# ON THE IDENTIFICATION OF COEFFICIENTS OF SEMILINEAR PARABOLIC EQUATIONS

LIU ZHENHAI (刘振海)

(*Department of Mathematics, Changsha University of Electric Power, Changsha* 450000, *China*)

**Abstract**

In this paper, a problem of identifying possibly discontinuous diffusion coefficients in parabolic equations is considered. General theorems on existence are proved in $L^1$ setting. A necessary condition is given for the solution of the parameter estimation problem.

**Key words:** Parameter identification, inverse problem, semilinear parabolic equations

## 1. Introduction

We consider the following system:

$$\begin{aligned} &\partial_t u - \sum_{i=1}^{n} \partial_{x_i}\big(a(x)\partial_{x_i} u\big) = f(x,t,u), && (x,t) \in Q_T, \\ &u(x,0) = u_0(x), && c \in \Omega, \\ &u(x,t) = 0, && (x,t) \in S_T, \end{aligned} \tag{1}$$

where $\Omega$ is a bounded domain in $R^n$ $(n \geq 1)$, $T$ is a constant with $0 < T < \infty$, $Q_T = \{(x,t) : x \in \Omega,\ t \in (0,T)\}$, $S_T = \{(x,t) : x \in \partial\Omega,\ t \in (0,T)\}$. We will assume that $a(x) \in L^\infty(\Omega)$, and moreover, $a(x) \in A_{ad}$,

$$A_{ad} = \{a(x) \in L^\infty(\Omega) :\ 0 < v_1 \leq a(x) \leq v_2 \text{ a.e. in } \Omega\}. \tag{2}$$

The parameter estimation problem for (1) and (2) is to determine the coefficient $a(x)$ in such a way that the solution $u(x,t,a)$, which is the solution of the problem (1) corresponding to some $a(x) \in A_{ad}$, "matches" the observed information $u^*(x,t)$ of (1) in a prescribed sense (see [1–3] for general information).

To be precise, we say that the coefficient $\bar{a}(x) \in K_{ad} \subset A_{ad}$ solves the parameter estimation problem for the admissible set $K_{ad}$ if $J_0(\bar{a}) = \inf\{J_0(a) :\ a \in K_{ad}\}$ where the cost function

$$J_0(a) = \int_{Q_T} \big|u(x,t,a) - u^*(x,t)\big|^2 dxdt. \tag{3}$$

Received February 5, 1991. Revised May 17, 1993.

We also consider another cost function

$$J_\varepsilon(a) = \int_{Q_T} |u(x,t,a) - u^*(x,t)|^2 \, dxdt + \varepsilon \int_\Omega |a(x) - a^*(x)|^2 \, dx, \tag{4}$$

where $u^*(x,t)$, $a^*(x)$ are observed informations.

It is known that in parameter estimation problems the solution of these problems does not always exist. There are relevant examples in [4], for example.

A number of important physical identification problems fall within the above frame work. For example, the partial differential equation

$$\beta\partial_t u = \partial_x(\alpha\partial_x u) + \partial_y(\alpha\partial_y u) + q$$

is a basic model equation in oil reservoir simulation and ground water flow. The quantity $u$ represents pressure or "piezometric head", $q$ is a source term, and $\alpha, \beta$ are positive coefficients which are often referred to as the "transmissivity" and "storage coefficient", respectively. These coefficients are commonly taken as functions of the space coordinates $x$ and $y$[5].

The identification of $\alpha$ and $\beta$ using measured $u$ and $q$ values at well sites is an important inverse problem. In the papers [2, 3], in which the source term $f$ in (1) is independent on $u$, for the set

$$K_{ad} = \{a(x) \in H^1(\Omega) : \ 0, \ v_1 \le a(x), \ \|a\|_{H^1} \le \text{const.}\}$$

the parameter estimation problems are considered. In the paper [6], the author deals with the parameter identification problem of the system of geothermal reservoir exploitation, in which the set

$$K_{ad} = \{a(x) \in C^{1+\alpha}[0,X], \ 0 < v_1 \le a(x), \ \text{for any } x \in [0,X], \ \|a\|_{1+\alpha} \le v_2\}$$

is considered. But the system of geothermal reservoir exploitation can be transformed into our system (1) (see [6] for details). Of course, the sufficient smoothness of the involved coefficients $a(x)$ having been considered in the papers [2, 3, 6] is reasonable. However the subject of this paper is the identification of parameters in (1)–(3) that refer to a different physical situation. Suppose, for example, that $\Omega = (0,1) \times (0,1)$, $\Omega_1 = (0, \frac{1}{2}) \times (0,1)$, and $\Omega_2 = (\frac{1}{2}, 1) \times (0,1)$, $a(x) = v_1$ if $x \in \Omega_1$, and $a(x) = v_2$ if $x \in \Omega_2$. Thus the medium consists of two regions with different permeabilities and we would like to develop a method for identification of discontinuous coefficients $a(x)$ in $\Omega$. Therefore the admissible set $K_{ad}$ must include the coefficients of the type described above. However this coefficient $a(x)$ does not belong to $H^1(\Omega)$ and therefore a different set $K_{ad}$ should be considered.

