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The flow in and around a droplet or bubble submerged in an unbound arbitrary velocity field

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With 3 figures

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1. Introduction

A knowledge of the flow field in and around a droplet submerged in an unbounded fluid is of considerable practical interest. It is usual to regard the fluid as being unbounded if the radius of the droplet is much smaller than that of the containing vessel (order of 10^{-3} or less), otherwise one has to consider hydrodynamic forces due to the walls of the vessel.

Theoretical analysis of the flow field around a droplet in an unbounded fluid have been limited in the past to cases when some particular velocity profile is postulated at a large distance from the droplet. Examples for such solutions are the solution of *Hadamard* (1) – *Rybczynski* (2), for uniform velocity profile and the solutions of *G. I. Taylor* (3), for shear flows.

In this work the flow fields interior to a droplet and exterior to it are solved for the case when the velocity profile at a large distance from the droplet is quite *arbitrary*.

The term 'droplet' is used throughout this work, even though the solutions are applicable to spherical solid particles, droplets or bubbles.

2. Statement of the problem

Consider a droplet which is submerged in an unbounded fluid. The droplet moves with constant velocity $\bar{U}$. The undisturbed velocity field $\bar{V}_\infty$ is *Stokes*ian, but other than that is quite arbitrary.

A spherical coordinate system is used, whose center coincides with the center of mass of the droplet, and moves with the constant velocity $\bar{U}$.

The fluids involved are homogeneous, isothermal, *Newton*ian and of constant densities. Surface-active-agents are assumed to be absent. The flow around the droplet is creeping, namely, the inertia terms in the equations of motion are negligible.

With these suppositions, the equations of motion and the equations of continuity are as follows:

for the continuous fluid

$$\nabla^2 \bar{v} = \frac{1}{\mu} \nabla p , \qquad [1]$$

and

$$\nabla \cdot \bar{v} = 0 , \qquad [2]$$

for the interior of the droplet

$$\nabla^2 \bar{v}' = \frac{1}{\mu'} \nabla p' , \qquad [3]$$

and

$$\nabla \cdot \bar{v}' = 0 , \qquad [4]$$

where $\bar{v}$ and $\bar{v}'$ are the velocities exterior to the droplet and interior to it, respectively. p and p' are the pressures and μ and μ' are the respective viscosities.

The boundary conditions for an observer moving with the center of mass of the droplet are as follows:

at $r = \infty$

$$\bar{v}_\infty = \bar{V}_\infty - \bar{U} \qquad [5]$$

on the interface

$$\bar{v}^* = \bar{v}'^* \qquad [6a]$$

$$\bar{\pi}^*_{(n)} - \bar{\tau}'^*_{(n)} = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \bar{i}_n \qquad [6b]$$

$$\bar{v}'^*_n = 0 , \qquad [6c]$$

where the star indicates that the functions are evaluated at the interface; $\bar{i}_n$ is a unit vector normal to the interface: $\bar{\pi}_{(n)}$ and $\bar{\tau}'_{(n)}$ are the normal stress vectors: based on the velocities $\bar{v}$ and $\bar{v}'$ respectively, $R_1(\theta, \varphi)$ and $R_2(\theta, \varphi)$ are the principal radii and σ is the surface tension.

3. The solution

The solution of the basic equations subject to the boundary conditions, has to yield explicitly the flow fields interior to the droplet and exterior to it, and the general equation of the interface. However, the mathematical treatment of solving simultaneously the flow fields and the equation of the interface is excessively difficult. Therefore, an iterative procedure is adopted here. First the droplet is postulated to be spherical and the flow fields are determined using only six of the boundary conditions. Later, the seventh boundary condition is used for determining the deviation of the interface from sphericity and its dependence on the various flow parameters. The interface thus determined can be used for determining a second iteration of the flow fields.

3.1. *The first iteration*

Based on experimental observation it is known that small droplets or bubbles are nearly spherical. Therefore, we begin the calculations by assuming a spherical droplet. Using *Lamb*'s general solution for *Stokes*ian flows, the velocity and pressure fields are as follows:

Outside the droplet:

$$\bar v = \bar v_\infty + \bar V \qquad [7]$$

where

$$\bar V = \sum_{n=1}^{\infty}\Big\{\nabla x(\bar r\,\chi_{-n-1}) + \nabla\Phi_{-n-1} - \frac{(n-2)}{2n(2n-1)\mu} r^2\nabla p_{-n-1} + \frac{n+1}{n(2n-1)\mu}\bar r\, p_{-n-1}\Big\}, \qquad [8]$$

and

$$p = \sum_{n=0}^{\infty} p_{-n-1}. \qquad [9]$$

Inside the droplet:

$$\bar v' = \sum_{n=1}^{\infty}\Big\{\nabla x(\bar r\,\chi_n) + \nabla\Phi_n + \frac{n+3}{2(n+1)(2n+3)\mu'} r^2\nabla p_n - \frac{n}{(n+1)(2n+3)\mu'}\bar r\, p_n\Big\} \qquad [10]$$

and

$$p' = \sum_{n=0}^{\infty} p_n, \qquad [11]$$

where χ_{-n-1}, Φ_{-n-1}, p_{-n-1}, χ_n, Φ_n and p_n are solid spherical harmonics of degree $-n-1$ and n, respectively, which have to be determined.

The spherical harmonics are conveniently determined by transforming the boundary conditions in the following way (Appendix A):

$$\bar v'^* \cdot \bar i_r = \bar V^* \cdot \bar i_r + \bar v^*_\infty \cdot \bar i_r = 0 \qquad [12a, b]$$

$$\left[r\frac{\partial v'_r}{\partial r}\right]^* - \left[r\frac{\partial V_r}{\partial r}\right]^* = \left[r\frac{\partial v_{\infty r}}{\partial r}\right]^* \qquad [12c]$$

$$[\bar r\cdot\nabla x\,\bar v']^* - [\bar r\cdot\nabla x\,\bar V]^* = [\bar r\cdot\nabla x\,\bar v_\infty]^* \qquad [12d]$$

$$[\bar r\cdot\nabla x(\bar r\,x\,\bar\tau'_{(r)})]^* - [\bar r\cdot\nabla x(\bar r\,x\,\bar\tau_{(r)})]^* = [\bar r\cdot\nabla x(\bar r\,x\,\bar\tau_{\infty(r)})]^* \qquad [12e]$$

$$[\bar r\cdot\nabla x\,\bar\tau'_{(r)}]^* - [\bar r\cdot\nabla x\,\bar\tau_{(r)}]^* = [\bar r\cdot\nabla x\,\bar\tau_{\infty(r)}]^* \qquad [12f]$$

where $\bar\tau_{(r)}$ and $\bar\tau'_{(r)}$ are based on $\bar V$ and $\bar v'$, respectively and $\bar\tau_{\infty(r)}$ is based on the unperturbed velocity $\bar v_\infty$.

The seventh boundary condition is used in section 3.3 and is discussed there.

Since the unperturbed velocity field $\bar v_\infty$ is a solution of the *Stokes*' equation, it can be expressed by the general solution of *Lamb*, i. e.

$$\bar v_\infty = \sum_{n=-\infty}^{\infty}\Big\{\nabla x(\bar r\,\chi_n^\infty) + \nabla\Phi_n^\infty + \frac{n+3}{2(n+1)(2n+3)\mu} r^2\nabla p_n^\infty - \frac{n}{(n+1)(2n+3)\mu}\bar r\,p_n^\infty\Big\}, \qquad [13]$$

where χ_n^∞, Φ_n^∞ and p_n^∞ are yet undetermined solid spherical harmonics.

Substitution of eqs. [7], [8], [10] and [13] into eqs. [12], one obtains (Appendix B):

From eq. [12a]:

$$\sum_{n=1}^{\infty}\left[\frac{n\,a}{2(2n+3)\mu'}p_n^* + \frac{n}{a}\Phi_n^*\right] = 0. \qquad [14a]$$

From eq. [12b]:

$$\sum_{n=1}^{\infty}\left[\frac{a(n+1)}{2\mu(2n-1)}p^*_{-n-1} - \frac{n+1}{a}\Phi^*_{-n-1}\right] = -\sum_{n=-\infty}^{\infty}\left[\frac{n\,a}{2\mu(2n+3)}p_n^{\infty *} + \frac{n}{a}\Phi_n^{\infty *}\right]. \qquad [14b]$$

From eq. [12c]:

$$\sum_{n=1}^{\infty}\left[\frac{n(n+1)a}{2\mu'(2n+3)}p_n^* + \frac{n(n-1)}{a}\Phi_n^* + \frac{n(n+1)a}{2\mu(2n-1)}p^*_{-n-1} - \frac{(n+1)(n+2)}{a}\Phi^*_{-n-1}\right] = \sum_{n=-\infty}^{\infty}\left[\frac{n(n+1)a}{2\mu(2n+3)}p_n^{\infty *} + \frac{n(n-1)}{a}\Phi_n^{\infty *}\right]. \qquad [14c]$$

