THE GRAVITATIONAL ATTRACTION OF A RIGHT VERTICAL CIRCULAR CYLINDER AT POINTS EXTERNAL TO IT

by MISAC N. NABIGHIAN (*)

Summary — In this paper a general and exact expression of the gravitational attraction of a right vertical circular cylinder at points external to it is developed. This expression is derived in terms of complete elliptic integrals of the first and second kind and the Neumann's Lambda function. Since the solution involves only tabulated functions, it is well suited for rapid desk calculations with any degree of accuracy at any points, including the points in the plane of the cylinder (outcroping cylinder). For this case, the corresponding master curve is given. Finally, a relation between the abscissa of the inflexion point of the Δg curve and the depth of the cylinder is established.

List of Symbols.

r, φ, z = polar coordinates,

R = the radius of the cylinder,

a_0 = the depth of the cylinder,

$$a = \frac{a_0}{R},$$

x_0 = the horizontal distance between the axis of the cylinder and the point of computation,

$$x = \frac{x_0}{R},$$

$$d_0^2 = x_0^2 + a_0^2,$$

$$d^2 = x^2 + a^2,$$

$$k^2 = \frac{4Rx_0}{(R + x_0)^2 + a_0^2} = \frac{4x}{(1 + x)^2 + a^2}$$ = the modulus of the elliptic integrals,

(*) Academia R.P.R., Sectia V, Centrul de Cercetari Geofizice, Calea Victoriei 125, *Bucuresti*, Romînia. - Now: Rue Hamade Zarub el Sidani, Imm. Assaf Dib, Zokak el Blatt, *Beirut*, Lebanon.

$$\left.\begin{aligned} n^2 &= \frac{2x_0}{d_0 - x_0} = \frac{2x}{d - x} \\ m^2 &= \frac{2x_0}{d_0 + x_0} = \frac{2x}{d + x} \end{aligned}\right\} = \text{parameters of the elliptic integrals of the third kind,}$$

$K(k)$ = complete elliptic integral of the first kind,

$E(k)$ = complete elliptic integral of the second kind,

$\Pi(-n^2, k)$ and $\Pi(m^2, k)$ = complete elliptic integrals of the third kind,

$\Lambda_0(\varphi, k)$ = Neumann's Lambda function,

G = universal gravitational constant,

δ = density of the cylinder.

Introduction.

Frequently, either for explaining observed anomalies or for applying corrections to them, an estimate of the gravitational attraction of a right vertical cylinder at points external to it is needed. But, at the present moment, in spite of the apparent simplicity, this problem at least as we know it has not a general exact solution.

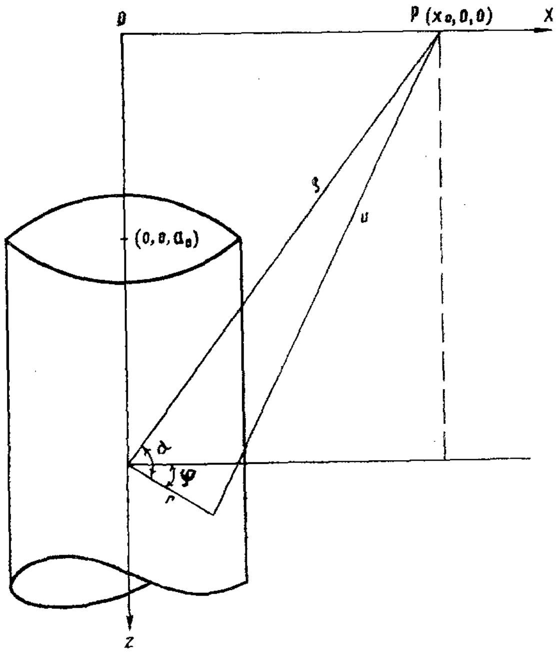

Fig. 1 - *Notation used to obtain the attraction of a vertical cylinder.*

Recently, Gravity Prospectors turned their attention to a paper published by D. S. PARASNIS [1] which developed a new expression, under the form of an infinite series, for the calculation of the gravitational attraction of a vertical cylinder, indicating at the same time the errors made in applying the existing approximative formulae. Unfortunately, on the one hand, the Δg expression due to PARASNIS does not represent correctly the situation at $a = 0$ and for small values of « a » (cases often met in practice) and, on the other hand, the calculations are laborious, especially when a great accuracy is needed.

In this paper, a general and exact solution, in finite form, of the gravitational attraction of a right vertical cylinder at points external to it is developed in terms of complete elliptic integrals of the first and second kind and the NEUMANN's Lambda function. Since the solution involves only tabulated functions, the calculations can be rapidly made with any degree of accuracy and at any point, including the points in the plane of the cylinder.