## 2. Initial-boundary Valued Problem and Functional Continuity Results

First of all we consider the linear initial-boundary value problem

$$\begin{aligned} &\partial_t u - \sum_{i=1}^n \partial_{x_i}\big(a(x)\partial_{x_i} u\big) = f_0(x,t) + \sum_{i=1}^n \partial_{x_i} f_i(x,t), && (x,t) \in Q_T. \\ &u(x,0) = u_0(x), && x \in \Omega, \\ &u(x,t) = 0, && (x,t) \in S_T. \end{aligned} \tag{5}$$

We consider the (weak) solution of the problem (5) from the Banach space $\overset{0}{V}{}_2^{1,0}(Q_T)$ with norm [7, p.6]

$$|u|_{Q_T} = \max_{0\le t\le T} \|u(x,t)\|_{2,\Omega} + \|\nabla u\|_{2,Q_T},$$

where

$$\|u(x,t)\|_{2,\Omega}^2 = \int_\Omega |u(x,t)|^2\,dx,$$

$$\|\nabla u\|_{2,Q_T}^2 = \int_{Q_T} |\nabla u|^2\,dxdt \equiv \int_{Q_T} \sum_{i=1}^n |\partial_{x_i} u|^2\,dxdt.$$

We say that function $u(x,t)$ is the (weak) solution of the problem (5), if $u(x,t)$ belongs to the Banach space $\overset{0}{V}{}_2^{1,0}(Q_T)$, and satisfies the equation [7, p.136]

$$\int_\Omega u(x,t)g(x,t)\,dx - \int_0^t\int_\Omega u(x,t)\partial_t g(x,t)\,dxdt + \int_0^t\int_\Omega a(x)\sum_{i=1}^n \partial_{x_i}u\partial_{x_i}g\,dxdt$$
$$= \int_0^t\int_\Omega \left[f_0(x,t)g(x,t) - \sum_{i=1}^n f_i(x,t)\partial_{x_i}g(x,t)\right]dxdt + \int_\Omega u_0(x)g(x,0)\,dx \tag{6}$$

for all $g(x,t) \in \overset{0}{W}{}_2^{1,1}(Q_T)$.

If $u_0(x) \in L^2(\Omega)$, $f_i(x,t) \in L^2(Q_T)$, $i = 0,1,\cdots,n$ and $a(x) \in A_{ad}$, then it is well known (see, e.g. [7, p.160]) that the system (5) has a unique (weak) solution $u(x,t)$.

In the following, we shall state that under some assumptions on the data, the semilinear problem (1) for any $a(x) \in A_{ad}$ has a unique (weak) solution.

Assumption (H$_1$) Function $f(x,t,u)$ is measurable with respect to $(x,t) \in Q_T$ for all $u \in R^1 = (-\infty,+\infty)$, and $f(x,t,0) \in L^2(Q_T)$, furthermore, $f(x,t,u)$ satisfies uniformly the Lipschitz condition in $u$, that is, there exists a positive constant $L$, such that

$$\big|f(x,t,u) - f(x,t,v)\big| \le L|u-v| \qquad \text{for almost all } (x,t) \in Q_T.$$

**Theorem 1** (See [7, Chap. 3]). Suppose $a(x) \in A_{ad}$, $u_0(x) \in L^2(\Omega)$, $f(x,t,u)$ satisfies (H$_1$). Then there exists a unique (weak) solution $u(x,t,a)$ to the semilinear Problem (1).

**Theorem 2.** Suppose $u_0 \in L^2(\Omega)$, $f$ satisfies (H$_1$). Then the operator $u: a \mapsto u(a) \equiv u(x,t,a)$, which is the solution of the semilinear Problem (1) corresponding to $a(x) \in A_{ad}$, takes strong convergence in $L^1(\Omega)$ into strong convergence in $\overset{0}{V}{}_2^{1,0}(Q_T)$.

*Proof.* We consider a sequence $\{a_m\}$ in $A_{ad}$ such that $a_m \to a$ strongly in $L^1(\Omega)$. Let $u_m = u(a_m)$, $m = 1,2,\cdots$, $u = u(a)$. Then the function $\bar u_m \equiv u_m - u$ is the solution of the following initial-boundary value problem

$$\begin{aligned}
&\partial_t\bar u_m - \sum_{i=1}^n \partial_{x_i}(a_m\partial_{x_i}\bar u_m) = \sum_{i=1}^n \partial_{x_i}[(a_m-a)\partial_{x_i}u] + f(x,t,u_m) - f(x,t,u), && (x,t)\in Q_T,\\
&\bar u_m(x,0) = 0, && x\in\Omega,\\
&\bar u_m(x,t) = 0, && (x,t)\in S_T.
\end{aligned} \tag{7}$$

Since $a_m$, $a \in A_{ad}$, $\partial_{x_i}u \in L^2(Q_T)$ and $|f(x,t,u_m) - f(x,t,u) \le L|u_m - u|$, so $(a_m - a)\partial_{x_i}u \in L^2(Q_T)$, $f(x,t,u_m) - f(x,t,u) \in L_{2,1}(Q_T)$, i.e. the coefficients and free terms of Problem

(7) satisfy all the assumptions (1.2)–(1.6) in [7, Chap. 3]. Therefore by virtue of the relation [7, p.142], we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\bar{u}_m|^2 dx + \int_0^t \int_\Omega a_m(x)|\nabla \bar{u}_m|^2 dxdt\\ =&\int_0^t \int_\Omega [f(x,t,u_m)-f(x,t,u)]\bar{u}_m\, dxdt + \int_0^t \int_\Omega (a-a_m)\sum_{i=1}^n \partial_{x_i} u \partial_{x_i}\bar{u}_m\, dxdt\\ \le& L\int_0^t \int_\Omega |\bar{u}_m|^2\, dxdt + \int_0^t \int_\Omega |a-a_m|\,|\nabla u|\,|\nabla \bar{u}_m|\, dxdt.\end{aligned}$$

