Applied Mathematics and Mechanics
(English Edition, Vol. 11, No. 5, May 1990)

Published by SUT,
Shanghai, China

THE SOLUTION OF RECTANGULAR PLATES WITH LARGE DEFLECTION BY SPLINE FUNCTIONS

Pan Li-zhou (潘立宙) Chen Wei-zhong (陈为众)

(Shanghai University of Technology; Shanghai Institute of Applied Mathematics and Mechanics, Shanghai)

(Received Sept. 20, 1989)

Abstract

In this paper, Von Kármán's set of nonlinear equations for rectangular plates with large deflection is divided into several sets of linear equations by perturbation method, the dimensionless center deflection being taken as a perturbation parameter. These sets of linear equations are solved by the spline finite-point (SFP) method and by the spline finite element (SFE) method. The solutions for rectangular plates having any length-to-width ratios under a uniformly distributed load and with various boundary conditions are presented, and the analytical formulas for displacements and deflections are given in the paper. The computer programs are worked out by ourselves. Comparison of the results with those in other papers indicates that the results of this paper are satisfactorily better.

I. Introduction

Elastic thin plates are widely used in aviation, chemical industry and shipbuilding. If the transversal displacement, i.e. deflection, of a thin plate is far less than its thickness, the results calculated on the basis of the Kirchhoff's theory are considerably satisfactory. However, when a thin plate of metal is used, and the ratio of its deflection to thickness is nearly unity, even greater than unity, the calculated results certainly quite differ from the parctical situation if the theory of small deflection is still used. Therefore, the formulas and charts for designing or checking this kind of metal thin plates in their engineering use must be derived from the theory of large deflection.

The spline function method, as a method of numerical fitting, was first presented by I. J. Schoenberg, and up to now a great number of papers have been published. Especially, the rapid development of electronic computers and numerical techniques, as powerful tools, has been greatly supporting the wide application of spline functions.

II. Construction of Spline Base Functions

A piecewise expression of cubic spline function is

$$\varphi_3(x)=\frac{1}{6}\begin{cases}(x+2)^3 & (-2<x\leqslant -1)\\ (x+2)^3-4(x+1)^3 & (-1<x\leqslant 0)\\ (2-x)^3-4(1-x)^3 & (0<x\leqslant 1)\\ (2-x)^3 & (1<x<2)\\ 0 & (|x|\geqslant 2)\end{cases}$$

Suppose that the trial displacement function expressed in terms of cubic spline base functions is

$$S(x)=\sum_{i=-1}^{N+1} a_i\Phi_i(x)$$

where a_i are coefficients to be determined and $\Phi_i(x)$ a set of base functions associated with cubic splines. In order to express boundary conditions in a simpler form, $\Phi_i(x)$ are defined in the interval $[x_0,x_N]$ with $h_x=(x_N-x_0)/N$ by the following functions:

$$\Phi_{-1}(x)=\varphi_3\left(\frac{x-x_0}{h_x}+1\right)$$

$$\Phi_0(x)=\varphi_3\left(\frac{x-x_0}{h_x}\right)-4\varphi_3\left(\frac{x-x_0}{h_x}+1\right)$$

$$\Phi_1(x)=\varphi_3\left(\frac{x-x_0}{h_x}-1\right)-\frac{1}{2}\varphi_3\left(\frac{x-x_0}{h_x}\right)+\varphi_3\left(\frac{x-x_0}{h_x}+1\right)$$

$$\Phi_2(x)=\varphi_3\left(\frac{x-x_0}{h_x}-2\right)$$

......

$$\Phi_{N-2}(x)=\varphi_3\left(\frac{x-x_0}{h_x}-N+2\right)$$

$$\Phi_{N-1}(x)=\varphi_3\left(\frac{x-x_0}{h_x}-N+1\right)-\frac{1}{2}\varphi_3\left(\frac{x-x_0}{h_x}-N\right)$$

$$+\varphi_3\left(\frac{x-x_0}{h_x}-N-1\right)$$

$$\Phi_N(x)=\varphi_3\left(\frac{x-x_0}{h_x}-N\right)-4\varphi_3\left(\frac{x-x_0}{h_x}-N-1\right)$$