Solution of the Problem.

In cylindrical coordinates, with the symbols used in Fig. 1, and taking into account the relation

$$\rho \cos \alpha = x_0 \cos \varphi$$

the gravitational attraction of a vertical seminfinite cylinder at the point $P(x_0, 0, 0,)$ is

$$\Delta g = G\delta \int_0^{2\pi} \int_0^{R} \int_{a_0}^{\infty} \frac{z r dr d\varphi dz}{[r^2 + \rho^2 - 2 r x_0 \cos \varphi]^{3/2}} .$$

After a first integration with respect to z and a second integration with respect to r, and after applying the substitution

$$\varphi = \pi + 2\beta$$

it results

$$\text{(1)} \qquad \Delta g = 4G\delta \int_0^{\pi/2} \left\{ \sqrt{R^2 + d_0^2 + 2Rx_0 \cos 2\beta} - \right.$$

$$\left. - d_0 + x_0 \cos 2\beta \ln \frac{d_0 + x_0 \cos 2\beta}{\sqrt{R^2 + d_0^2 + 2Rx_0 \cos 2\beta} + R + x_0 \cos 2\beta} \right\} d\beta$$

or

$$\text{(2)} \qquad \Delta g = 4G\delta \left[- \frac{\pi}{2} d_0 + I_1 + I_2 + I_3 \right]$$

where

$$I_1 = \int_0^{\pi/2} \sqrt{R^2 + d_0^2 + 2Rx_0 \cos 2\beta} \, d\beta$$

$$I_2 = \int_0^{\pi/2} x_0 \cos 2\beta \ln [d_0 + x_0 \cos 2\beta] \, d\beta$$

$$I_3 = - \int_0^{\pi/2} x_0 \cos 2\beta \ln [\sqrt{R^2 + d_0^2 + 2Rx_0 \cos 2\beta} + R + x_0 \cos 2\beta] \, d\beta .$$

a) *The calculation of the integral* I_1.

(3) $$I_1 = \int_0^{\pi/2} \sqrt{R^2 + d_0^2 + 2Rx_0 (1 - 2\sin^2\beta)}\, d\beta = \frac{2\sqrt{Rx_0}}{k} E(k) .$$

b) *The calculation of the integral* I_2.

(4) $$I_2 = x_0 \left\{ \frac{1}{2} \sin 2\beta \ln (d_0 + x_0 \cos 2\beta) \Big|_0^{\pi/2} + x_0 \int_0^{\pi/2} \frac{\sin^2 2\beta}{d_0 + x_0 \cos 2\beta} d\beta \right\} =$$

$$= \frac{\pi}{2} (d_o - a_o) .$$

c) *The calculation of the integral* I_3.

If we introduce the abbreviation $R^2 + d_0{}^2 + 2Rx_0 \cos 2\beta = A$, the integral I_3 becomes

$$I_3 = - x_0 \left\{ \frac{1}{2} \sin 2\beta \ln [\sqrt{A} + R + x_0 \cos 2\beta] \Big|_0^{\pi/2} + \right.$$

$$\left. + x_0 \int_0^{\pi/2} \sin^2 2\beta \frac{R + \sqrt{A}}{\sqrt{A}(\sqrt{A} + R + x_0 \cos 2\beta)} d\beta \right\} .$$

By taking into account that $(\sqrt{A} + R + x_0 \cos 2\beta)(\sqrt{A} - R - x_0 \cos 2\beta) = = d_0{}^2 - x_0{}^2 \cos^2 2\beta$ it results

(5) $$I_3 = - x_0^2 \int_0^{\pi/2} \sin^2 2\beta \frac{d_0^2 + Rx_0 \cos 2\beta - \sqrt{A}\, x_0 \cos 2\beta}{\sqrt{A}(d_0^2 - x_0^2 \cos^2 2\beta)} d\beta =$$

$$= - x_0^2 \int_0^{\pi/2} \frac{\sin^2 2\beta [d_0^2 + Rx_0 \cos 2\beta]}{\sqrt{A}(d_0^2 - x_0^2 \cos^2 2\beta)} d\beta$$

because

$$\int_0^{\pi/2} \frac{\sin^2 2\beta \cdot \cos 2\beta}{d_0^2 - x_0^2 \cos^2 2\beta} d\beta = 0 .$$

The expression (5) after calculations that do not present any difficulty, becomes