Using Young's inequality, we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\bar{u}_m|^2 dx + v_1\int_0^t \int_\Omega |\nabla \bar{u}_m|^2 dxdt\\ \le& L\int_0^t \int_\Omega |\bar{u}_m|^2 dxdt + \frac{1}{2v_1}\int_0^t \int_\Omega |a-a_m|^2\,|\nabla u|^2\, dxdt + \frac{v_1}{2}\int_0^t \int_\Omega |\nabla \bar{u}_m|^2 dxdt,\end{aligned}$$

that is,

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\bar{u}_m|^2 dx + \frac{v_1}{2}\int_0^t \int_\Omega |\nabla \bar{u}_m|^2 dxdt\\ \le& L\int_0^t \int_\Omega |\bar{u}_m|^2 dxdt + \frac{1}{2v_1}\int_0^t \int_\Omega |a-a_m|^2\,|\nabla u|^2\, dxdt.\end{aligned} \tag{8}$$

In particular,

$$\int_\Omega |\bar{u}_m|^2 dx \le 2L\int_0^t \int_\Omega |\bar{u}_m|^2 dxdt + \frac{1}{v_1}\int_0^t \int_\Omega |a-a_m|^2|\nabla u|^2\, dxdt.$$

Using Gronwall's inequality [7, p.94], we obtain

$$\int_\Omega |\bar{u}_m|^2 dx \le \frac{1}{v_1}e^{2Lt}\int_0^t \int_\Omega |a-a_m|^2\,|\nabla u|^2 dxdt. \tag{9}$$

On the right side of the inequality (8) we replace $\int_\Omega |\bar{u}_m|^2 dx$ with the right side of (9), and we have

$$\begin{aligned}&\max_{0\le t\le T}\int_\Omega |\bar{u}_m|^2 dx + v_1\int_0^t \int_\Omega |\nabla \bar{u}_m|^2 dxdt\\ \le& \left[\frac{2LT}{v_1}e^{2LT} + \frac{1}{v_1}\right]\int_0^t \int_\Omega |a-a_m|^2|\nabla u|^2 dxdt.\end{aligned} \tag{10}$$

Because $a_m \to a$ strongly in $L^1(\Omega)$, it is easy to know that the sequence $\{a_m\}$ converges in measure to $a(x)$. At the same time, taking into consideration the fact that

$$|a_m - a|^2|\nabla u|^2 \le 4v_2^2|\nabla u|^2,$$

by Lebesgue's convergence theorem we obtain

$$\lim_{m\to\infty}\int_0^t \int_\Omega |a_m - a|^2|\nabla u|^2 dxdt = 0. \tag{11}$$

From (10) and (11), we know

$$\lim_{m\to\infty} |\bar{u}_m|_{Q_T} = 0,$$

which has proved Theorem 2.

## 3. Existence of the Optimal Coefficients

From Theorem 2, we know that the mapping $a \mapsto u(a)$ is continuous from $L^1(\Omega)$ to $\overset{0}{V}{}_2^{1,0}(Q_T)$. Therefore the cost function $J_0(a) = \int_0^T \int_\Omega |u(x,t,a) - u^*(x,t)|^2 dxdt$ is continuous on $A_{ad} \subset L^1(\Omega)$. Hence it attains its minimum on any compact set $K_{ad} \subset A_{ad}$.

**Remark 1.** If we let $K_{ad} = \{a(x) \in H^1(\Omega) : 0 < v_1 \le a(x),\ \Omega \subset R^n,\ n = 1,2,3,\ \|a\|_{H^1} \le \text{const.}\}$, by the imbedding theorems $K_{ad}$ is compact in $L^1(\Omega)$. Therefore, the functional $J_0(a)$ attains a minimum on it (see [2,3,6]).

Now we consider functions of bounded variation. Let $\Omega$ be an open bounded set in $R^n$ with a Lipschitz continuous boundary $\partial\Omega$. Following [8], define the variation $\int_\Omega |Df|$ of a function $f \in L^1(\Omega)$ by

$$\int_\Omega |Df| = \sup \left\{ \int_\Omega f \operatorname{div} g \, dx : \ g = (g_1, \cdots, g_n) \in C_0^1(\Omega, R^n) \right.$$
$$\left. \text{and } |g(x)| = \sqrt{\sum_{i=1}^n g_i^2} \le 1 \text{ for } x \in \Omega \right\},$$

where $\operatorname{div} g = \sum_{i=1}^n \partial_{x_i} g_i$.

If the variation of $f$ is finite, that is, $\int_\Omega |Df| < \infty$, we say that $f$ has a bounded variation. The space of all functions $f \in L^1(\Omega)$ with bounded variation is denoted by $BV(\Omega)$. Under the norm $\|f\|_{BV} = \|f\|_{L^1} + \int_\Omega |Df|$, $BV(\Omega)$ is a Banach space [8, Th.1.12].

**Remark 2.** A lot of rather general discontinuous functions belong to $BV(\Omega)$. For example, let $f \in L^1(\Omega)$ be continuously differentiable on $\Omega_i$, $1 \le i \le p$ and can be continuously extended in every $\overline{\Omega}_i$, where $\overline{\Omega}_i \subset \Omega$, $\Omega_i \cap \Omega_j = \emptyset$ for $i \ne j$ and $\cup_{i=1}^p \overline{\Omega}_i \supset \Omega$, and the boundary $\partial\Omega$, $\partial\Omega_i$ are Lipschitz continuous and each $\Omega_i$ satisfies the conditions of the divergence (Gauss-Green) theorem. Then $f \in BV(\Omega)$ (see [8] for details).