$$\Phi_{N+1}(x)=\varphi_3\left(\frac{x-x_0}{h_x}-N-1\right)$$

With the base functions presented above, when $x=x_0$, we have $\Phi_i(x_0)=0$ $(i\neq-1)$, and $\Phi_i'(x_0)=0$ $(i\neq-1,\ 0)$; when $x=x_N$, we have $\Phi_i(x_N)=0$ $(i\neq N+1)$, and $\Phi_i'(x_N)=0$ $(i\neq N,\ N+1)$ Thus, functions $\Phi_i(x)$ make it very convenient to treat displacement boundary conditions. For example, when an edge is free,we take all the functions $\Phi_i(x)$; when an edge is simply supported. $\Phi_{-1}(x)$ (or $\Phi_{N+1}(x)$)is to be deleted; when an edge is clamped or built in, $\Phi_{-1}(x)$ and $\Phi_0(x)$ (or $\Phi_N(x)$ and $\Phi_{N+1}(x)$) are to be deleted.

III. The Functional Form of Perturbation Equations

A metal thin plate of length $2a$,width $2b$,and thickness h, as shown in Fig. 1, will produce its interior displacement, deformation and traction under the action of uniform transversal load q. $U(x,y)$, $V(x,y)$ and $W(x,y)$ represent the displacements of points of the mid-plane in x, y, and z directions, respectively. The displacement in z direction is also referred to as deflection. These displacements are functions of the coordinates x and y only, so they are independent of the coordinate z.

Von Kármán's dimensionless equations for large deflection of a rectangular plate in terms of displacements are

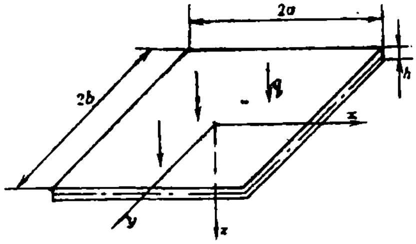

Fig. 1

$$
\left.
\begin{aligned}
&2\frac{\partial^2 u}{\partial \xi^2}+(1-\mu)\lambda^2\frac{\partial^2 u}{\partial \eta^2}+(1+\mu)\lambda\frac{\partial^2 v}{\partial \xi\partial \eta}\\
&\quad=-(1-\mu)\frac{\partial w}{\partial \xi}\left(\frac{\partial^2 w}{\partial \xi^2}+\lambda^2\frac{\partial^2 w}{\partial \eta^2}\right)\\
&\quad\quad-\frac{1+\mu}{2}\frac{\partial}{\partial \xi}\left[\left(\frac{\partial w}{\partial \xi}\right)^2+\lambda^2\left(\frac{\partial w}{\partial \mu}\right)^2\right]\\
&2\lambda^2\frac{\partial^2 v}{\partial \eta^2}+(1-\mu)\frac{\partial^2 v}{\partial \xi^2}+(1+\mu)\lambda\frac{\partial^2 u}{\partial \xi\partial \eta}\\
&\quad=-(1-\mu)\lambda\frac{\partial w}{\partial \eta}\left(\frac{\partial^2 w}{\partial \xi^2}+\lambda^2\frac{\partial^2 w}{\partial \eta^2}\right)\\
&\quad\quad-\frac{1+\mu}{2}\lambda\frac{\partial}{\partial \eta}\left[\left(\frac{\partial w}{\partial \xi}\right)^2+\lambda^2\left(\frac{\partial w}{\partial \eta}\right)^2\right]\\
&\frac{\partial^4 w}{\partial \xi^4}+2\lambda^2\frac{\partial^4 w}{\partial \xi^2\partial \eta^2}+\lambda^4\frac{\partial^4 w}{\partial \eta^4}\\
&\quad=Q+\frac{\partial^2 w}{\partial \xi^2}\left(\frac{\partial u}{\partial \xi}+\mu\lambda\frac{\partial v}{\partial \eta}\right)+\lambda^2\frac{\partial^2 w}{\partial \eta^2}\left(\lambda\frac{\partial v}{\partial \eta}+\mu\frac{\partial u}{\partial \xi}\right)\\
&\quad\quad+\lambda(1-\mu)\frac{\partial^2 w}{\partial \xi\partial \eta}\left(\lambda\frac{\partial u}{\partial \eta}+\frac{\partial v}{\partial \xi}\right)+\frac{1}{2}\frac{\partial^2 w}{\partial \xi^2}\\
&\quad\quad\left[\left(\frac{\partial w}{\partial \xi}\right)^2+\mu\lambda^2\left(\frac{\partial w}{\partial \eta}\right)^2\right]+\frac{1}{2}\lambda^2\frac{\partial^2 w}{\partial \eta^2}\left[\lambda^2\left(\frac{\partial w}{\partial \eta}\right)^2\right.\\
&\quad\quad\left.+\mu\left(\frac{\partial w}{\partial \xi}\right)^2\right]+\lambda^2(1-\mu)\frac{\partial w}{\partial \xi}\frac{\partial w}{\partial \eta}\frac{\partial^2 w}{\partial \xi\partial \eta}
\end{aligned}
\right\}\qquad(3.1)
$$

