DIRECTIONAL PERMEABILITY OF POROUS MEDIA TO HOMOGENEOUS FLUIDS (*)

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Summary — Investigations have shown that in groundwater- and oil-bearing strata there are preferential directions of flow that are often maintained over wide areas. JOHNSON & HUGHES (1948, see Ref.) analysed a series of oil well cores by cutting them into small horizontal plugs and they obtained directional permeabilities which they plotted in the form of polar graphs. They were not able to give a physical explanation of this phenomenon. On the other hand, there exists a theory of permeability in which the latter is represented as a symmetric tensor. This theory has been developed by FERRANDON (1948, see Ref.), but no experimental substantiation of it seems ever to have been attempted.

In the present paper, the author undertakes to compare the two sets of findings. From FERRANDON's theory, the directional permeabilities (denoted by k) corresponding to the experiments of JOHNSON & HUGHES are calculated and it is shown that $k^{-1/2}$ if plotted as polar graph, should form an ellipse. The data of JOHNSON & HUGHES are then are drawn. In this manner, a substantiation of the tensor theory of FERRANDON is obtained.

Zusammenfassung — Untersuchungen der Permeabilität von Grundwasser und Erdöl führenden Gesteinsschichten haben gezeigt, dass dieselbe in vielen Fällen richtungsabhängig ist. Hierbei bleibt die Richtung extremaler Permeabilität oft über weite Gebiete konstant. JOHNSON & HUGHES untersuchten eine Reihe von Bohrkernen von Oelquellen auf Richtungsabhängigkeit der Permeabilität. Hierzu schnitten sie aus denselben kleine, waagrechte Stücke, bestimmten deren Permeabilität und stellten das Ergebnis ihrer Messungen in der Form von Permeabilitätspolardiagrammen dar. Sie waren nicht im Stande, eine theoretische Erklärung der erhaltenen Kurven zu geben. Auf der anderen Seite existiert eine Theorie der Permeabilität, wobei die letztere als symmetrischer Tensor behandelt wird. Diese Theorie wurde von FERRANDON vorgeschlagen; es scheint aber, dass keine experimentelle Bestätigung davon je versucht worden ist.

In der vorliegenden Arbeit vergleicht der Verfasser die zwei Typen von Untersuchungen. Nach der FERRANDON'schen Theorie wird die « gerichtete » Permeabilität (mit k bezeichnet), die den Experimenten von JOHNSON & HUGHES entspricht, berechnet. Es wird gezeigt, dass $k^{-1/2}$, als Polardiagramm dargestellt, die Gestalt einer Ellipse

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haben sollte. Die Resultate von JOHNSON & HUGHES werden dann in die Form von $k^{-1/2}$ umgerechnet und als entsprechende Polardiagramme dargestellt. In dieser Weise wird eine experimentelle Bestätigung der Tensortheorie von FERRANDON erhalten.

1. *Introduction* — The study of flow of fluids through porous media has recently become important in several branches of applied geophysics. The dislocation of fluids in water and in oil-bearing strata is fully conditioned by the mechanics of such flow. Thus, the latter is fundamental for ground water hydrology as well as for the science of petroleum production.

The fundamental law of flow through porous media has been postulated a great numer of years ago by DARCY. However, the law of DARCY, in its usual form, surmises the porous medium to be isotropic. Unfortunately it has been found that natural porous media are more often that not anisotropic, i.e., that fluids may move more easily in one direction than in another. This fact is well known from studies in groundwater hydrology. Similarly, JOHNSON et al. (JOHNSON & HUGHES, 1948; JOHNSON & BRESTON, 1950) analysed a series of oil well cores by cutting them into small horizontal plugs and observed that the permeability varies with the direction in which the plug is cut.

It is therefore necessary that the theory of DARCY be extended to cases where anisotropy occurs. Such an extension has been developed by FERRANDON (1948) upon theoretical grounds, in which permeability is represented as a symmetric tensor. An evaluation of this theory as presented in the present paper, yields that directional permeabilities (denoted by k) should not form a simple figure if they are plotted as a polar graph, but that a polar graph of $k^{-1/2}$ should yield a simple curve, namely an ellipse.

Furthermore, an analysis of the measurements of JOHNSON et al. in the light of the theory of FERRANDON, leads to a substantiation of the tensor theory of permeability. The data published by JOHNSON et al. can be recalculated and, for thirty samples, the values of $k^{-1/2}$ can be determined. A method can then be devised to plot the best-fitting ellipses through those values whereupon it is seen that the curves obtained in this manner coincide reasonably well with the actually measured values. Accordingly, a substantiation of the tensor theory of FERRANDON is obtained.

2. *The Permeability Tensor* — The flow of fluids through porous media is commonly thought to be subject to DARCY's law, viz.

$$\boldsymbol{q} = (k/\mu)\,\mathrm{grad}\,p \tag{2.1}$$

where $\boldsymbol{q}$ is the vector of the seepage velocity of the fluid, k is the permeability of the porous medium, μ is the viscosity of the fluid and p is the pressure in the fluid. Herein, it has been assumed that gravity can be neglected, otherwise one would have to replace p by a function of the hydrostatic head and the density of the fluid.

Equation (2.1) describes the flow in an isotropic porous medium. In the case of anisotropic media, equation (2.1) has to be generalized. Such a generalization has been proposed by FERRANDON (1948), and similarly, by LITWINISZYN (1950). A further review of the subject has been given by IRMAY (1951). LITWINISZYN arrived at his equation by analogy with the process of diffusion, whereas FERRANDON gave an actual theoretical derivation of the formulas. Both theories are essentially identical.

The theory of FERRANDON assumes that the contribution to the quantity q_n of flow through unit area in the direction $\boldsymbol{n}$ (components n_i) from elementary flow tubes parallel to the direction $\boldsymbol{m}$ (components m_i) whose combined cross-sectional area is equal to $\gamma d\omega$ ($d\omega$ denoting the solid angle), is proportional to the gradient of the pressure in the direction of $\boldsymbol{m}$. Thus one has

$$dq_n = \Sigma_{ij} k \, \gamma n_i \, m_i \, (\partial p/\partial x_j) \, m_j \, d\omega/\mu \, . \tag{2.2}$$

Here, k and γ are, of course, functions of m_i, such that one can set upon integration:

$$q_n = \Sigma_{ij} \, n_i \, \partial p/\partial x_j \, k \, \gamma \, m_i \, m_j \, d\omega/\mu = \Sigma_{ij} \, n_i \, (k_{ij}/\mu) \, (\partial p/\partial x_j) \tag{2.3}$$

where $k_{il} = k_{ji}$. This can be written in vectorial form as follows:

$$\boldsymbol{q} = (\bar{k}/\mu) \text{ grad } p \tag{2.4}$$

where $\bar{k}$ is a symmetric tensor consisting of the components k_{ij}. It can properly be referred to as the « permeability-tensor » of the porous medium.