**Theorem 3.** The set $K_c = \{a(x) \in A_{ad} : \int_\Omega |Da| \le c\}$ is compact in $L^1(\Omega)$ for any $c > 0$.

*Proof.* If $\int_\Omega |Da| \le c$, then $\|a\|_{BV(\Omega)} \le c + v_2|\Omega|$. Thus $K_c$ is precompact in $L^1(\Omega)$ by [8, Th.1.19]. If $a_m \to a$ in $L^1(\Omega)$ as $m \to \infty$ and $\int_\Omega |Da_m| \le c$, then $\int_\Omega |Da| \le c$ by [8, Th.1.9], so $K_c$ is closed in $L^1(\Omega)$.

According to Theorem 2 and Theorem 3, we obtain

**Theorem 4.** Suppose that $f(x,t,u)$ satisfies ($\mathrm{H}_1$), $u_0(x) \in L^2(\Omega)$. Then the cost function $J_0(a)$ attains a minimum on $K_c$.

It is obvious that $A_{ad} = \{a(x) \in L^\infty(\Omega) : 0 < v_1 \le a(x) \le v_2 \text{ a.e. in } \Omega\}$ is not a compact subset of $L^1(\Omega)$. But it is a closed subset of $L^1(\Omega)$. In order to obtain the existence theorem for the optimal control on $A_{ad}$, we need the following well-known result.

**Theorem 5** (See [9]). Let $X$ be a reflexive Banach space, $S$ a closed subset of $X$, and $I : S \to R^1$ a functional bounded below and lower semicontinuous. Then the set of those $\xi$ of $X$ for which there is a minimum on $S$ of the functional $x \mapsto I(x) + \|x - \xi\|_x$ is dense in $X$.

**Theorem 6.** For almost all $a^*(x) \in L^2(\Omega)$, $u^*(x,t) \in L^2(Q_T)$ and any $\varepsilon > 0$, the cost function $J_\varepsilon(a)$ attains a minimum on $A_{ad}$.

To prove Theorem 6, it is sufficient in the conditions of Theorem 5 to put $X \equiv L^2(\Omega)$, $S \equiv A_{ad}$, $I(a) \equiv \frac{1}{\varepsilon}\|u(x,t,a) - u^*(x,t)\|^2_{L^2(Q_T)}$. Because the topologies of $L^1(\Omega)$ and $L^2(\Omega)$ coincide on $A_{ad}$, therefore the functional $I(a)$ is bounded below and is lower semicontinuous (even continuous).

**Remark 3.** Similarly, if we consider the functional

$$\bar{J}_0(a) = \int_\Omega |u(x,T,a) - u^*(x,T)|^2\,dx$$

and

$$\bar{J}_\varepsilon(a) = \int_\Omega |u(x,T,a) - u^*(x,T)|^2\,dx + \varepsilon \int_\Omega |a - a^*|^2\,dx$$

we will have the same results as Theorems 4 and 6.

## 4. The Necessary Condition of the Optimal Control

Now we consider the operator $u :\mapsto u(a) \equiv u(x,t,a)$, where $a(x) \in K$, $K = \{a(x) \in L^\infty(\Omega) : \frac{v_1}{2} < a(x) < 2v_2 \text{ a.e. in } \Omega\}$. For any $\delta a \in L^\infty(\Omega)$, according to Theorem 2, we have

$$\lim_{s\to 0} \left|u(x,t,a+s\delta a) - u(x,t,a)\right|_{Q_T} = 0,$$

that is,

$$\lim_{s\to 0}\left[\max_{0\le t\le T} \|u(x,t,a+s\delta a) - u(x,t,a)\|_{2,\Omega} + \|\nabla u(x,t,a+s\delta a) - \nabla u(x,t,a)\|_{2,Q_T}\right] = 0. \tag{12}$$

In order to obtain a necessary condition of the optimal control, let us introduce the following hypothesis:

($H_2$) Suppose $f(x,t,u)$ satisfies ($H_1$). Also assume $f(x,t,u)$ has partial derivative $\partial_u f(x,t,u)$ and $\partial_u f(x,t,u)$ satisfies the Carathéodory condition, that is,

1) for almost all $(x,t) \in Q_T$, $\partial_u f(x,t,u)$ is continuous in $u$;

2) for any $u \in R^1$, $\partial_u f(x,t,u)$ is measurable in $Q_T$.

**Theorem 7.** Suppose $f(x,t,u)$ satisfies ($H_2$), $u_0(x) \in L^2(\Omega)$. Then the Gateaux derivative of $u$ exists at $a(x) \in K$ and the Gateaux derivative $u'(a)\delta a \equiv \hat{u}(x,t)$ where $\hat{u}(x,t)$ is the unique (weak) solution of the following linear problem:

$$\begin{cases} \partial_t \hat{u} - \sum_{i=1}^n \partial_{x_i}\big(a(x)\partial_{x_i}\hat{u}\big) = \partial_u f\big(x,t,u(a)\big)\hat{u} + \sum_{i=1}^n \partial_{x_i}\big(\delta a \partial_{x_i} u(a)\big), & (x,t) \in Q_T, \\ \hat{u}(x,0) = 0, & x \in \Omega, \\ \hat{u}(x,t) = 0, & (x,t) \in S_T, \end{cases} \tag{13}$$

where $u(a) \equiv u(x,t,a)$.