(6) $$I_3 = a_0^2 \int_0^{\pi/2} \frac{d_0^2 + Rx_0 \cos 2\beta}{\sqrt{A}(d_0^2 - x_0^2 \cos 2\beta)} d\beta - \int_0^{\pi/2} \frac{d_0^2 + Rx_0 \cos 2\beta}{\sqrt{A}} d\beta =$$

$$(6)\quad \begin{aligned} &= a_0^2 \left\{ \frac{1}{2} \frac{d_0 + R}{d_0 - x_0} \int_0^{\pi/2} \frac{d\beta}{(1 + n^2 \sin^2 \beta) \sqrt{A}} + \right. \\ &\left. + \frac{1}{2} \frac{d_0 - R}{d_0 + x_0} \int_0^{\pi/2} \frac{d\beta}{(1 - m^2 \sin^2 \beta) \sqrt{A}} \right\} - \int_0^{\pi/2} \frac{d_0^2 + R x_0 \cos 2\beta}{\sqrt{A}} d\beta = \\ &= \frac{1}{2k \sqrt{R x_0}} \left\{ [2 R x_0 - k^2 d_0^2 - k^2 R x_0] K(k) - 2 R x_0 E(k) + \right. \\ &\left. + \frac{k^2}{2} [(d_0 + R)(d_0 + x_0) \Pi(-n^2, k) + (d_0 - R)(d_0 - x_0) \Pi(m^2, k)] \right\} . \end{aligned}$$

Adding now the expressions (3), (4) and (6) and introducing the values x and a, the expression (2) becomes

$$(7)\quad \begin{aligned} \Delta g = 2 G \delta R &\left\{ \frac{(2 - k^2) x - k^2 d^2}{k \sqrt{x}} K(k) + \frac{2 \sqrt{x}}{k} E(k) + \right. \\ &\left. + k \sqrt{x} \left[\frac{d + 1}{m^2} \Pi(-n^2, k) + \frac{d - 1}{n^2} \Pi(m^2, k) \right] - \pi a \right\} . \end{aligned}$$

By expressing the elliptic integrals of the third kind in terms of NEUMANN's Lambda function (circular cases) (²):

$$\Pi(-n^2, k) = \frac{k^2}{k^2 + n^2} K(k) + \frac{\pi}{2} \Lambda_0(\beta, k) \sqrt{\frac{n^2}{n^2 + k^2}} \cdot \frac{1}{\sqrt{1 + n^2}}$$

$$\left(\sin \beta = \sqrt{\frac{n^2}{n^2 + k^2}} \right)$$

and

$$\Pi(m^2, k) = \frac{\pi}{2} \Lambda_0(\theta, k) \sqrt{\frac{m^2}{m^2 - k^2}} \cdot \frac{1}{\sqrt{1 - m^2}}$$

$$\left(\sin \theta = \sqrt{\frac{m^2 - k^2}{m^2 (1 - k^2)}} \right)$$

and by taking into account the addition formula for Lambda function

$$\Lambda_0(\beta, k) + \Lambda_0(\theta, k) = \Lambda_0(\varphi, k) + \frac{2}{\pi} (1 - k^2) \sin \beta \sin \theta \sin \varphi K(k)$$

with

$$\cos \varphi = \frac{\cos \beta \cos \theta - \sin \beta \cdot \sin \theta \sqrt{(1 - k'^2 \sin^2 \beta)(1 - k'^2 \sin^2 \theta)}}{1 - k'^2 \sin^2 \theta \sin^2 \beta} \qquad (k'^2 = 1 - k^2)$$

the expression (7) finally becomes (*)

$$(8) \qquad \Delta g = 2G\delta R \left\{ \frac{1 - x^2}{\sqrt{(1+x)^2 + a^2}} K(k) + \sqrt{(1+x)^2 + a^2}\, E(k) + \right.$$

$$\left. + \frac{\pi}{2} a \Lambda_0 (\varphi, k) - \pi a \right\}$$

where

$$\sin \varphi = \frac{a}{\sqrt{(1-x)^2 + a^2}} \, .$$

For $x = 0$ $[\Lambda_0(\varphi, 0) = \sin \varphi]$ the formula (8) gives

$$\Delta g|_{x=0} = 2\pi G \delta R \left(\sqrt{1 + a^2} - a \right) = 2\pi G \delta \left(\sqrt{R^2 + a_0^2} - a_0 \right)$$

the well known expression of the gravitational attraction on the axis of the cylinder.

For $a = 0$ (outcropping cylinder)

$$(9) \qquad \Delta g|_{a=0} = 2G\delta R \left\{ (1 - x) K(k) + (1 + x) E(k) \right\} .$$

This expression appears for the first time, to our knowledge, in the publications of applied geophysics.

The master curve for this case is shown in Fig. 2 and the corresponding numerical values are given in the table 1.