The fact that the permeability of an anisotropic porous medium can be represented as a symmetric tensor, leads immediately to the following conclusions:

(*i*) In general, the pressure gradient (grad p) and the seepage velocity $\boldsymbol{q}$ do not have the same direction.

(*ii*) There are three orthogonal axes in space along which the pressure gradient and the seepage velocity do have the same direction. These axes are termed the « principal axes » of the permeability tensor. Their direction is that of the eigenvectors of the permeability tensor.

The task remains to relate the permeability tensor to what physically might be called « directional permeability » of a porous medium. The « directional permeability » may be obtained by cutting a pencil-shaped piece parallel to the direction $\boldsymbol{n}$ out of the medium and measuring its permeability in the ordinary manner (i.e. based upon DARCY's law). In this case, the seepage velocity $\boldsymbol{q}$ must obviously be parallel to $\boldsymbol{n}$; let it be denoted by q_n. The pressure drop along the « pencil », denoted by p_n, is then given by Eq. (2.4) as follows:

$$p_n = \boldsymbol{n} \, \text{grad} \, p = \mu \, \boldsymbol{n} \, \overline{K} \, \boldsymbol{n} \, q_n \, , \tag{2.5}$$

where $\overline{K}$ is the inverse tensor of $\bar{k}$, such that

$$\overline{K} \, \bar{k} = \bar{k} \, \overline{K} = \bar{I} \, , \tag{2.6}$$

$\bar{I}$ denoting the unit tensor.

The directional permeability k_n in the direction $\boldsymbol{n}$ is then given by

$$k_n = \mu \, q_n/p_n = 1/(\boldsymbol{n}\overline{K}\boldsymbol{n}) \, . \tag{2.7}$$

An alternative method to define directional permeability physically is by choosing a system in which the pressure drop is given by the boundary conditions, and by measuring that component of the velocity which is parallel to the pressure gradient. The directional permeability is thus defined by DARCY's law from the given pressure and the velocity component parallel to it. Thus, denoting the pressure gradient by p_n, it being in the direction of $\boldsymbol{n}$, and the velocity component parallel to this direction by $q_n = \boldsymbol{nq}$, one has

$$k_n = \mu \, q_n/p_n = \mu \boldsymbol{n}\boldsymbol{q}/p_n = \boldsymbol{n}\bar{k}\boldsymbol{n} \, .$$

However, this last expression is identical to that obtained in (2.7) because one has, owing to the fact that the distributive law holds in matrix multiplication:

$$\boldsymbol{n}\,\overline{\boldsymbol{k}}\,\boldsymbol{n}\,\boldsymbol{n}\,\overline{\boldsymbol{K}}\,\boldsymbol{n} = \boldsymbol{n}\,\overline{\boldsymbol{k}}\,\overline{\boldsymbol{K}}\,\boldsymbol{n} = \boldsymbol{n}\,\overline{\boldsymbol{I}}\,\boldsymbol{n} = 1\,,$$

and thus:

$$\boldsymbol{n}\,\overline{\boldsymbol{k}}\,\boldsymbol{n} = 1/(\boldsymbol{n}\,\overline{\boldsymbol{K}}\,\boldsymbol{n})\,.$$

Let us now choose the principal axes of the permeability tensor as co-ordinate axes (the corresponding permeabilities being k_1, k_2, k_3) and denote the angles of $\boldsymbol{n}$ with those axes by α, β, γ. Then Eq. (2.7) yields

$$1/k_n = \cos^2\alpha/k_1 + \cos^2\beta/k_2 + \cos^2\gamma/k_3 \tag{2.8}$$

which is the central equation of an ellipsoid if

$$r = k_n^{-1/2} \tag{2.9}$$

is plotted on the corresponding directions of $\boldsymbol{n}$. One has thus the following theorem:

If the inverse square root of the directional permeability is plotted on all of the corresponding directions in a point of an anisotropic porous medium, then one obtains an ellipsoid. The axes of the latter are in the direction of the principal axes of permeability, their length being equal to the inverse square root of the principal permeabilities (i.e. of the eigenvalues of the permeability tensor). The ellipsoid is called the « ellipsoid of permeability ».

Now, if any plane section of a porous medium be taken and the directional permeability be determined around 180°, then the values of $k^{-1/2}$, if plotted on the directions to which they apply in form of a polar diagram, lie on an ellipse. In a polar diagram of k instead of $k^{-1/2}$, the resulting curve will not be as simple as an ellipse, but given by the equation

$$k = k_1 k_2/(k_2 \sin^2\alpha + k_1 \cos^2\alpha)\,. \tag{2.10}$$

3. *Theory of Directional Permeability Measurements* — In practice, it is possible to measure directional permeabilities of a porous medium for various directions. The problem is then to properly plot those values, to test how well they conform to the theory postulating the existence of a permeability tensor, and to calculate the best-fitting such tensor.

The subsequent analysis will be confined to the two- dimensional case. The extension to three dimensions would be quite straightforward.

The proper representation of directional permeability k for a porous medium is obviously by plotting $k^{-1/2}$ as a polar diagram. The resulting figure should be an ellipse. If this is the case, then the measurements conform to the theory. In practice, if a series of points are measured, there will be some scattering around the ellipse. In order to find the permeability tensor, one is faced with the task of drawing the best-fitting ellipse.

It is a very tedious job to construct a best-fitting ellipse through a series of points, owing to the fact that the equation of the latter is quadratic. On can, however, approach the problem from a different angle.