From eq. [12d]:

$$\sum_{n=1}^{\infty}[n(n+1)(\chi_n^* - \chi^*_{-n-1})] = \sum_{n=-\infty}^{\infty}[n(n+1)\chi_n^{\infty *}]. \qquad [14d]$$

From eq. [12e]:

$$\sum_{n=1}^{\infty}\left[\frac{2n(n+1)(n+2)}{a}\Phi^*_{-n-1}-\frac{(n+1)^2(n-1)a}{\mu(2n-1)}p^*_{-n-1}+\frac{2\lambda}{a}(n-1)n(n+1)\Phi^*_n+\frac{n^2(n+2)\lambda a}{\mu'(2n+3)}p^*_n\right]$$
$$=\sum_{n=-\infty}^{\infty}\left[\frac{2}{a}(n-1)n(n+1)\Phi_n^{\infty *}+\frac{n^2(n+2)a}{\mu(2n+3)}p_n^{\infty *}\right]. \quad [14e]$$

From eq. [12f]:

$$\sum_{n=1}^{\infty}\{n(n+1)[\lambda(n-1)\chi^*_n+(n+2)\chi^*_{-n-1}]\}=\sum_{n=-\infty}^{\infty}[(n-1)n(n+1)\chi_n^{\infty *}], \quad [14f]$$

where $\lambda = \mu'/\mu$.

The solution is now continued by defining the solid spherical harmonics in the following way:

$$p_n = A_n\,\mu'\,a^{-n-1}\,r^n\,S_n(\theta,\varphi);$$
$$p_{-n-1} = A_{-n-1}\,\mu\,a^n\,r^{-n-1}\,S_n(\theta,\varphi);$$
$$\Phi_n = B_n\,a^{-n+1}\,r^n\,S_n(\theta,\varphi);$$
$$\Phi_{-n-1} = B_{-n-1}\,a^{n+2}\,r^{-n-1}\,S_n(\theta,\varphi);$$
$$\chi_n = C_n\,a^{-n}\,r^n\,S_n(\theta,\varphi);$$
$$\chi_{-n-1} = C_{-n-1}\,a^{n+1}\,r^{-n-1}\,S_n(\theta,\varphi), \quad [15]$$

where S_n are surface harmonics of order n, 'a' is the radius of the spherical droplet and A_n, B_n, C_n etc. are yet undetermined coefficients. The notation used above is an abbreviation, but one must keep in mind that the product $A_n S_n(\theta, \varphi)$, for example, represents $2n+1$ terms, that is:

$$A_n S_n(\theta,\varphi)=\sum_{m=0}^{n}(A_n^m\cos m\varphi+\hat{A}_n^m\sin m\varphi)P_n^m(\cos\theta). \quad [16]$$

In an analogous manner, the solid spherical harmonics of the unperturbed flow field $\bar{v}_\infty$, namely those in eq. [13], can be defined:

$$p_n^{\infty}=\frac{2(2n+3)}{n}\alpha_n\,\mu\,a^{-n-1}\,r^n\,S_n(\theta,\varphi)$$
$$\Phi_n^{\infty}=\frac{1}{n}\beta_n\,a^{-n+1}\,r^n\,S_n(\theta,\varphi)$$
$$\chi_n^{\infty}=\frac{1}{n(n+1)}\gamma_n\,a^{-n}\,r^n\,S_n(\theta,\varphi), \quad [17]$$

where $\alpha_n S_n(\theta,\varphi)$, $\beta_n S_n(\theta,\varphi)$ etc., are an abbreviation of $2n+1$ terms, analogue to eq. [16], and where $n=-\infty$ to $+\infty$. Notice that p_0^∞, Φ_0^∞, χ_0^∞ and χ_{-1}^∞ cannot be determined from this definition. This, however, is no limitation since these particular spherical harmonics do not contribute to the unperturbed velocity in eq. [13].

The coefficients A_n^m, B_n^m, C_n^m, etc. of the solid spherical harmonics in eq. [15] are determined by substitution of eqs. [15] through [17] into eqs. [14]. Using the orthogonality properties of the surface harmonics, there are obtained six linearly independent equations from which the coefficients are determined:

$$A_n^m=\frac{(2n-1)(2n+3)}{n(1+\lambda)}\left[\frac{(2n+3)}{(2n-1)}\alpha_n^m+\beta_n^m\right] \quad [18a]$$

$$B_n^m=-\frac{A_n^m}{2(2n+3)} \quad [18b]$$

$$A_{-n-1}^m=\frac{(1-2n)}{(n+1)}\times\left\{2\alpha_{-n-1}^m+\frac{\lambda[\alpha_n^m(2n+3)+\beta_n^m(2n+1)]+2\beta_n^m}{(1+\lambda)}\right\} \quad [18c]$$