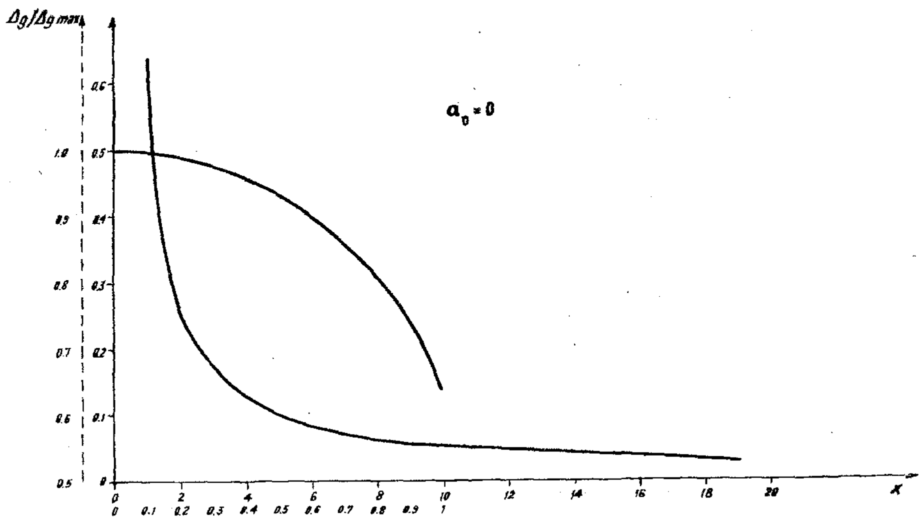

Fig. 2 - *Master curve for an outcropping vertical cylinder.*

(*) For numerical computations, one must note the relations

$$x < 1 \rightarrow 0 \leq \varphi \leq \frac{\pi}{2} \, ; \qquad x > 1 \rightarrow \frac{\pi}{2} \leq \varphi \leq \pi \, ; \qquad \Lambda_0 (\pi - \varphi, k) = 2 - \Lambda_0 (\varphi, k) \, .$$

TABLE 1.

$x = \frac{x_0}{R}$	k^2	$\Delta g/\Delta g_{max}$	$x = \frac{x_0}{R}$	k^2	$\Delta g/\Delta g_{max}$
0	0.00	1.000000	1	1.00	0.636619
1/19	0.19	0.999305	11/9	0.99	0.457258
1/9	0.36	0.996905	3/2	0.96	0.355930
3/17	0.51	0.992165	13/7	0.91	0.280246
1/4	0.64	0.984185	7/3	0.84	0.219579
1/3	0.75	0.971612	3	0.75	0.169083
3/7	0.84	0.952364	4	0.64	0.125999
7/13	0.91	0.923003	17/3	0.51	0.088579
2/3	0.96	0.877325	9	0.36	0.055641
9/11	0.99	0.801886	19	0.19	0.026321

Further it is easy to calculate the first and the second derivatives of the expression (8). It results

$$\frac{\partial(\Delta g)}{\partial x} = 2G\delta R \left\{ - \frac{1 + x^2 + a^2}{x\sqrt{(1+x)^2 + a^2}} K(k) + \frac{\sqrt{(1+x)^2 + a^2}}{x} E(k) \right\}$$

and

$$(10) \quad \frac{\partial^2(\Delta g)}{\partial x^2} = \frac{2G\delta R}{x^2[(1-x)^2 + a^2]\sqrt{(1+x)^2 + a^2}} \{(1 + a^2)[(1-x)^2 + a^2] K(k) -$$

$$- [(a^2 + 1)^2 + x^2(a^2 - 1)] E(k)\} .$$

This last expression gives a relation between the abscissa of the inflexion point of the Δg curve and the depth a of the cylinder by means of the equation

$$(1 + a^2)[(1-x)^2 + a^2] K(k) = [(a^2 + 1)^2 + x^2(a^2 - 1)] E(k) .$$

In a next paper, the master curves for a right vertical circular cylinder for $0 \leq x \leq 20$ and $0 \leq a \leq 20$ will be given and the expression (8) will be discussed from the point of view of the second derivative.

Acknowledgements.

The author wishes to thank Professor SABBA STEFANESCU, of the Institute of Oil, Gas and Geology, and Engineer RADU BOTEZATU, chief of Gravity Department of the Geological Comitee, for their advices and for active encouraging of these investigations.

REFERENCES

(1) PARASNIS D. S. (1961): Geophysical Prospecting, IX, number 3, pp. 382-398. — (2) BYRD P. F. & FRIEDMAN M. D. (1954): *Handbook of elliptic integrals for Engineers and Physicists*, Springer Verlag, Berlin.

(*Received on* 15*th May* 1962)