Each directional permeability value measured — denoted by k_i, — should be dependent on the corresponding angle α_i according to Eq. (2.7). This means

$$K_i = 1/k_i = K_{11}\cos^2\alpha_i + 2K_{12}\cos\alpha_i\sin\alpha_i + K_{22}\sin^2\alpha_i \tag{3.1}$$

where K_{ij} are the components of the inverse permeability tensor.
Denoting the measured values of $1/k_i$ by ρ_i, the « theoretical » values of $1/k_i$ by K_i (i.e. the values which correspond to the assumption that permeability *is* a tensor), one has to determine the coefficients K_{ij} such that one obtains a best fit between « theoretical » and « measured » values. According to the usual procedures, this is done by minimizing the sum of the square-deviations of measured from theoretical values, i.e. by requesting (δ denoting the variation):

(3.2) $$\delta \Sigma (K_i - \rho_i)^2 = 0 \; .$$

Inserting (3.1) yields

(3.4) $$2\Sigma (K_{11} \cos^2 \alpha_i + 2K_{12} \cos \alpha_i \sin \alpha_i + K_{22} \sin^2 \alpha_i - \rho_i) (\cos^2 \alpha_i \, \delta K_{11} + 2 \cos \alpha_i \sin \alpha_i \, \delta K_{12} + \sin^2 \alpha_i \, \delta K_{22}) = 0 \; .$$

This leads to

(3.5) $$\begin{cases} K_{11} \Sigma \cos^4 \alpha i + K_{12} \Sigma 2 \cos^3 \alpha_i \sin \alpha_i + K_{22} \Sigma \sin^2 \alpha_i \cos^2 \alpha_i = \Sigma \rho_i \cos^2 \alpha_i \\ K_{11} \Sigma \cos^2 \alpha_i \sin^2 \alpha_i + K_{12} \Sigma 2 \cos \alpha_i \sin^3 \alpha_i + K_{22} \Sigma \sin^4 \alpha_i = \Sigma \rho_i \sin^2 \alpha_i \\ K_{11} \Sigma \cos^3 \alpha_i \sin \alpha_i + K_{12} \Sigma 2 \cos^2 \alpha_i \sin^2 \alpha_i + K_{22} \Sigma \cos \alpha_i \sin^3 \alpha_i = \Sigma \rho_i \cos \alpha_i \sin \alpha_i . \end{cases}$$

This is a system of 3 *linear* equations for the three unknowns. It can be solved without too much trouble. However, if the measurements are made at equal angle intervals between 0° and 180°, then all the terms containing only odd powers of the goniometric functions will cancel out and the system reduces to:

(3.6) $$\begin{cases} K_{11} \Sigma \cos^4 \alpha_i + K_{22} \Sigma \sin^2 \alpha_i \cos^2 \alpha_i = \Sigma \rho \cos^2 \alpha_i \\ K_{11} \Sigma \cos^2 \alpha_i \sin^2 \alpha_i + K_{22} \Sigma \sin^4 \alpha_i = \Sigma \rho_i \sin^2 \alpha_i \\ K_{12} \Sigma 2 \cos^2 \alpha_i \sin^2 \alpha_i = \Sigma \rho_i \cos \alpha i \sin \alpha_i \; . \end{cases}$$

In particular, if the angles α_i chosen are 0°, 30°, 60° ....., one obtains

(3.7) $$\begin{cases} K_{11} = (1/2) \Sigma \; \rho_i \cos^2 \alpha_i - (1/6) \Sigma \rho_i \sin^2 \alpha_i \\ K_{22} = (1/2) \Sigma \rho_i \sin^2 \alpha_i - (1/6) \Sigma \rho_i \cos^2 \alpha_i \\ K_{12} = (2/3) \Sigma \rho_i \cos \alpha_i \sin \alpha_i \, , \end{cases}$$

and if the angles are 0, 22° 30′, 45° ....., one obtains:

(3.8) $$\begin{cases} K_{11} = (3/8) \Sigma \rho_i \cos^2 \alpha_i - (1/8) \Sigma \rho_i \sin^2 \alpha \\ K_{22} = (3/8) \Sigma \rho_i \sin^2 \alpha_i - (1/8) \Sigma \rho_i \cos^2 \alpha \\ K_{12} = (1/2) \Sigma \rho_i \cos \alpha_i \sin \alpha_i \; . \end{cases}$$

With the knowledge of K_{ij} — the inverse permeability tensor —, it is easy to calculate the position of the principal axes of the ellipse of permeability. One obtains

(3.9) $$\tan \varphi = \left\{ K_{22} - K_{11} \pm [(K_{11} - K_{22})^2 + 4K_{12}^2]^{1/2} \right\} / (2K_{12})$$

where φ is the angle of position of the principal axes.

Furthermore, the length of the axes of the ellipse of permeability are simply given by the square root of the eigenvalues λ of the inverse permeability tensor K. The latter are the roots of the equation

(3.10) $$| \overline{K} - \lambda \overline{I} \, | = 0$$

i.e. of

(3.11) $$(K_{11} - \lambda) (K_{22} - \lambda) - K_{12}^2 = 0 \; .$$

Explicitly, the eigenvalues are

$$\lambda = (1/2)\left\{ K_{11} + K_{22} \pm [(K_{11} - K_{22})^2 + 4K_{12}^2)]^{1/2} \right\}. \tag{3.12}$$

The « theoretical » ellipse of permeability can therefore be plotted and can be compared with the actually measured points.

4. *Analysis of Experimental Data* — The task remains to compare the theory outlined in Secs. 2 and 3 with experiments that have actually been performed. In this istance, it may be noted that JOHNSON and coworkers (JOHNSON & HUGHES, 1948; JOHNSON & BRESTON, 1950) devised a method of measuring directional permeabilities. They also reported a great number of results of such measurements. We shall endeavor to analyse these results and to compare them with the theory outlined above. In referring to the papers of JOHNSON and coworkers, the first paper, by JOHNSON & HUGHES (1948) will be noted as « *I* », and the second, by JOHNSON & BRESTON (1950), by « *II* ».

The one method used by JOHNSON et al. consists of cutting pencil-shaped sections at various angles from well cores of fluid-bearing strata which were obtained by diamond drilling. The permeability of the pencil-shaped sections is then measured by an ordinary permeability apparatus. It is obvious that this manner of measuring directional permeability corresponds to the first possibility of mathematical definition of the latter suggested in Section 2.

An alternative method of measuring directional permeability was also employed by JOHNSON et al. In this second method, a hole was drilled down the center of a cylindrical piece of an anisotropic porous medium (again a well core) whose faces were previously made parallel. The equipment used, then, consisted of a system of clamps, mounted with bearings which allowed the porous cylinder to be rotated to any position for flow measurement while fluid was continuously being flowed from the center to the outside of the cylinder. A collecting head, which was clamped to the cylinder, was used to collect the fluid which flowed from that portion of the cylinder which is under observation. The directional permeability was then calculated according to DARCY's law from the volume of fluid flowing in the given direction and collected by the head. It is obvious that this manner of measuring directional permeability corresponds to the second mathematical definition of the latter outlined in Section 2.