where:

$$
\left.
\begin{aligned}
&\lambda=\frac{a}{b},\quad \xi=\frac{x}{a},\quad \eta=\frac{y}{b};\quad u=\frac{12aU}{h^2},\quad v=\frac{12aV}{h^2}\\
&w=2\sqrt{3}\frac{W}{h},\quad Q=\frac{24\sqrt{3}(1-\mu^2)qa^4}{Eh^4}
\end{aligned}
\right\}\qquad(3.2)
$$

in which E is Young's modulus and μ is Poisson's ratio of the material of the plate.

The dimensionless center deflection of the rectangular plate is

$$w_0 = w(0,0) = 2\sqrt{3}W_0/h \tag{3.3}$$

Taking this center deflection as a perturbation parameter, we assume the following series expansions:

$$\left.\begin{aligned}
Q &= a_1 w_0 + a_3 w_0^3 + a_5 w_0^5 + \cdots \\
u &= s_2(\xi,\eta) w_0^2 + s_4(\xi,\eta) w_0^4 + \cdots \\
v &= t_2(\xi,\eta) w_0^2 + t_4(\xi,\eta) w_0^4 + \cdots \\
w &= w_1(\xi,\eta) w_0 + w_3(\xi,\eta) w_0^3 + \cdots
\end{aligned}\right\} \tag{3.4}$$

where the coefficients in the expansion of w must satisfy the following conditions:

$$w_1(0,0) = 1, \qquad w_3(0,0) = w_5(0,0) = \cdots = 0 \tag{3.5}$$

Substituting series expansions (3.4) into equations (3.1) and equating the coefficients of like powers of w_0, we obtain the following successive sets of equations:

1. The first set of equation

$$\frac{\partial^4 w_1}{\partial \xi^4} + 2\lambda^2 \frac{\partial^4 w_1}{\partial \xi^2 \partial \eta^2} + \lambda^4 \frac{\partial^4 w_1}{\partial \eta^4} = a_1 \tag{3.6}$$

2. The second set of equations

$$\left.\begin{aligned}
&2\frac{\partial^2 s_2}{\partial \xi^2} + (1-\mu)\lambda^2 \frac{\partial^2 s_2}{\partial \eta^2} + (1+\mu)\lambda \frac{\partial^2 t_2}{\partial \xi \partial \eta} \\
&\quad = -(1-\mu)\frac{\partial w_1}{\partial \xi}\left(\frac{\partial^2 w_1}{\partial \xi^2} + \lambda^2 \frac{\partial^2 w_1}{\partial \eta^2}\right) \\
&\qquad - \frac{1+\mu}{2} \frac{\partial}{\partial \xi}\left[\left(\frac{\partial w_1}{\partial \xi}\right)^2 + \lambda^2 \left(\frac{\partial w_1}{\partial \eta}\right)^2\right] \\
&2\lambda^2 \frac{\partial^2 t_2}{\partial \eta^2} + (1-\mu)\frac{\partial^2 t_2}{\partial \xi^2} + (1+\mu)\lambda \frac{\partial^2 s_2}{\partial \xi \partial \eta} \\
&\quad = -(1-\mu)\lambda \frac{\partial w_1}{\partial \eta}\left(\frac{\partial^2 w_1}{\partial \xi^2} + \lambda^2 \frac{\partial^2 w_1}{\partial \eta^2}\right) \\
&\qquad - \frac{1+\mu}{2}\lambda \frac{\partial}{\partial \eta}\left[\left(\frac{\partial w_1}{\partial \xi}\right)^2 + \lambda^2 \left(\frac{\partial w_1}{\partial \eta}\right)^2\right]
\end{aligned}\right\} \tag{3.7}$$