*Proof.* According to ($H_2$), $\|\partial_u f(x,t,u)\|_{L^\infty(Q_T)} \le L$. Therefore for any $\delta a \in L^\infty(\Omega)$, by using [7, Chap. 3 Th.4.2] there exists a unique (weak) solution $\hat{u}$ of linear Problem (13).

Let $a(x) \in K$, $\delta a \in L^\infty(\Omega)$. If the real number $s$ ($s \neq 0$) is sufficiently small, we have $a(x) + s\delta a \in K$. Then $W_s = \frac{1}{s}[u(a+s\delta a) - u(a)]$ satisfies

$$\begin{aligned} \partial_t W_s - \sum_{i=1}^n \partial_{x_i}(a(x)\partial_{x_i} W_s) &= \frac{1}{s}\left[f(x,t,u(a+s\delta a)) - f(x,t,u(a))\right] \\ &\quad + \sum_{i=1}^n \partial_{x_i}(\delta a \partial_{x_i} u(a+s\delta a)), \qquad (x,t) \in Q_T, \\ W_s(x,0) = 0, &\qquad x \in \Omega, \\ W_s(x,t) = 0, &\qquad (x,t) \in S_T. \end{aligned}$$

According to the definition of Gateaux derivative, we should prove $\lim_{s\to 0}|W_s-\hat{u}|_{Q_T}=0$.

Let $\bar{u}=W_s-\hat{u}$. Then $\bar{u}$ satisfies

$$\begin{aligned}
&\partial_t\bar{u}-\sum_{i=1}^{n}\partial_{x_i}\big(a(x)\partial_{x_i}\bar{u}\big)=\frac{1}{s}\Big[f\big(x,t,u(a+s\delta a)\big)-f\big(x,t,u(a)\big)\Big]\\
&\qquad-\partial_u f\big(x,t,u(a)\big)\hat{u}+\sum_{i=1}^{n}\partial_{x_i}\Big[\delta a\partial_{x_i}\big(u(a+s\delta a)-u(a)\big)\Big], && (x,t)\in Q_T, \quad (14)\\
&\bar{u}(x,0)=0, && x\in\Omega,\\
&\bar{u}(x,t)=0, && (x,t)\in S_T.
\end{aligned}$$

Since $u(a+s\delta a),\ u(a),\ \hat{u}\in V_2^{1,0}(Q_T)$, and by ($H_2$),

$$\begin{aligned}
&\frac{1}{s}\Big|f\big(x,t,u(a+s\delta a)\big)-f\big(x,t,u(a)\big)\Big|\le\frac{1}{s}L\big|u(a+s\delta a)-u(a)\big|,\\
&\big|\partial_u f(x,t,u(a))\hat{u}\big|\le L|\hat{u}|,
\end{aligned}$$

therefore, for any fixed $s>0$,

$$\begin{aligned}
&\frac{1}{s}\Big[f\big(x,t,u(a+s\delta a)\big)-f\big(x,t,u(a)\big)\Big]-\partial_u f\big(x,t,u(a)\big)\hat{u}\in L_{2,1}(Q_T),\\
&\delta a\partial_{x_i}\big(u(a+s\delta a)-u(a)\big)\in L^2(Q_T).
\end{aligned}$$

Therefore the coefficients and free terms of Problem (14) satisfy all the assumptions (1.2)–(1.6) in [7, Chap. 3]. So linear Problem (14) has a unique solution by [7, Chap. 3, Th.4.2], and we can use the relation [7, p.142] to obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega|\bar{u}|^2\,dx+\frac{v_1}{2}\int_0^t\int_\Omega|\nabla\bar{u}|^2\,dxdt\\
\le&\int_0^t\int_\Omega\left|\frac{1}{s}\Big[f\big(x,t,u(a+s\delta a)\big)-f\big(x,t,u(a)\big)\Big]-\partial_u f\big(x,t,u(a)\big)\hat{u}\right|\,|\bar{u}|\,dxdt\\
&+\int_0^t\int_\Omega|\delta a|\,\Big|\nabla\big(u(a+s\delta a)-u(a)\big)\Big|\,|\nabla\bar{u}|\,dxdt\\
\le&\frac{1}{2}\int_0^t\int_\Omega\left|\frac{1}{s}\Big[f\big(x,t,u(a+s\delta a)\big)-f\big(x,t,u(a)\big)\Big]-\partial_u f\big(x,t,u(a)\big)\hat{u}\right|^2\,dxdt\\
&+\frac{1}{2}\int_0^t\int_\Omega|\bar{u}|^2\,dxdt+\frac{1}{v_1}\int_0^t\int_\Omega|\delta a|^2\Big|\nabla\big(u(a+s\delta a)-u(a)\big)\Big|^2\,dxdt\\
&+\frac{v_1}{4}\int_0^t\int_\Omega|\nabla\bar{u}|^2\,dxdt,
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_\Omega|\bar{u}|^2\,dx+\frac{v_1}{2}\int_0^t\int_\Omega|\nabla\bar{u}|^2\,dxdt\\
\le&\int_0^t\int_\Omega|\bar{u}|^2\,dxdt+\int_0^t\int_\Omega\left|\frac{1}{s}\Big[f\big(x,t,u(a+s\delta a)\big)-f\big(x,t,u(a)\big)\Big]-\partial_u f\big(x,t,u(a)\big)\hat{u}\right|^2\,dxdt
\end{aligned}$$

$$+\frac{2}{v_1}\int_0^t\int_\Omega|\delta a|^2\Big|\nabla\big(u(a+s\delta a)-u(a)\big)\Big|^2\,dxdt. \tag{15}$$