JOHNSON et al. employed the first method only in one example in order to test whether the two methods would yield identical experimental results. As this was satisfactorily the case, the second method was employed for most of their measurements as it is much simpler to carry out. They did not realize that the tensor theory of permeability, if correct, requires the two types of measurements to be identical.

The results of JOHNSON et al. were reported in the form of polar diagrams of the directional permeability k_n against the polar angle α. The results were published in graphical form. No attempt was made by JOHNSON et al. to either confirm or refute the tensor theory of permeability of whose existence these authors do not seem to have been aware. The prime object of JOHNSON's investigations was to show that in one geological stratum or even in a set of strata the directions of maximum permeability are aparellel over wide areas. The direction of maximum permeability was determined in a rather haphazard manner by inspecting the graphs. Furthermore, the graphs were drawn by strictly plotting the experimental results without making any attempts to understand their physical significance. Thus, it may often be noted in the graphs that permeabilities in directions differing by 180° are not identical. This is most certainly due to inhomogeneities in the pieces of porous materials used, as it is quite inconceivable that the permeability would vary upon reversal of the direction of flow. Such an effect would be equivalent to a valve-action of the porous medium which, although not a priori impossibile, would destroy any basis for DARCY's law to hold, — to say the least. Until such an effect is much more definitely established, the writer is therefore inclined to ascribe the observed discrepancies to inhomogeneities in the material,

One particular well core (I, Fig. 7) was most extensively analysed experimentally by JOHNSON & HUGHES. It will be demonstrated on this example how the enperimental data can be recalculated so as to permit to draw physical conclusions from them.

JOHNSON & HUGHES represented the results of their experiments on this

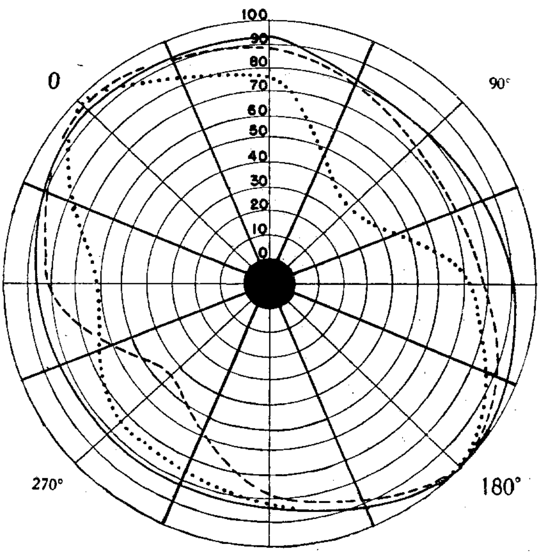


Fig. 1

particular piece of porous material in a diagram which is reproduced in Fig. 1. The method of measuring directional permeabilities employed in this case was that of cutting pencil-shaped pieces out of the medium. The three curves in Fig. 1 represent a quasipolar (as zero is not in the center!) diagram of 3 sets of « pencils » cut at various angles from 3 different parallel layers out of the medium. The layers are only a few centimeters apart.

The curves shown by JOHNSON have physically obviously not much meaning. They show essentially the variations of permeability that can be expected in the various directions due to small inhomogeneities in a natural porous medium. The fact that JOHNSON's data have been referred to « 100 » for the maximum permeability in each layer further helps to obliterate any physical meaning of the curves. Obviously, the first task is therefore to restore the original permeability values in the various directions for the 3 curves. This has been done in Table I. the corresponding results are plotted in Fig. 2.

The next step is to average all the values and to symmetrize the curves with respect to the origin of the diagram in order to obtain an average directional permeability value for each horizontal direction in the porous medium under conside-

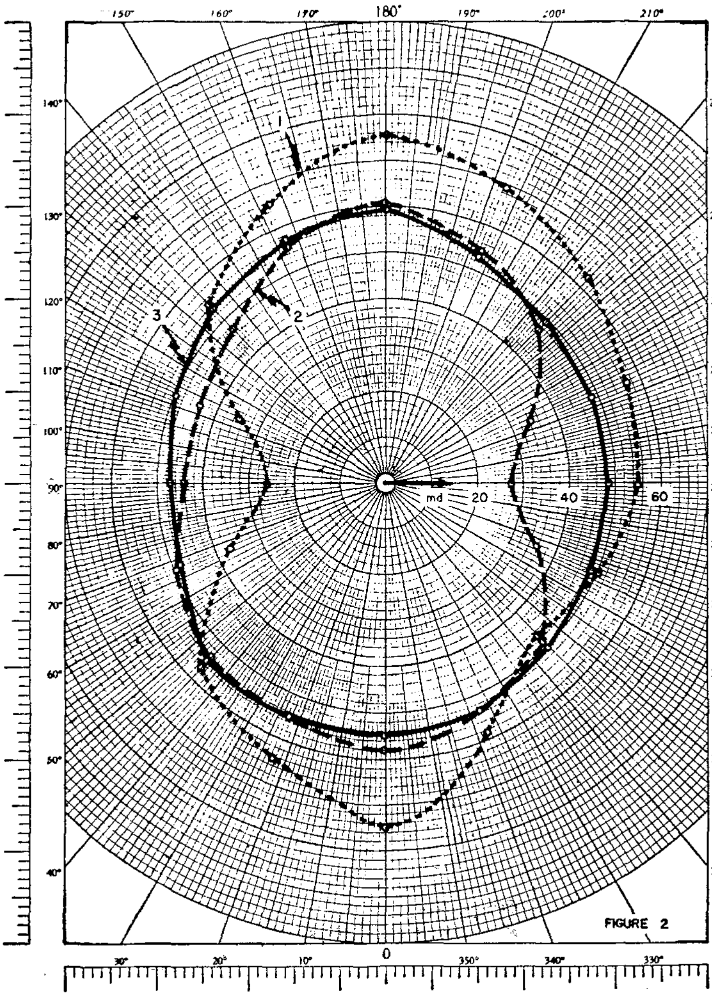


Fig. 2

ration. Finally, one can calculate $k^{-1/2}$ for each direction. The results of these calculations are also shown in Table I. The corresponding values are plotted in Fig. 3.