3. The third set of equations

$$\begin{aligned}
&\frac{\partial^4 w_3}{\partial \xi^4} + 2\lambda^2 \frac{\partial^4 w_3}{\partial \xi^2 \partial \eta^2} + \lambda^4 \frac{\partial^4 w_3}{\partial \eta^4} \\
&\quad = a_3 + \frac{\partial^2 w_1}{\partial \xi^2}\left(\frac{\partial s_2}{\partial \xi} + \mu\lambda \frac{\partial t_2}{\partial \eta}\right) + \lambda^2 \frac{\partial^2 w_1}{\partial \eta^2}\left(\lambda \frac{\partial t_2}{\partial \eta} + \mu \frac{\partial s_2}{\partial \xi}\right) \\
&\qquad + \frac{1}{2}\frac{\partial^2 w_1}{\partial \xi^2}\left[\left(\frac{\partial w_1}{\partial \xi}\right)^2 + \mu\lambda^2 \left(\frac{\partial w_1}{\partial \eta}\right)^2\right] \\
&\qquad + \frac{1}{2}\lambda^2 \frac{\partial^2 w_1}{\partial \eta^2}\left[\lambda^2 \left(\frac{\partial w_1}{\partial \eta}\right)^2 + \mu \left(\frac{\partial w_1}{\partial \xi}\right)^2\right] \\
&\qquad + (1-\mu)\lambda \frac{\partial^2 w_1}{\partial \xi \partial \eta}\left(\lambda \frac{\partial s_2}{\partial \eta} + \frac{\partial t_2}{\partial \xi}\right) + (1-\mu)\lambda^2 \frac{\partial w_1}{\partial \xi} \frac{\partial w_1}{\partial \eta} \frac{\partial^2 w_1}{\partial \xi \partial \eta}
\end{aligned} \tag{3.8}$$

and so on.

We shall solve the first three sets of equations only, as they will give us the results with sufficient accuracy for engineering purposes. It is rather difficult to solve analytically these equations along with certain boundary conditions. In this paper, approximate solutions are carried out with the help of variational principles. To do this we transform these perturbation equations into corresponding functionals as follows:

1. The functional of the first order

$$\Pi_1=\int_{-1}^{1}\int_{-1}^{1}\left[\frac{1}{2}\left(\frac{\partial^2 w_1}{\partial\xi^2}\right)^2+\lambda^2\left(\frac{\partial^2 w_1}{\partial\xi\partial\eta}\right)^2+\frac{1}{2}\lambda^4\left(\frac{\partial^2 w_1}{\partial\eta^2}\right)^2-\alpha_1 w_1\right]d\xi d\eta \qquad (3.9)$$

2. The functional of the second order

$$\begin{aligned}\Pi_2=-\int_{-1}^{1}\int_{-1}^{1}\Bigg\{&\left(\frac{\partial s_2}{\partial\xi}\right)^2+\frac{1}{2}(1-\mu)\lambda^2\left(\frac{\partial s_2}{\partial\eta}\right)^2\\&+\frac{1}{2}(1+\mu)\lambda\left(\frac{\partial s_2}{\partial\xi}\frac{\partial t_2}{\partial\eta}+\frac{\partial t_2}{\partial\xi}\frac{\partial s_2}{\partial\eta}\right)\\&+\frac{1}{2}(1-\mu)\left(\frac{\partial t_2}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial t_2}{\partial\eta}\right)^2\\&-\left[(1-\mu)\frac{\partial w_1}{\partial\xi}\left(\frac{\partial^2 w_1}{\partial\xi^2}+\lambda^2\frac{\partial^2 w_1}{\partial\eta^2}\right)+\frac{1}{2}(1+\mu)\frac{\partial}{\partial\xi}\left[\left(\frac{\partial w_1}{\partial\xi}\right)^2\right.\right.\\&\left.\left.+\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2\right]\right]s_2-\left[(1-\mu)\lambda\frac{\partial w_1}{\partial\eta}\left(\frac{\partial^2 w_1}{\partial\xi^2}+\lambda^2\frac{\partial^2 w_1}{\partial\eta^2}\right)\right.\\&\left.+\frac{1}{2}(1+\mu)\lambda\frac{\partial}{\partial\eta}\left[\left(\frac{\partial w_1}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2\right]\right]t_2\Bigg\}d\xi d\eta\end{aligned} \qquad (3.10)$$