Now we estimate the second integration on the right side of (15). First we let

$$F(x,t,s)=\begin{cases}\dfrac{f\big(x,t,u(a+s\delta a)\big)-f\big(x,t,u(a)\big)}{u(a+s\delta a)-u(a)}, & \text{if } (x,t)\in\big\{u(x,t,a+s\delta a)\neq u(x,t,a)\big\},\\ \partial_u f(x,t,u(a)), & \text{if } (x,t)\in\big\{u(x,t,a+s\delta a)=u(x,t,a)\big\}.\end{cases}$$

Then $|F(x,t,s)|\leq L$ and

$$\begin{aligned}&\frac{1}{s}\Big[f\big(x,t,u(a+s\delta a)\big)-f\big(x,t,u(a)\big)\Big]\\ =&F(x,t,s)\frac{u(a+s\delta a)-u(a)}{s}=F(x,t,s)W_s,\end{aligned} \tag{16}$$

$$\begin{aligned}&\int_0^t\int_\Omega\left|\frac{1}{s}\Big[f\big(x,t,u(a+s\delta a)\big)-f\big(x,t,u(a)\big)\Big]-\partial_u f\big(x,t,u(a)\big)\hat{u}\right|^2\,dxdt\\ =&\int_0^t\int_\Omega\Big|F(x,t,s)W_s-\partial_u f\big(x,t,u(a)\big)\hat{u}\Big|^2\,dxdt\\ =&\int_0^t\int_\Omega\Big|F(x,t,s)\bar{u}+\big[F(x,t,s)-\partial_u f\big(x,t,u(a)\big)\big]\hat{u}\Big|^2\,dxdt\\ \leq&2L^2\int_0^t\int_\Omega|\bar{u}|^2\,dxdt+2\int_0^t\int_\Omega\Big|F(x,t,s)-\partial_u f\big(x,t,u(a)\big)\Big|^2|\hat{u}|^2\,dxdt.\end{aligned} \tag{17}$$

Replacing the second integration of the right side of (15) by the right side of (17), we obtain

$$\begin{aligned}&\int_\Omega|\bar{u}|^2dx+\frac{v_1}{2}\int_0^t\int_\Omega|\nabla\bar{u}|^2\,dxdt\\ \leq&(1+2L^2)\int_0^t\int_\Omega|\bar{u}|^2dxdt+2\int_0^t\int_\Omega\Big|F(x,t,s)-\partial_u f\big(x,t,u(a)\big)\Big|^2|\hat{u}|^2\,dxdt\\ &+\frac{2}{v_1}\int_0^t\int_\Omega|\delta a|^2\Big|\nabla\big(u(a+s\delta a)-u(a)\big)\Big|^2\,dxdt.\end{aligned} \tag{18}$$

In particular, we have

$$\begin{aligned}\int_\Omega|\bar{u}|^2\,dx\leq(1+2L^2)\int_0^t\int_\Omega|\bar{u}|^2\,dxdt+2\int_0^t\int_\Omega\Big|F(x,t,s)-\partial_u f\big(x,t,u(a)\big)\Big|^2|\hat{u}|^2\,dxdt\\ +\frac{2}{v_1}\int_0^t\int_\Omega|\delta a|^2\Big|\nabla\big(u(a+s\delta a)-u(a)\big)\Big|^2\,dxdt.\end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned}\int_\Omega|\bar{u}|^2dx\leq e^{[1+2L^2]t}\Big[&2\int_0^t\int_\Omega\Big|F(x,t,s)-\partial_u f\big(x,t,u(a)\big)\Big|^2|\hat{u}|^2\,dxdt\\ &+\frac{2}{v_1}\int_0^t\int_\Omega|\delta a|^2\Big|\nabla\big(u(a+s\delta a)-u(a)\big)\Big|^2\,dxdt\Big].\end{aligned}$$

Therefore, from (18) we have

$$\begin{aligned}&\int_\Omega |\bar{u}|^2\,dx+\frac{v_1}{2}\int_0^t\int_\Omega|\nabla\bar{u}|^2\,dxdt\\ \le&\Big[(1+2L^2)Te^{[1+2L^2]T}+1\Big]\Big[2\int_0^t\int_\Omega\Big|F(x,t,s)-\partial_u f\big(x,t,u(a)\big)\Big|^2|\hat{u}|^2\,dxdt\\ &+\frac{2\|\delta a\|^2_{L^\infty}}{v_1}\int_0^t\int_\Omega\Big|\nabla\big(u(a+s\delta a)-u(a)\big)\Big|^2\,dxdt\Big].\end{aligned}\tag{19}$$

According to the definition of $F(x,t,s)$ and the mean theorem, we have

$$F(x,t,s)=\partial_u f\Big[x,\ t,\ u(x,t,a)+\vartheta\big(u(x,t,a+s\delta a)-u(x,t,a)\big)\Big],$$

in which $\vartheta$ is a function of variables $(x,t,s)$ with $0\le\vartheta\le 1$. Making use of (12), we have

$$\lim_{s\to 0}\int_0^t\int_\Omega\Big|\nabla\big[u(x,t,a+s\delta a)-u(x,t,a)\big]\Big|^2\,dxdt=0\tag{20}$$

and $u(x,t,a+s\delta a)$ converges in measure to $u(x,t,a)$ on $Q_T$ as $s\to 0$.