In order to test the theory, one now has to compare the measured values of $k^{-1/2}$ with the best-fitting ellipse. The best fitting ellipse can be calculated accor-

TABLE 1: *Permeabilities in Millidarcies*

Angle	Curve 1	Curve 2 — — —	Curve 3 ———	Average k	$1/\sqrt{k}$
0	74.5	57.9	55.1	63.9	0.125
22.5	64.1	55.2	55.1	58.5	0.131
45	57.2	53.8	55.1	54.2	0.136
67.5	36.4	48.8	48.4	46.6	0.148
90	26.0	44.2	47.0	41.4	0.156
112.5	34.6	44.2	49.7	43.8	0.151
135	55.4	46.9	53.7	50.6	0.141
157.5	65.8	55.2	56.4	57.2	0.132
180	76.2	60.7	59.1	63.9	0.125
202.5	69.3	53.8	53.7	58.5	0.131
225	62.4	46.9	49.7	54.2	0.136
247.5	57.2	34.5	48.4	45.6	0.148
270	55.4	27.6	48.4	41.4	0.156
192.5	50.2	35.8	48.4	43.8	0.151
315	46.8	49.7	51.1	50.6	0.141
337.5	58.4	53.8	53.7	57.2	0.132
Max. Perm. . . .	76.2	60.7	59.1	—	—

ding to formulas (3.8), since the measurements have been taken at equal angle intervals of 22.5°. One obtains, after a straight forward calculation

$$K_{11} = \quad 0.0148 \text{ md}^{-1}$$
$$K_{22} = \quad 0.0237 \text{ md}^{-1}$$
$$K_{12} = -0.0006 \text{ md}^{-1}\,.$$

Consequently, one obtains for the principal directions:

$$\tan\varphi = \quad 15.26 \therefore \varphi = \quad 86^\circ$$
$$\tan\varphi = -\ 0.06 \therefore \varphi = -\ 4^\circ$$

and for the eigenvalues:

$$\lambda = 0.0239 \text{ md}^{-1} \therefore \text{axis} = 0.154 \text{ md}^{-1/2}$$
$$\lambda = 0.0146 \text{ md}^{-1} \therefore \text{axis} = 0.121 \text{ md}^{-1/2}\,.$$

The principal permeabilities turn out to be 42 md and 68 md, respectively. The corresponding curves are shown in Fig. 3 and it is seen that the coincidence with the measured points is excellent.

A similar analysis has been made with the remaining 29 polar diagrams of directional permeability measurements that have been published by JOHNSON et al. The results of this analysis are tabulated in Table 2. Therein, column 1 assigns a number to each sample, col. 2 gives the figure corresponding to the sample in JOHNSON's papers, 3 gives the key used for the sample by JOHNSON 4 gives the position of the « top » in JOHNSON's figure as compared with our choice of zero angle, and 5, 6 and 7 give the position of the principal axes, their ratios, and the principal permeability ratio, respectively. Fig. 4-10 represent the results graphi-

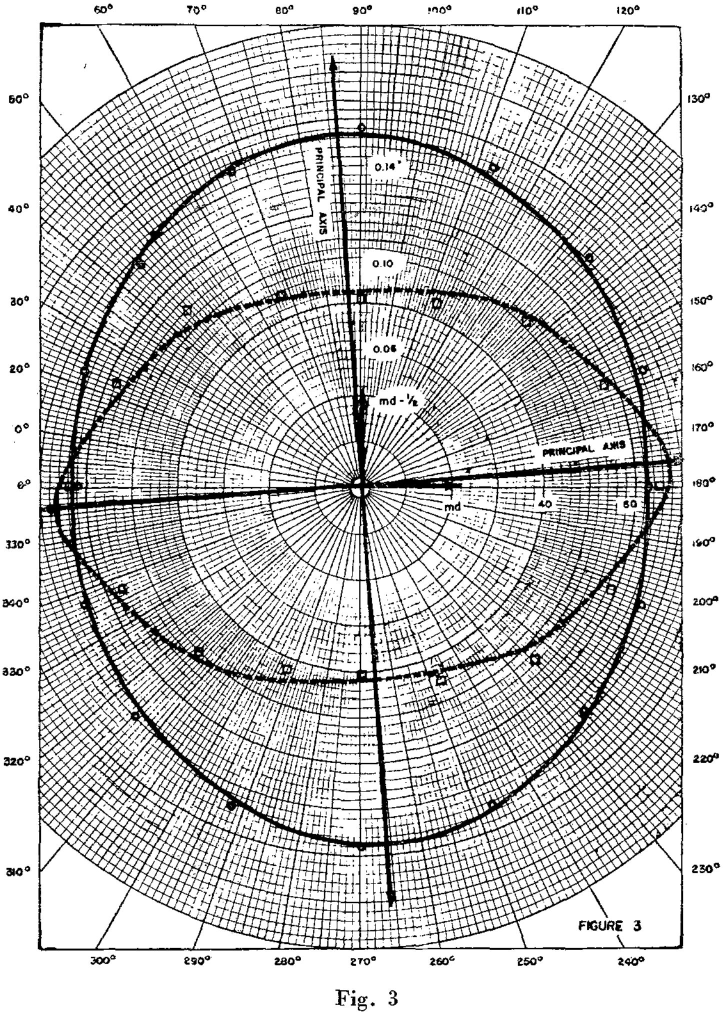


Fig. 3

cally. The first plot for each sample is JOHNSON's plot, the second is the same after symmetrization, and the third is a plot of $k^{-1/2}$ together with the best-fitting ellipse. All values are referred to as « 100 » for the maximum value of the quantity in question that was obtained.

The calculation of the position of the principal axes as well as of the eigenvalues has been made by using formulas (3.7), since the values were measured at equal intervals of 30°. The calculations are somewhat lengthy, but by no means difficult. They can easily be set up for working out by untrained help. There is therefore no reason why they could not be made standard procedure in connection