3. The functional of the third order

$$\begin{aligned}\Pi_3=\int_{-1}^{1}\int_{-1}^{1}\Bigg\{&\frac{1}{2}\left(\frac{\partial^2 w_3}{\partial\xi^2}\right)^2+\lambda^2\left(\frac{\partial^2 w_3}{\partial\xi\partial\eta}\right)^2+\frac{1}{2}\lambda^4\left(\frac{\partial^2 w_3}{\partial\eta^2}\right)^2-\alpha_3 w_3\\&-\left[\frac{\partial^2 w_1}{\partial\xi^2}\left(\frac{\partial s_2}{\partial\xi}+\mu\lambda\frac{\partial t_2}{\partial\eta}\right)+\lambda^2\frac{\partial^2 w_1}{\partial\eta^2}\left(\lambda\frac{\partial t_2}{\partial\eta}+\mu\frac{\partial s_2}{\partial\xi}\right)\right.\\&+(1-\mu)\lambda\frac{\partial^2 w_1}{\partial\xi\partial\eta}\left(\lambda\frac{\partial s_2}{\partial\eta}+\frac{\partial t_2}{\partial\xi}\right)+\frac{1}{2}\frac{\partial^2 w_1}{\partial\xi^2}\left[\left(\frac{\partial w_1}{\partial\xi}\right)^2\right.\\&\left.+\mu\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2\right]+\frac{1}{2}\lambda^2\frac{\partial^2 w_1}{\partial\eta^2}\left[\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2+\mu\left(\frac{\partial w_1}{\partial\xi}\right)^2\right]\\&\left.+(1-\mu)\lambda^2\frac{\partial w_1}{\partial\xi}\frac{\partial w_1}{\partial\eta}\frac{\partial^2 w_1}{\partial\xi\partial\eta}\right]w_3\Bigg\}d\xi d\eta\end{aligned} \qquad (3.11)$$

These functionals require the trial displacement functions to be of C^2 continuity only.

IV. Solution by the Spline Finite-Point (SFP) Method

The spline finite-point method is based on splines, orthogonal functions and variational principles. The trial functions for deflections w_1 and w_3 are linear combinations of products of splines and orthogonal functions, which are taken from the mode functions of a vibrating beam. And the trial displacement functions, s_2 and t_2, are linear combinations of products of splines and trigonometric functions.

Suppose that the trial displacement functions of the rectangular thin plate are

$$\left.\begin{aligned} w_1(\xi,\eta)&=\sum_{m=1}^{r}X_m(\xi)[\Phi(\eta)]\{a\}_m \\ s_2(\xi,\eta)&=\sum_{m=1}^{r}S_m(\xi)[\Phi(\eta)]\{b\}_m \\ t_2(\xi,\eta)&=\sum_{m=1}^{r}S_m(\xi)[\Phi(\eta)]\{c\}_m \\ w_3(\xi,\eta)&=\sum_{m=1}^{r}X_m(\xi)[\Phi(\eta)]\{d\}_m \end{aligned}\right\} \tag{4.1}$$

where $X_m(\xi)$ are mode functions of a vibrating beam,

$S_m(\xi)$ are trigonometric functions,

$$[\Phi(\eta)]=[\Phi_{-1}\ \Phi_0\ \Phi_1\ \cdots\ \Phi_N\ \Phi_{N+1}],$$

$$\{a\}_m=[a_{-1,m}\ a_{0,m}\ a_{1,m}\ \cdots\ a_{N,m}\ a_{N+1,m}]^T$$

......

