5) we see that for all $h \leq h_0(\gamma)$ $z^*_m, y^*_m \in G_\gamma$ for all m, i.e. $z^*_m \equiv z_m$, $y^*_m \equiv y_m$ and (5.4) holds. ■

REMARK 5.1. The estimate (5.4) holds for all $\varepsilon \leq C^*h$. Hairer/Lubich/Roche ([5]) show for Rosenbrock methods applied to (1.1) by consideration of higher index problems of algebraic differential equations for methods of classical order $p \geq 2$ the following estimates for $\varepsilon \leq C^*h$:

$$z_m - z(t_m) = O(h^r) + O(\varepsilon h), \quad y_m - y(t_m) = O(h^r) + O(\varepsilon h^2)$$

where r denotes the differential algebraic order. For $\varepsilon \approx h$ this yields for the stiff components at most $z_m - z(t_m) = O(h^2)$.

6. Partitioned linearly implicit Runge-Kutta methods for differential-algebraic equations.

We consider for system (1.1) the limiting case $\varepsilon = 0$, i.e. the system of differential-algebraic equations (1.3). Using the abbreviations

$$r^{(i)} := \lim_{\xi \to \infty} R_0^{(i)}(\xi), \quad \alpha_{ij} := \lim_{\xi \to \infty} \xi A_{ij}(\xi), \quad \beta_j := \lim_{\xi \to \infty} \xi B_j(\xi) \tag{6.1}$$

we get from method (2.3) the partitioned linearly implicit Runge-Kutta method for differential-algebraic equations

$$z^{(1)}_{m+1} = z_m, \quad y^{(1)}_{m+1} = y_m$$

$$z^{(i)}_{m+1} = r^{(i)} z_m + (r^{(i)} - 1) T_1^{-1} T_2 y_m + c_i^{-1} \sum_{j=1}^{i-1} \alpha_{ij} G^*_j - h \sum_{j=1}^{i-1} a_{ij} T_1^{-1} T_2 f_j$$

$$y^{(i)}_{m+1} = y_m + h \sum_{j=1}^{i-1} a_{ij} f_j, \quad i = 2(1)s \tag{6.2}$$

$$z_{m+1} = r^{(s+1)} z_m + (r^{(s+1)} - 1) T_1^{-1} T_2 y_m + \sum_{j=1}^{i-1} \beta_j G^*_j - h \sum_{j=1}^{s} b_j T_1^{-1} T_2 f_j$$

$$y_{m+1} = y_m + h \sum_{j=1}^{s} b_j f_j$$

with $G_j^* := T_1^{-1} g_j - z_{m+1}^{(j)} - T_1^{-1} T_2 y_{m+1}^{(j)}$.

Note that for the computation of $z_{m+1}^{(i)}$, linear systems with the same coefficient matrix T_1 have to be solved. By replacing K_i, K_{s+1} by

$$\bar{K}_i = \{j : 1 \leq j \leq i-1, \alpha_{ij} \neq 0\}, i = 2(1)s; K_{s+1} = \{j : 1 \leq j \leq s, \beta_j \neq 0\}$$

and (3.2a) by

$$(6.3) \qquad \sum_{j=1}^{i-1} \alpha_{ij} = c_i(r^{(i)} - 1); \quad \sum_{j=1}^{i-1} \alpha_{ij} c_j^l = -c_i^{l+1}, \quad l = 1(1)\bar{q}_{z,i}$$

we obtain analogous results to theorem 3.1 and theorem 3.2. Further we get for the local error

THEOREM 6.1. *Let the conditions* (3.6*b*) *be satisfied and let*

$$(6.4) \qquad \sum_{j=1}^{s} \beta_j = r^{(s+1)} - 1; \quad \sum_{j=1}^{s} \beta_j c_j^l = -1, \quad l = 1(1)\bar{q}_z.$$

Then it holds for the full local error the estimate (3.7) *with* $q_y = \min\{q_{z,j} + 1, q_{y,j} + 1, \bar{q}_y, \quad j \in \bar{K}_{s+1}\}$

a) $q_z = \min\{q_{z,j} + 1, q_{y,j}, \bar{q}_z, \bar{q}_y, \quad j \in \bar{K}_{s+1}\}$ *for* T_2 *an arbitrary matrix*

b) $q_z = \min\{q_{z,j} + 1, q_{y,j} + 1, \bar{q}_z^*, \bar{q}_y, j \in \bar{K}_{s+1}\}$, $\bar{q}_z^* = \max(1, \bar{q}_z)$ *for autonomous systems and* T_2 *according to* (2.6).