$$B_{-n-1}^m=\frac{1}{n+1}\left\{\beta_{-n-1}^m+\frac{1}{1+\lambda}\left\{\alpha_n^m-\frac{\lambda}{2}[\alpha_n^m(2n+1)+(2n-1)\beta_n^m]\right\}\right\} \quad [18d]$$

$$C_n^m=\gamma_n^m\frac{(2n+1)}{n(n+1)[\lambda(n-1)+(n+2)]} \quad [18e]$$

$$C_{-n-1}^m=-\frac{\gamma_{-n-1}^m}{n(n+1)}+\frac{\gamma_n^m(n-1)(1-\lambda)}{n(n+1)[\lambda(n-1)+(n+2)]} \quad [18f]^{*)}$$

where $n=1$ to ∞. The coefficients with the carat, e. g. $\hat{A}_n^m$, are similar to the above, with $\hat{\alpha}_n^m$ replacing α_n^m etc.

In order to proceed with the solution, the coefficients α_n, β_n and γ_n have to be determined from the known unperturbed velocity distribution at infinity, $\bar{v}_\infty$. For this purpose we multiply eq. [13] by a unit vector in the radial direction $\bar{l}_r$, viz.

$$\bar{v}_\infty\cdot\bar{l}_r=\sum_{n=-\infty}^{\infty}\left\{\frac{n}{2\mu(2n+3)}r\,p_n^\infty+\frac{n}{r}\Phi_n^\infty\right\} \quad [19]$$

and

$$\bar{r}\cdot\nabla x\,\bar{v}_\infty=\sum_{n=-\infty}^{\infty}\{n(n+1)\chi_n^\infty\}. \quad [20]$$

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*) Substitution of $n=1$ in eq. [18f], and recalling that $\gamma_{-2}^m=0$, yields $C_{-2}^m=0$. This implies that there is no torque acting on a droplet, since the torque is given by $T=-8\pi\mu\,\nabla(r^3\chi_{-2})$.

For large values of λ, one obtains $C_{-2}^m=-\frac{1}{2}\gamma_1^m$, since one has to substitute first $\lambda=\infty$ in eq. [18f], and then $n=1$.

Substitution of eqs. [17] in eqs. [19] and [20], and recalling that $S_{-n-1}(\theta,\varphi) = S_n(\theta,\varphi)$, yields:

$$\bar{v}_\infty \cdot \bar{i}_r = \sum_{n=0}^{\infty}\left[\alpha_n\left(\frac{r}{a}\right)^{n+1} + \beta_n\left(\frac{r}{a}\right)^{n-1} + \alpha_{-n-1}\left(\frac{r}{a}\right)^{-n} + \beta_{-n-1}\left(\frac{r}{a}\right)^{-n-2}\right] S_n(\theta,\varphi) \quad [21]$$

and

$$\bar{r}\cdot\nabla x\,\bar{v}_\infty = \sum_{n=1}^{\infty}\left[\gamma_n\left(\frac{r}{a}\right)^{n} + \gamma_{-n-1}\left(\frac{r}{a}\right)^{-n-1}\right] S_n(\theta,\varphi). \quad [22]$$

The coefficients α_n, β_n, γ_n, α_{-n-1}, β_{-n-1} and γ_{-n-1} can readily be determined from eqs. [21] and [22] by using the orthogonality properties of the *Legendre* polynomials.

Thus, the coefficients in eqs. [15] have been determined in terms of the known coefficients of the unperturbed velocity distribution.

This completes the solution for the velocity distribution outside the droplet (eqs. [7, 8, 15 and 18]) and inside it (eqs. [10, 15 and 18]).

3.2. *The settling velocity*

The settling velocity can now be determined by writing a force balance on the droplet:

$$\frac{4\pi}{3} a^3(\varrho' - \varrho)\, g\, \bar{k} - 4\pi\, \nabla(r^3 p_{-2}) = 0\,, \quad [23]$$

where the first expression is the buoyancy force and the second one is the drag [*Happel* and *Brenner* (5), p. 65]. The expression for p_{-2} is, in general:

$$p_{-2} = \mu\, a\, r^{-2}[A^0_{-2} P_1(\cos\theta) + A^1_{-2}\cos\varphi\, P^1_1(\cos\theta) + \hat{A}^1_{-2}\sin\varphi\, P^1_1(\cos\theta)]. \quad [24]$$

By substituting eq. [24] in eq. [23] and equating the corresponding vectors we obtain:

$$A^0_{-2} = \frac{a^2 g}{3\mu}(\varrho' - \varrho);\quad A^1_{-2} = \hat{A}^1_{-2} = 0\,. \quad [25]$$