Table 2

Sample No.	Source	Key	Top in source	Positions of Princ. axes	Ratio of axes of ellipse	Permeab. Ratio
1	I, Fig. 7	. . . . .	90°	73°	.94	.81
2	I, Fig. 7	– – –	90°	88°	.87	.76
3	I, Fig. 7	———	90°	73°	.92	.85
4	I, Fig. 7	———	60°	4°	.82	.67
5	I, Fig. 8	· – – –	60°	73°	.92	.84
6	I, Fig. 8	- - - - -	60°	20°	.87	.78
7	I, Fig. 8	+ + + + + + +	60°	68°	.91	.83
8	I, Fig. 9	———	120°	8°	.92	.85
9	I, Fig. 9	——— ———	120°	86°	.77	.59
10	I, Fig. 9	– – – –	120°	4°	.81	.65
11	I, Fig. 9	- - - - -	120°	5°	.85	.73
12	I, Fig. 10	————	30°	1°	.80	.64
13	I, Fig. 10	- - - -	30°	4°	.82	.67
14	I, Fig. 10	→ → →	30°	2°	.87	.76
15	I, Fig. 10	– – –	30°	1°	.87	.76
16	I, Fig. 10	–·–·–·	30°	1°	.93	.87
17	I, Fig. 10	+ + + + + + +	30°	3°	.84	.71
18	I, Fig. 11	– – – – –	60°	5°	.93	.87
19	I, Fig. 11	+ + + + + + +	60°	4°	.85	.72
20	I, Fig. 7	———	60°	22°	.92	.85
21	I, Fig. 12	——————	30°	10°	.82	.67
22	II, Fig. 4	——————	30°	0°	.80	.64
23	II, Fig. 4	- - - - - - -	30°	89°	.82	.68
24	II, Fig. 4	– – – – – – –	30°	6°	.84	.71
25	II, Fig. 4	+ + + + + + +	30°	4°	.85	.72
26	II, Fig. 5	PB54-1B	210°	20°	.96	.92
27	II, Fig. 5	PB54-2B(1/2S)	90°	16°	.94	.88
28	II, Fig. 5	PB54-2B(NS)	60°	86°	.94	.88
29	II, Fig. 5	PB54-1C	120°	4°	.92	.85

with routine directional permeability determinations. In this manner, at least, a definite value for the principal permeability directions as well as for the corresponding principal permeabilities is obtained, which is much better than reading those values off a diagram in a manner which is little more but haphazard.