The conditions (6.3) and (6.4) are satisfied if the conditions (3.2a) and (3.6a) are fulfilled for the corresponding method (2.3). Therefore, we see from theorem 6.1 that q_z and q_y in the error estimates of method (6.2) are at least as large as those in the error estimates of the corresponding method (2.3). However, it is possible that q_z and q_y of method (6.2) are larger than those of method (2.3).

EXAMPLE. Consider the two-stage method (4.1) with T_2 according to (2.6) for autonomous systems. In this case we have $q_z = 1$ and $q_y = 2$ (see chapter 4). The corresponding two-stage method for differential-algebraic equations is characterized by the following tableau

$$\begin{array}{c|cc} c_2 & c_2(r^{(2)} - 1) & \\ \hline & r^{(3)} - 1 + c_2^{-1} & -c_2^{-1}. \end{array}$$

For the special choice $c_2 = 1$ the conditions (6.4) are satisfied up to $l = 2$ and therefore, we get with theorem 6.1 $q_z = 2$ and $q_y = 2$.

Because for $\varepsilon = 0$ the assumption $\varepsilon \leq C^* h$ is fulfilled for all h we get from lemma 5.2 the estimate (5.3) with

$$r = |r^{(s+1)}| = |R_0^{(s+1)}(\infty)|.$$

Analogously to theorem 5.1 we get

THEOREM 6.2. *Let* (3.7) *hold for the local error of method* (6.2) *with* $q_z \geq q - 1$, $q_y = q$ *and let* $|r^{(s+1)}| < 1$. *Then it holds for* $h \leq h_0$

$$z_m - z(t_m) = O(h^q), \quad y_m - y(t_m) = O(h^q), \quad m \geq 0,$$

i.e. the convergence order is q.

REMARK 6.1. A convergence theorem for general one-step methods for differential-algebraic equations is given in [1]. Roche [10] gives order results for (nonpartitioned) ROW-methods derived directly for differential-algebraic systems.

7. Numerical examples.

In this section we give some numerical tests which confirm the estimates given in theorem 5.1 and 6.2. We used the methods:

(M1): method (4.1) with $c_2 = \dfrac{1}{2}$ and the stability functions

$$R_0^{(2)}(c_2\xi) = \frac{1 + (c_2 - \gamma)\xi}{1 - \gamma\xi}, \quad R_0^{(3)}(\xi) = \frac{1 + (1 - 2\gamma)\xi}{(1 - \gamma\xi)^2}, \quad \gamma = 1 + \sqrt{2}/2.$$

(M2): method (4.2) with $c_2 = \dfrac{1}{2}$, $c_3 = 1$, $c_4 = \dfrac{2}{3}$ and

$$R_0^{(i)}(c_i\xi) = \frac{1 + (c_i - 2\gamma)\xi + (c_i^2/2 - 2\gamma c_i + \gamma^2)\xi^2}{(1 - \gamma\xi)^2}, \qquad i = 2, 3, 4$$

$$R_0^{(5)}(\xi) = \frac{1 + (1 - 3\gamma)\xi + (1/2 - 3\gamma + 3\gamma^2)\xi^2}{(1 - \gamma\xi)^3}, \qquad \gamma = 0.4358665215084592.$$

Both methods are L-stable, i.e. $r^{(s+1)} = 0$. Method (M1) has the classical order $p = 2$, method (M2) the classical order $p = 3$.