According to [10, Chap. 1 Lemma 1.3] and ($H_2$), we know $\partial_u f\big[x,t,u(a)+\vartheta\big(u(x,t,a+s\delta a)-u(x,t,a)\big)\big]$ converges in measure to $\partial_u f(x,t,u(a))$ on $Q_T$ as $s\to 0$. On the other hand,

$$\Big|F(x,t,a)-\partial_u f\big(x,t,u(a)\big)\Big|\le 2L.$$

By virtue of Lebesgue's theorem, we obtain

$$\lim_{s\to 0}\int_0^t\int_\Omega|F(x,t,s)-\partial_u f(x,t,u(a))|^2|\hat{u}|^2\,dxdt=0.\tag{21}$$

Using (20) and (21), from (19) we have

$$\lim_{s\to 0}|\bar{u}|_{Q_T}=0,$$

which has proved Theorem 7.

**Theorem 8.** If the cost function $J_0(a)$ at $a_0(x)$ attains a minimum on $K_c$, it is necessary that the variational inequality be satisfied,

$$\int_0^t\int_\Omega\big(a(x)-a_0(x)\big)\nabla u(x,t,a_0)\nabla z(x,t,a_0)\,dxdt\ge 0,\qquad \text{for any } a(x)\in K_c,\tag{22}$$

where $z(x,t,a)$ is the (weak) solution of the adjoint system

$$\begin{aligned}&-\partial_t z-\sum_{i=1}^n\partial_{x_i}\big(a(x)\partial_{x_i}z\big)=\partial_u f(x,t,u(a))z-(u(a_0)-u^*), && (x,t)\in Q_T,\\ &z(x,T)=0, && x\in\Omega,\\ &z(x,t)=0, && (x,t)\in S_T.\end{aligned}\tag{23}$$

*Proof.* Let $t_1 = T - t$, $z_1(x,t_1) = z(x, T - t_1)$. Then Problem (23) is changed into the following problem:

$$\begin{aligned}
&\partial_{t_1} z_1 - \sum_{i=1}^{n} \partial_{x_i}\big(a(x)\partial_{x_i} z_1\big) = \partial_u f(x, T - t_1, u(a)) z_1 - (u(a_0) - u^*), && (x,t) \in Q_T,\\
&z_1(x,0) = 0, && x \in \Omega,\\
&z_1(x,t_1) = 0, && (x,t_1) \in S_T.
\end{aligned}$$

By using [7, Chap. 3 Th.4.2], the problem above has a unique (weak) solution $z_1(x,t_1) \in \overset{0}{V}{}_2^{1,0}(Q_T)$. Therefore the adjoint System (23) has a unique (weak) solution $z(x,t) \in \overset{0}{V}{}_2^{1,0}(Q_T)$.

We introduce the notation

$$P_h(x,t) = \frac{1}{h}\int_T^{t+h} P(x,\tau)\,d\tau, \qquad h > 0.$$

From (13) and the relation [7, p.142], we have

$$\begin{aligned}
&\int_0^{t_1}\int_\Omega \big[\partial_t(\hat{u}_h) g + (a\nabla\hat{u})_h \nabla g\big]\,dxdt\\
&= \int_0^{t_1}\int_\Omega \big(\partial_u f(x,t,u(a))\hat{u}\big)_h g\,dxdt - \int_0^{t_1}\int_\Omega (\delta a \nabla u)_h \nabla g\,dxdt
\end{aligned} \tag{24}$$

for any function $g(x,t)$ from $\overset{0}{V}{}_2^{1,0}(Q_{t_1})$ when $t_1 < T - h$.

In (24), taking $g(x,t) = z_h(x,t)$ we have

$$\begin{aligned}
&\int_0^{t_1}\int_\Omega \big[\partial_t(\hat{u}_h) z_h + (a\nabla\hat{u})_h \nabla z_h\big]\,dxdt\\
&= \int_0^{t_1}\int_\Omega \big(\partial_u f(x,t,u(a))\hat{u}\big)_h z_h\,dxdt - \int_0^{t_1}\int_\Omega (\delta a \nabla u)_h \nabla z_h\,dxdt.
\end{aligned} \tag{25}$$

Similarly from (23), we have

$$\begin{aligned}
&\int_0^{t_1}\int_\Omega \big[-\partial_t(z_h)\hat{u}_h + (a\nabla z)_h \nabla \hat{u}_h\big]\,dxdt\\
&= \int_0^{t_1}\int_\Omega \big(\partial_u f(x,t,u(a)) z\big)_h \hat{u}_h\,dxdt - \int_0^{t_1}\int_\Omega \big(u(a_0) - u^*\big)_h \hat{u}_h\,dxdt.
\end{aligned} \tag{26}$$