$$X_m(\xi)=c_1\sin\frac{\mu_m}{2}(1+\xi)+c_2\cos\frac{\mu_m}{2}(1+\xi)$$

$$+c_3\mathrm{sh}\frac{\mu_m}{2}(1+\xi)+c_4\mathrm{ch}\frac{\mu_m}{2}(1+\xi)$$

in which the parameter μ_m and the coefficients c_1, c_2, c_3 and c_4 will be determined by boundary conditions. If we take $S_m(\xi)=\sin\frac{m\pi}{2}(1+\xi)$, then we have $s_2=t_2=0$ at $\xi=\pm 1$

Substituting the expression w_1 from equation (4.1) into the functional (3.9), we obtain

$$\Pi_1=2^{-1}\{a\}^T[G]\{a\}-\alpha_1\{a\}^T\{f\}$$

Taking the first variation of this functional, letting it be equal to zero, namely $\partial\Pi_1/\partial\{a\}=\{0\}$ and considering the central condition $w_1(0,0)=1$, we can write the result in matrix form as follows:

$$\begin{bmatrix}[G] & -\{f\}\\ [g] & 0\end{bmatrix}\begin{bmatrix}\{a\}\\ \alpha_1\end{bmatrix}=\begin{bmatrix}\{0\}\\ 1\end{bmatrix} \tag{4.2}$$

where:

$$[g]=[[g]_1\ [g]_2\ \cdots\ [g]_r],$$

$$[g]_m=[X_m(0)\Phi_{-1}(0)\ X_m(0)\Phi_0(0)\ \cdots\ X_m(0)\Phi_{N+1}(0)],$$

$$\{f\}=[\{f\}_1^T\ \{f\}_2^T\ \cdots\ \{f\}_r^T]^T,$$

$$\{f\}_m=\int_{-1}^{1}\int_{-1}^{1}X_m(\xi)[\Phi(\eta)]^T d\xi d\eta,$$

$$\{a\}=[\{a\}_1^T\ \{a\}_2^T\ \cdots\ \{a\}_r^T]^T,$$

$$\{a\}_m=[a_{-1,m}\ a_{0,m}\ a_{1,m}\ \cdots\ a_{N+1,m}]^T$$

$$[G]=G_{mn}=A_\xi F_\eta+2\lambda^2 B_\xi B_\eta+\lambda^4 F_\xi A_\eta,$$

$$A_\xi=\int_{-1}^{1}X_m''X_n''d\xi,\quad B_\xi=\int_{-1}^{1}X_m'X_n'd\xi,\quad F_\xi=\int_{-1}^{1}X_mX_nd\xi,$$

$$A_\eta=\int_{-1}^{1}[\Phi'']^T[\Phi'']d\eta,\quad B_\eta=\int_{-1}^{1}[\Phi']^T[\Phi']d\eta,\quad F_\eta=\int_{-1}^{1}[\Phi]^T[\Phi]d\eta.$$

Having formed matrix $[G]$, we are to introduce boundary conditions in η direction. In doing this,we have to delete some related rows and columns.

On solving the system of linear algebraic equations (4.2), we obtain the required coefficients $\{a\}$ and the required load factor α_1 . Hence we get the first order approximate solution w_1

In like manner, we substitute the expressions s_2 and t_2 from equation (4.1), whose coefficients are to be determined, and the equation w_1 just found into functional (3.10), and then we take the first variation of the resulting functional and let it be equal to zero. A system of linear algebraic equations in unknowns $\{b\}$ and $\{c\}$ is thus obtained. Solution of this system of equations will give us the required $s_2(\xi,\eta)$ and $t_2(\xi,\eta)$ hence the displacements u and v in the mid-plane of the plate to a second approximation.

Substituting the fourth expression w_3 from equation (4.1) and the equations s_2, t_2 and w_1 just obtained above into functional (3.11), taking the first variation of the resulting functional, letting it be equal to zero, and taking central condition $w_3(0,0)=0$ into account, we get a system of equations, whose unknowns are the coefficients $\{d\}$ and the load factor α_3 . The solution of this system will give us part of the deflection in third order of approximation, w_3 , and the load factor α_3..

Substituting the load factors α_1 and α_3 into the first expression of equation (3.4), we obtain a nonlinear relationship between the center deflection w_0 and the load Q.