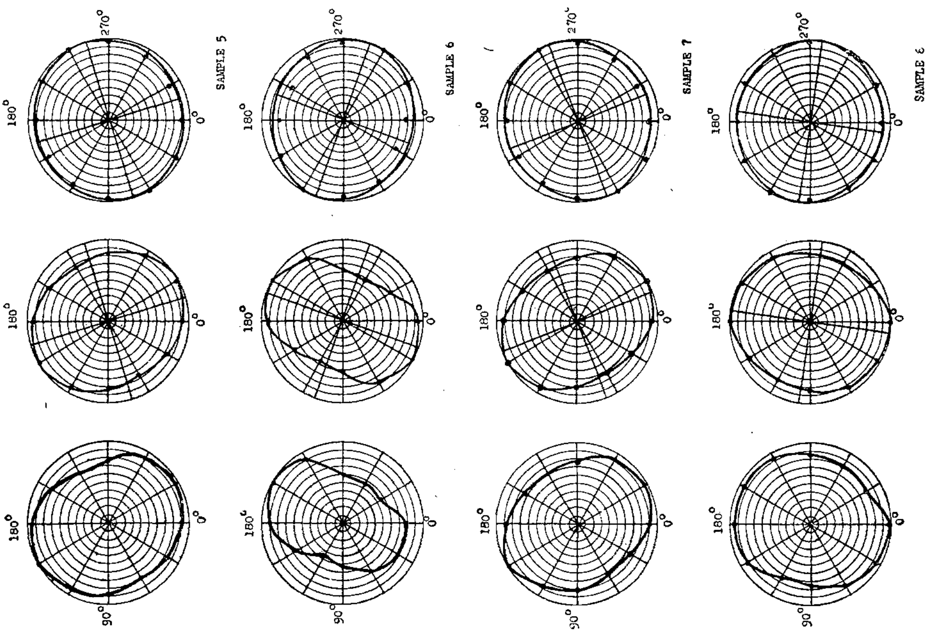

180°
0°
90°
270°
SAMPLE 5
SAMPLE 6
SAMPLE 7

Fig. 5

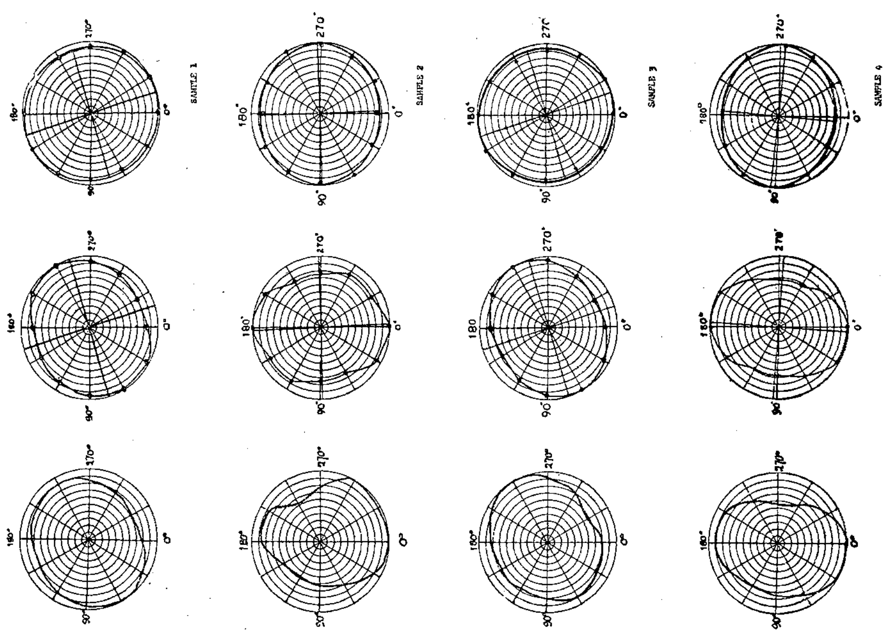
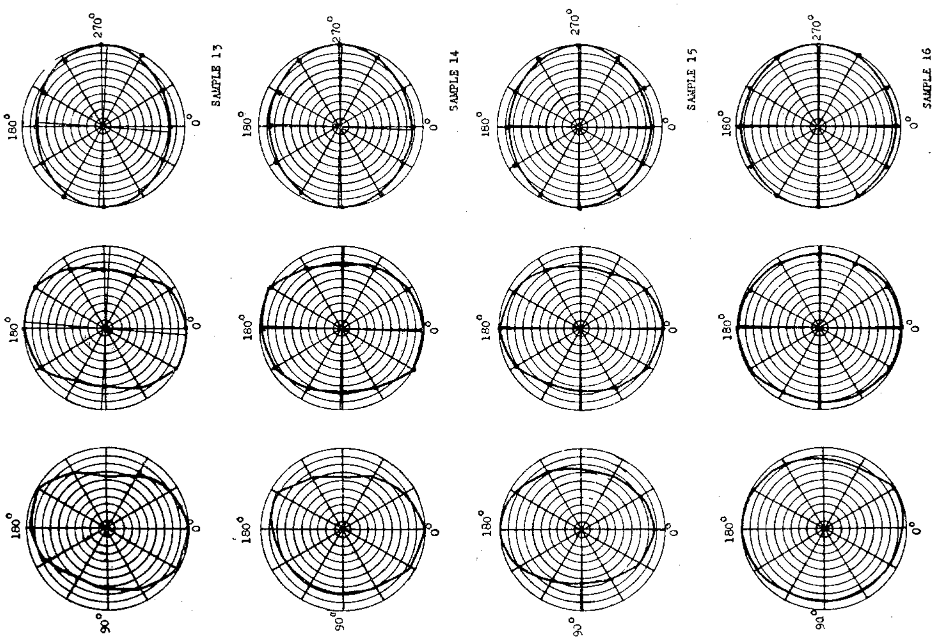
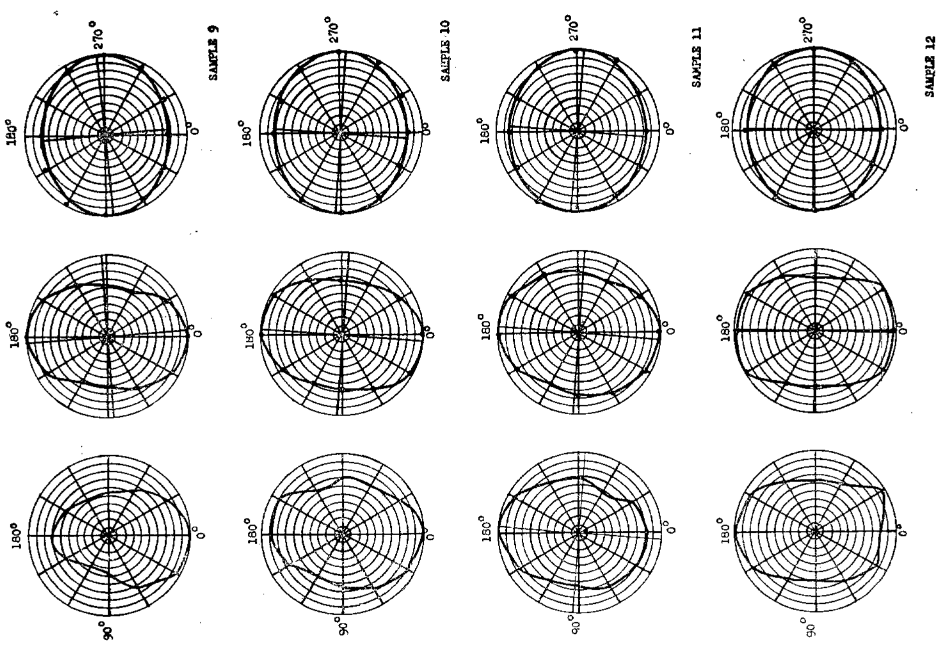