From (25) and (26), we obtain

$$\begin{aligned}
&\int_0^{t_1}\int_\Omega \partial_t(z_h\hat{u}_h)\,dxdt\\
&= \int_0^{t_1}\int_\Omega \Big[\big(\partial_u f(x,t,u(a))\hat{u}\big)_h z_h - \big(\partial_u f(x,t,u(a))z\big)_h \hat{u}_h\Big]\,dxdt\\
&\quad + \int_0^{t_1}\int_\Omega \big(u(a_0) - u^*\big)_h \hat{u}_h\,dxdt - \int_0^{t_1}\int_\Omega (\delta a\nabla u)_h \nabla z_h\,dxdt,
\end{aligned}$$

that is,

$$\int_\Omega z_h \hat{u}_h \, dx \Big|_{t=0}^{t=t_1}$$
$$= \int_0^{t_1} \int_\Omega \Big[ (\partial_u f(x,t,u(a))\hat{u})_h z_h - \big(\partial_u f(x,t,u(a))z\big)_h \hat{u}_h \Big] \, dxdt$$
$$+ \int_0^{t_1} \int_\Omega \big(u(a_0) - u^*\big)_h \hat{u}_h \, dxdt - \int_0^{t_1} \int_\Omega (\delta a \nabla u)_h \nabla z_h \, dxdt.$$

Let $h$ tend to zero. This gives

$$\int_\Omega z\hat{u} \, dx \Big|_{t=0}^{t=t_1} = \int_0^{t_1} \int_\Omega \big(u(a_0) - u^*\big)\hat{u} \, dxdt - \int_0^{t_1} \int_\Omega (\delta a \nabla u) \nabla z \, dxdt. \tag{27}$$

Because $z(x,t)$, $\hat{u}(x,t) \in \overset{0}{V}{}_2^{1,0}(Q_T)$, we know that $z(x,t)$ and $\hat{u}(x,t)$ are continuous in $t$ in the norm of $L^2(\Omega)$. According to the initial conditions of $z(x,t)$ and $\hat{u}(x,t)$, it is easy to know

$$\int_\Omega z(x,t)\hat{u}(x,t) \, dx \Big|_{t=0}^{t=T} = 0.$$

From (27), we obtain

$$\int_0^T \int_\Omega \delta a \nabla u \nabla z \, dxdt = \int_0^T \int_\Omega \big(u(a_0) - u^*\big)\hat{u} \, dxdt. \tag{28}$$

Now we will consider the necessary condition (22). It is obvious that the set $K_c$ is a convex set. Therefore for any $a \in K_c$, $s \in [0, 1]$, $a_0 + s(a - a_0) \in K_c$. Since $J_0(a_0) = \inf\{J_0(a) : a(x) \in K_c\}$, the following (one-sided) directional derivative has to be nonnegative:

$$\lim_{s \to 0^+} \frac{J_0(a_0 + s(a - a_0)) - J_0(a_0)}{s} \geq 0 \qquad \text{for any } a(x) \in K_c.$$

Let $\phi(s) = J_0(a_0 + s\delta a)$ where $\delta a = a - a_0$. Then $\phi'(0^+) \geq 0$ and this implies

$$\int_0^T \int_\Omega \big(u(a_0) - u^*\big)\hat{u}(x,t) \, dxdt \geq 0 \qquad \text{for any } a(x) \in K_c,$$

where $\hat{u}(x,t)$ is the solution of (13). According to (28), we obtain

$$\int_0^T \int_\Omega (a - a_0) \nabla u \nabla z \, dxdt \geq 0 \qquad \text{for any } a(x) \in K_c,$$

which is the inequality (22).

Similarly, we have

**Theorem 9.** If the cost function $J_\varepsilon(a)$ at $a_0(x)$ attains a minimum on $A_{ad}$, it is necessary that the following variational inequality be satisfied:

$$\int_0^T \int_\Omega (a - a_0) \nabla u(a_0) \nabla z(a_0) \, dxdt + \varepsilon \int_\Omega (a_0 - a^*)(a - a_0) \, dx \geq 0$$

for any $a(x) \in A_{ad}$, where $z(x,t,a)$ is the (weak) solution of Problem (23).

**Acknowledgements.** The author wishes to thank Prof. Fang Aïnong for his valuable help on the work. The author is also grateful to the referee for his careful reading and constructive criticism of the original manuscript.

## References

[1] Banks H.T. On a Variational Approach to Some Parameter Estimation Problem. ICASE Report No. 85-32 Nasa Langley Research Center, Hampton, VA, June, 1985.

[2] Kravaris C. & Seinfield J.H. Identification of Parameters in Distributed Parameter Systems by Regularization. ßSIAM J. Control Optim., 1985, 23: 217–241.

[3] Kunisch K. & White L. The Parameter Estimation Problem for Parabolic Equations and Discontinuous Observation Operators. ßSIAM J. Control Optim., 1985, 23: 900–927.

[4] Murat F. Contre-examples Pourdivers Problems ou le Controle Intervient dans les coefficients. ßAnn. Math. Pura. et Appl., 1977, 112: 49–68.

[5] Jacob Bear. Dynamics of Fluids in Porous Media. American Elsevier, New York, 1972, 214–215.

[6] Liu Xiyuan. Identification Problem of a Parameter for Geothermal Reservoir Exploitation. ßPartial Differential Equations, 1988, 1(B) (1): 67–74 (in Chinese).

[7] Ladyzenskaja, O.A., Slolnnikov V.A. and Urallceva N.N. Linear and Quasi-linear Equations of Parabolic Type. Amer. Math. Soc., Providence, R.I., 1968.

[8] Giusti E. Minimal Surfaces and Functions of Bounded Variation. Birkhauser, Biston, Stutlgart, 1984.

[9] Baranger J. Existence de solutions pour des problemes d'optimisation nonconvexe. ßJ. Math. Pures. et Appl., 1973, 52(4): 377–405.

[10] Guo Dajun. Nonlinear Functional Analysis. Shandon Acad. Press, Jinan, 1985 (in Chinese).