REMARK 7.1. The conditions (3.2a) and (3.6a) yield linearly implicit Runge-Kutta methods in the form of adaptive Runge-Kutta methods (see [11]). For stability functions with one multiple real pole these methods are closely related to W-methods. So, with respect to a general system of ordinary differential equations method (M1) is equivalent to the 3-stage W-method

$$(I - \gamma hT)k_1 = q(t_m, u_m)$$

$$(I - \gamma hT)k_2 = q(t_m + c_2h, u_m + c_2hk_1) - c_2hTk_1$$

$$(I - \gamma hT)k_3 = q(t_m + c_2h, u_m + c_2hk_1) + hT(\gamma_{31}k_1 + \gamma_{32}k_2)$$

$$u_{m+1} = u_m + h(b_1k_1 + b_2k_2 + b_3k_3)$$

with

$$\gamma_{31} = \frac{(1/2 - \gamma)(1 - \gamma/c_2)}{b_3} - c_2, \quad \gamma_{32} = \frac{(1/2 - \gamma)\gamma}{b_3 c_2}$$

$b_1 = 1 - 1/(2c_2), b_2 = 1/(2c_2) - b_3$ and $b_3 \neq 0$ arbitrary.

The methods (M1) and (M2) have been applied to 2 "test-problems":

EXAMPLE 1. (for $\varepsilon = 0$ due to [9])

$$\begin{aligned}
\varepsilon y_1' &= -y_1^2 - y_3^2 + y_4^4/y_2 - \varepsilon y_3, & y_1(0) &= 1 \\
\varepsilon y_2' &= -y_2 + y_4^4 - 2\varepsilon y_2, & y_2(0) &= 1 \\
y_3' &= y_1, & y_3(0) &= 0 \\
y_4' &= -\tfrac{1}{2} y_2^{0.25}, & y_4(0) &= 1, \quad t \in [0, 1].
\end{aligned}$$

EXACT SOLUTION: $y_1(t) = \cos t, y_3(t) = \sin t$

$$y_2(t) = e^{-2t}, y_4(t) = e^{-0.5t}.$$

EXAMPLE 2. $\varepsilon y_1' = -(2 + y_1 y_2) y_1 + \cos^2 t \sin t + 2\cos t - \varepsilon y_2, \quad y_1(0) = 1$

$$y_2' = y_1 + y_2 - \sin t, \quad y_2(0) = 0$$

$t \in [0, 1]$.

EXACT SOLUTION: $y_1(t) = \cos t$, $y_2(t) = \sin t$.

The following tables show the Euclidean norm of the absolute error err(h) and $\text{err}\left(\frac{h}{2}\right)$ for various ε at the endpoint $t_e = 1$ obtained with constant stepsize $h = \frac{1}{100}$. Further, we give the numerically obtained order

$$q_{num} = \log_2 \frac{\text{err}(h)}{\text{err}\left(\frac{h}{2}\right)}$$

for $T_2 = g_y(t_m, y_m, z_m)$ and $T_2 = 0$.

For $\varepsilon = 1$ we get the classical order (M1: $q_{num} \approx 2$, M2: $q_{num} \approx 3$). For $\varepsilon \leq C^* h$ we receive for autonomous systems (example 1) and with $T_2 = g_y(y_m, z_m)$ $q_{num} \approx 2$ and $q_{num} \approx 3$ for M1 and M2, respectively. With $T_2 = 0$ we have only $q_{num} \approx 1$ and $q_{num} \approx 2$. For the nonautonomous system (example 2) we always get only $q_{num} \approx 1$ and $q_{num} \approx 2$ for M1 and M2, respectively. The errors for both cases of T_2 are comparable.

Table 7.1. *Results for example* 1.

method	ε	$T_2 = g_y(y_m, z_m)$ err(h	err$\left(\frac{h}{2}\right)$	q_{num}	$T_2 = 0$ err(h)	err$\left(\frac{h}{2}\right)$	q_{num}
M1	$1.e+0$	$3.7e-4$	$9.5e-5$	1.97	$4.6e-4$	$1.2e-4$	1.96
	$1.e-3$	$1.5e-5$	$3.1e-6$	2.26	$3.5e-3$	$1.9e-3$	0.91
	$1.e-6$	$2.8e-5$	$6.8e-6$	2.02	$3.2e-3$	$1.6e-3$	1.00
	0	$2.8e-5$	$6.8e-6$	2.02	$3.2e-3$	$1.6e-3$	1.00
M2	$1.e+0$	$9.4e-8$	$1.2e-8$	3.00	$1.3e-7$	$1.6e-8$	2.99
	$1.e-3$	$9.6e-7$	$6.6e-8$	3.86	$6.0e-5$	$1.1e-5$	2.39
	$1.e-6$	$2.2e-6$	$2.8e-7$	2.98	$6.9e-5$	$1.8e-5$	1.98
	0	$2.2e-6$	$2.8e-7$	2.97	$6.9e-5$	$1.8e-5$	1.98