180°
0°
90°
270°
SAMPLE 1
SAMPLE 2
SAMPLE 3
SAMPLE 4

Fig. 4



SAMPLE 13
SAMPLE 14
SAMPLE 15
SAMPLE 16

Fig. 7



SAMPLE 9
SAMPLE 10
SAMPLE 11
SAMPLE 12

Fig. 6

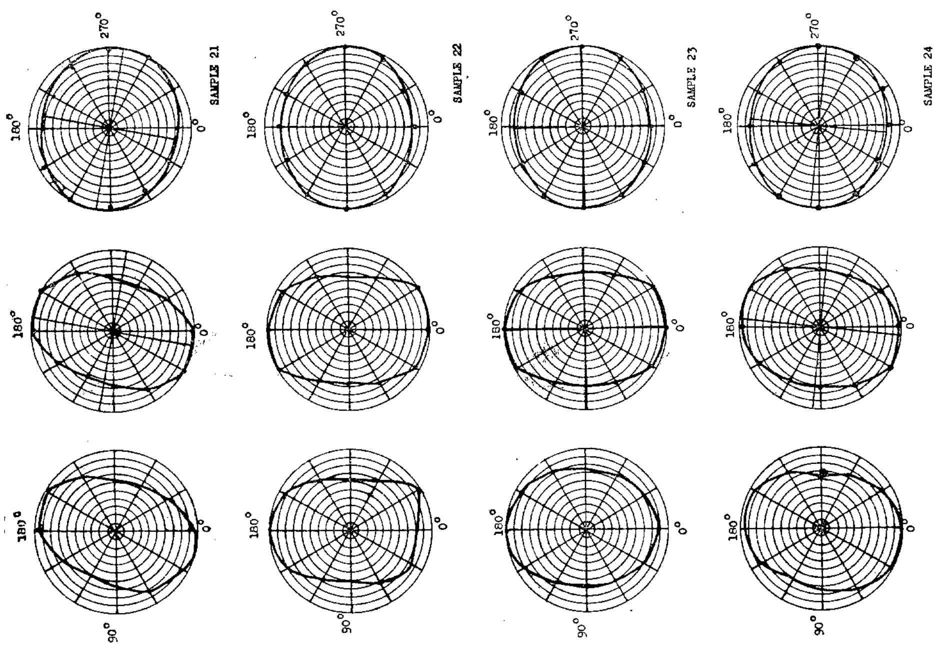

SAMPLE 21
SAMPLE 22
SAMPLE 23
SAMPLE 24
0°
90°
180°
270°

Fig. 9

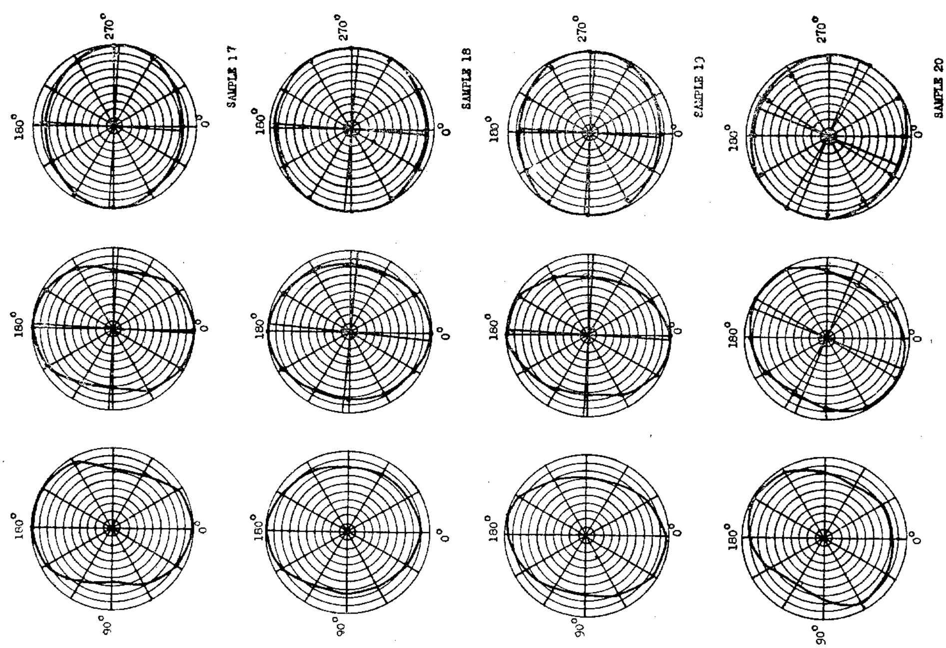

SAMPLE 17
SAMPLE 18
SAMPLE 20
0°
90°
180°
270°

Fig. 8

5. *Conclusions* — The final task is to investigate how well the points obtained from measurements of JOHNSON et al. do fit the best-fitting ellipse of the polar

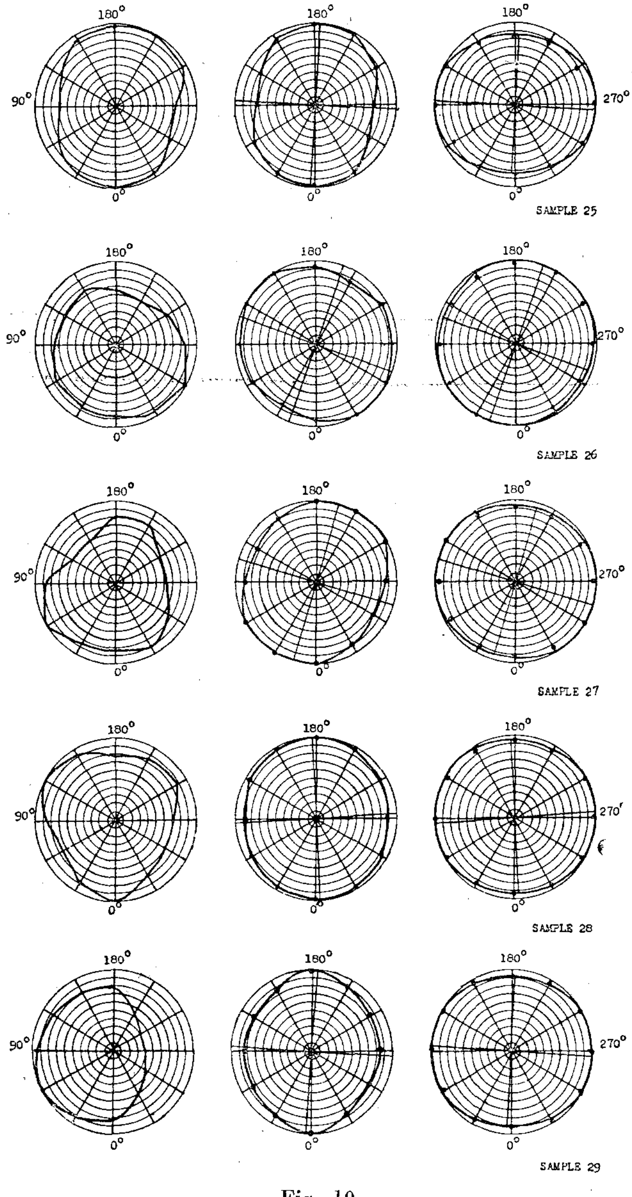


Fig. 10

diagrams in order to determine whether or not the tensor theory of permeability is a correct representation of the physical facts. In order to achieve this it is obviously necessary to obtain somehow a rough idea of the experimental confidence limits attached to the points as calculated from the data of JOHNSON et al.

Errors of the position of those points can arise from two sources. Firstly, there is the possibility of inaccuracies in the actual determination of each permeability value. Secondly, it must be held that the porous samples tested are in no case perfectly homogeneous such that the permeability values obtained in one direction from a small sample of the porous medium may be in error as compared with the average permeability in that direction if a larger piece of medium would have been used for the determination.

As so the first possible cause of error, JOHNSON et al. have already analysed it and have shown that these inaccuracies are very small, of the order of 1 per cent.

As to the second cause, no such evaluation has been made by the original authors. Therefore, one is faced with the task of supplying same. It is proposed to do this as follows:

In the first experiment, represented in our Fig. 1, JOHNSON & HUGHES, in fact, obtain six different permeability values for each direction. It must be assumed that all these values should be identical if the porous medium tested were perfectly homogeneous. The spread of permeability values around the mean, therefore, must be taken as an indication of the inhomogeneities encountered. It is, then, possible to calculate the standard deviation of the 48 permeability values listed in Table 1 around the means for the respective directions. This standard deviation expressed as a percentage, should be a good indication for the standard error (in per cent) of a directional permeability value to be expected in any one sample if only one measurement is made. From a straightforward evaluation of the data of Table 1, one obtains

$$\text{Standard permeability error} = 6.87 \text{ md} = 10.0\ \% \tag{5.1}$$

Therefore, the relative standard error of $1/k^{1/2}$ is roughly half that much, i.e. 5 %. Since each point in the graphs of $1/k^{1/2}$ in Figs. 4-10 is the average of two measurements (in opposite directions), this error is again lowered by a factor of $1/\sqrt{2}$. Since all measurements are referred to « 100 » as maximum values the percentage error as calculated above is also the absolute error of the graph. One obtains:

$$\text{Standard error of } 1/\sqrt{k} = 3.5\ \%\ . \tag{5.2}$$

In view of this result, it must be held that the experimental points in Figs. 4-10 do fit the « theoretical » ellipses. In most cases, the fit is even much better than within the standard error limits indicated by Eq. (5.2). It must be concluded, therefore, that the tensor theory of permeability is able to account for the data obtained by JOHNSON et al. in a most satisfactory manner.

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