V. Solution by the Spline Finite Element (SFE) Method

The spline finite element method is based on cubic splines and variational principles. Trial displacement functions are made up of linear combinations of products of cubic splines.

Suppose that the trial displacement functions of the rectangular thin plate are

$$\left.\begin{aligned}
w_1&=\sum_{i=-1}^{N+1}\sum_{j=-1}^{M+1}a_{ij}\Phi_i(\xi)\Psi_j(\eta)=[\Phi]\otimes[\Psi]\{A\}\\
s_2&=\sum_{i=-1}^{N+1}\sum_{j=-1}^{M+1}b_{ij}\Phi_i(\xi)\Psi_j(\eta)=[\Phi]\otimes[\Psi]\{B\}\\
t_2&=\sum_{i=-1}^{N+1}\sum_{j=-1}^{M+1}c_{ij}\Phi_i(\xi)\Psi_j(\eta)=[\Phi]\otimes[\Psi]\{C\}\\
w_3&=\sum_{i=-1}^{N+1}\sum_{j=-1}^{M+1}d_{ij}\Phi_i(\xi)\Psi_j(\eta)=[\Phi]\otimes[\Psi]\{D\}
\end{aligned}\right\}\qquad(5.1)$$

where:

$$[\Phi]=[\Phi_{-1}\ \Phi_0\ \Phi_1\ \cdots\ \Phi_{N+1}],$$
$$[\Psi]=[\Psi_{-1}\ \Psi_0\ \Psi_1\ \cdots\ \Psi_{M+1}],$$
$$\{A\}=[\{a\}_{-1}^T\ \{a\}_0^T\ \{a\}_1^T\ \cdots\ \{a\}_{N+1}^T]^T,$$
$$\cdots\cdots$$
$$\{a\}_i=[a_{i,-1}\ a_{i,0}\ a_{i,1}\ \cdots\ a_{i,M+1}]^T,$$
$$\cdots\cdots$$

The form of Ψ_j is quite the same as that of Φ_i except that i has been changed into j, ξ into η, h_ξ into h_η , and N into M. $\{A\}$, $\{B\}$, $\{C\}$, and $\{D\}$ are the same in their forms of construction. $[\Phi]\otimes[\Psi]$ denotes the *Kronecker* product of the matrices $[\Phi]$ and $[\Psi]$.

In a similar way to those presented in the preceding section, we substitute the expressions of equation (5.1) succesively into their respective functionals. From the stationary conditions of the functionals together with the central conditions, we can determine the coefficients $\{A\}$, $\{B\}$, $\{C\}$, and $\{D\}$ in the expressions for displacements and the load factors α_1 and α_3 . Thus we obtain once more the displacements u, v and the deflection w , and a nonlinear relationship between the load Q and center deflection w_0

VI. Numerical Examples

(1) Square plate with simply supported edges ($\mu=0.316$)

Boundary conditions:

$$U=V=W=\partial^2W/\partial x^2=0\ ,\quad \text{at}\quad x=\pm a\ ;$$
$$U=V=W=\partial^2W/\partial y^2=0\ ,\quad \text{at}\quad y=\pm a\ .$$

The computed results are shown in Tab. 1.

Tab. 1

		α_1	α_3
SFP method	$N=16$, $r=3$	15.368122	1.830996
SFE method	$N=M=4$	15.377132	1.823241
	$N=M=6$	15.383563	1.824830

(2) Rectangular plates with clamped edges ($\mu=1/3$)

Boundary conditions:

$$U=V=W=\partial W/\partial x=0\ ,\quad \text{at}\quad x=\pm a\ ;$$
$$U=V=W=\partial W/\partial y=0\ ,\quad \text{at}\quad y=\pm b\ .$$

The computed results are shown in Tab. 2.

(3) Rectangular plates with simply supported edges ($\mu=0.316$)

Boundary conditions:

$$U=V=W=\frac{\partial^2 W}{\partial x^2}=0\,, \quad \text{at} \quad x=\pm a\;;$$

$$U=V=W=\frac{\partial^2 W}{\partial y^2}=0\,, \quad \text{at} \quad y=\pm b\,.$$

The computed results are shown in Tab. 3.