Table 7.2. *Results for example* 2.

method	ε	$T_2 = g_y(t_m, y_m, z_m)$ err(h)	err$\left(\frac{h}{2}\right)$	q_{num}	$T_2 = 0$ err(h)	err$\left(\frac{h}{2}\right)$	q_{num}
M1	$1.e+0$	$1.5e-4$	$3.9e-5$	1.93	$1.3e-4$	$3.2e-5$	1.97
	$1.e-3$	$9.7e-4$	$4.8e-4$	1.02	$3.0e-3$	$1.5e-3$	1.03
	$1.e-6$	$9.9e-4$	$5.0e-4$	1.00	$3.1e-3$	$1.6e-3$	1.00
	0	$9.9e-4$	$5.0e-4$	1.00	$3.1e-3$	$1.6e-3$	1.00
M2	$1.e+0$	$5.9e-8$	$7.5e-9$	2.98	$1.2e-7$	$1.5e-8$	3.00
	1.e $-$ 3	4.4e $-$ 6	9.7e $-$ 7	2.16	8.0e $-$ 6	1.7e $-$ 6	2.22
	1.e $-$ 6	5.0e $-$ 6	1.3e $-$ 6	2.00	9.7e $-$ 6	2.4e $-$ 6	1.99
	0	5.0e $-$ 6	1.3e $-$ 6	2.00	9.7e $-$ 6	2.4e $-$ 6	1.99

The numerical tests confirm our theoretical results on the behaviour of the global error of partitioned linearly implicit Runge-Kutta methods on the classes F_ε and F_0.

REFERENCES

[1] Deuflhard, P., E. Hairer and J. Zugck, *One-step and extrapolation methods for differential-algebraic systems*, Numer. Math. 51, 501–516, 1987.

[2] Griepentrog, E. and R. März, *Differential-algebraic Equations and their Numerical Treatment*, BSB B.G. Teubner, Leipzig 1986.

[3] Hairer, E., Bader, G. and Ch. Lubich, *On the stability of semi-implicit methods for ordinary differential equations*, BIT 22 (1982), 211–232.

[4] Hairer, E. and Ch. Lubich, *Convergence of one-step methods at stiff differential equations*, University of Geneva, 1987.

[5] Hairer, E., Ch. Lubich and M. Roche, *Error of Rosenbrock methods for stiff problems studied via differential-algebraic equations*, BIT 29 (1989), 77–90.

[6] Hairer, E., S. P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I*, Springer-Verlag, Berlin-Heidelberg, 1987.

[7] Hundsdorfer, W. H., *The Numerical Solution of Nonlinear Stiff Initial Value Problems*, Centrum voor Wiskunde en Informatica, Amsterdam 1984.

[8] O'Malley, R. E., *Introduction to Singular Perturbations*, Academic Press, New York and London, 1974.

[9] Rentrop, P. and G. Steinebach, *The numerical solution of implicit ordinary differential equations arising in vehicle dynamic*, In: *Numerical Treatment of Differential Equaitions*, ed. by K. Strehmel, Teubner-Texte zur Mathematik, Leipzig 1988.

[10] Roche, M., *Rosenbrock methods for differential-algebraic equations*, Numer. Math., 52 (1988), 45–63.

[11] Strehmel, K. and R. Weiner, *Partitioned adaptive Runge-Kutta methods and their stability*, Numer. Math. 45 (1984), 283–300.

[12] Strehmel, K. and R. Weiner, *B-convergence results for linearly implicit one step methods*, BIT 27 (1987), 264–281.