Tab. 2

	$\lambda=\frac{a}{b}$	SFE method $N=M=6$	SFP method $N=8,\ r=2$	Ref. (7)
	1.0	49.418480	49.098549	49.611419
	1.1	60.707962	60.320473	60.951435
	1.2	75.177101	74.708984	75.498710
	1.3	93.436333	92.879036	93.868010
	1.4	116.174629	115.529625	116.752894
α_1	1.5	144.160263	143.452316	144.923864
	1.6	178.243362	177.532410	179.231295
	1.7	219.348343	218.741913	220.602573
	1.8	268.479095	268.145050	270.039454
	1.9	326.709623	326.900360	328.613813
	2.0	395.173431	396.231903	397.462501
	1.0	2.195729	1.829208	2.001669
	1.1	2.702238	2.370961	2.463083
	1.2	3.363337	3.073061	3.084275
	1.3	4.214519	3.968555	3.896871
	1.4	5.299982	5.096636	4.940128
α_2	1.5	6.674306	6.503717	6.265388
	1.6	8.404319	8.244557	7.939434
	1.7	10.570469	10.383200	10.044764
	1.8	13.267498	12.993663	12.675997
	1.9	16.603344	16.160093	15.933041
	2.0	20.695934	19.976143	19.913025

Tab. 3

	SFP method $N=M=4$		SFP method $N=4,\ r=2$	
$\lambda=\frac{a}{b}$	α_1	α_2	α_1	α_2
1.0	15.377132	1.823241	15.364720	1.806377
1.1	18.784475	2.244918	18.772064	2.225749
1.2	22.926159	2.796858	22.914036	2.790524
1.3	27.916729	3.506788	27.905188	3.467758
1.4	33.882942	4.407320	33.872261	4.345628
1.5	40.963902	5.536526	40.954395	5.438225
1.6	49.311134	6.938610	49.303204	6.783527
1.7	59.089729	8.664728	59.083546	8.424164
1.8	70.477333	10.773865	70.473366	10.407569
1.9	83.665421	13.333877	83.664063	12.786238
2.0	98.859383	16.422592	98.860779	15.617829

(4) Influence of different boundary conditions on the center deflection ($\lambda=1$, $\mu=1/3$)

The results computed by using SFE method ($N=M=6$) are shown in Fig. 2.

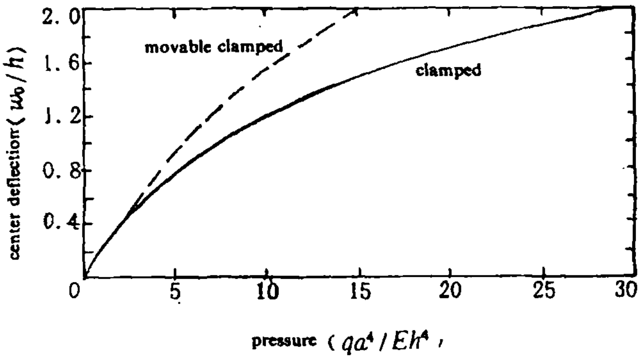

Fig. 2

(5) Square plate with clamped edges ($\mu=1/3$)

The results obtained with different divisions in using SFE method are shown in Tab. 4.

Tab. 4

N	M	a_1	a_3
4	4	49.558323	2.248114
6	6	49.418480	2.195729
8	8	49.401402	2.187941
12	12	49.396393	2.185711

VII. Conclusions

1. Through calculation, we see that the deflection of a plate is quite influenced by the displacements parallel to the mid-plane of the plate in the case of large deflection, although in the case of small deflection the influence is so little that it can be neglected.

2. The spline finite-point method and the spline element method used for analysing various structures in regular domain are superior to the finite element method and the semi-analytical finite strip method in that the former has fewer degrees of freedom and more accuracy, needs less computing work and less preparation of data, and gives us continuous values of stresses and bending moments.

3. The spline finite element method treats boundary conditions in a simpler manner than the spline finite-point method does. It is especially suitable for free edges or movable edges. Since the spline finite element is a piecewise polynomial, it can fit in more accurately with the displacement, and its computational accuracy is higher than the spline finite-point.

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