Since for the unperturbed flow $\nabla(r^3 p^\infty_{-2}) = 0$ we must have, by eq. [17] that $\alpha_{-2} = 0$, and eq. [18] results in

$$A^0_{-2} = -\frac{1}{2(1+\lambda)}[\lambda(5\alpha^0_1 + 3\beta^0_1) + 2\beta^0_1]$$

$$A^1_{-2} = -\frac{1}{2(1+\lambda)}[\lambda(5\alpha^1_1 + 3\beta^1_1) + 2\beta^1_1]$$

$$\hat{A}^1_{-2} = -\frac{1}{2(1+\lambda)}[\lambda(5\hat{\alpha}^1_1 + 3\hat{\beta}^1_1) + 2\hat{\beta}^1_1]. \quad [26]$$

We set now to determine the three components of the terminal settling velocity of the droplet, in terms of known coefficients and parameters.

First express eq. [5] as follows:

$$\bar{v}_\infty = \bar{V}_\infty - U = \bar{V}_\infty - (U_x \bar{i} + U_y \bar{j} + U_z \bar{k})$$

Multiplying the above by a unit vector in the radial direction $\bar{i}_r$

$$\bar{v}_\infty\cdot\bar{i}_r = \bar{V}_\infty\cdot\bar{i}_r - (U_x\cos\varphi\, P^1_1(\cos\theta) + U_y\sin\varphi\, P^1_1(\cos\theta) + U_z P_1(\cos\theta)]. \quad [27]$$

Expanding $\bar{V}_\infty\cdot\bar{i}_r$ in a series analogue to eq. [21], with coefficients $\underline{\alpha}_n$, $\underline{\beta}_n$, etc., substituting into eq. [27] and using the orthogonality properties of the *Legendre* polynomials, we obtain

$$U_z = \frac{2}{3}\frac{(\varrho' - \varrho)\,a^2 g}{\mu}\frac{1+\lambda}{2+3\lambda} + \underline{\beta}^0_1 + \frac{5\lambda}{2+3\lambda}\underline{\alpha}^0_1 \quad [28a]$$

$$U_y = \underline{\beta}^1_1 + \frac{5\lambda}{2+3\lambda}\hat{\underline{\alpha}}^1_1 \quad [28b]$$

$$U_x = \underline{\beta}^1_1 + \frac{5\lambda}{2+3\lambda}\underline{\alpha}^1_1. \quad [28c]$$

Note that in order to solve the coefficients $\underline{\alpha}^0_1$, $\underline{\beta}^0_1$, $\underline{\alpha}^1_1$, etc., one has merely to solve the following integrals:

$$\underline{\alpha}^0_1\left(\frac{r}{a}\right)^2 + \underline{\beta}^0_1 + \underline{\beta}^0_{-2}\left(\frac{r}{a}\right)^{-3} = \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}(\bar{V}_\infty\cdot\bar{i}_r)\cos\theta\sin\theta\, d\theta\, d\varphi \quad [29a]$$

$$\underline{\alpha}^1_1\left(\frac{r}{a}\right)^2 + \underline{\beta}^1_1 + \underline{\beta}^1_{-2}\left(\frac{r}{a}\right)^{-3} = \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}(\bar{V}_\infty\cdot\bar{i}_r)\sin^2\theta\cos\varphi\, d\theta\, d\varphi \quad [29b]$$

$$\hat{\underline{\alpha}}^1_1\left(\frac{r}{a}\right)^2 + \hat{\underline{\beta}}^1_1 + \hat{\underline{\beta}}^1_{-2}\left(\frac{r}{a}\right)^{-3} = \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}(\bar{V}_\infty\cdot\bar{i}_r)\sin^2\theta\sin\varphi\, d\theta\, d\varphi\,. \quad [29c]$$

The formulation presented above is quite general and is applicable also for the case when $\bar{v}_\infty = \infty$ at $r = 0$. In the event that $\bar{v}_\infty$ is finite everywhere in the field, eqs. [28] can be written in vectorial notation and the coefficients determined in a simplified manner as follows:

$$[\bar{V}_\infty]_{r=0} = [\nabla\Phi^\infty_1]_{r=0} \quad [30]$$

since all other solid spherical harmonics do not have any contribution. Similarly, we obtain:

$$[\nabla p_\infty]_{r=0} = [\nabla p^\infty_1]_{r=0}\,. \quad [31]$$

Substitution of eqs. [17] into eqs. [30] and [31] yields:

$$[\bar{V}_\infty]_{r=0} = \beta_0^1 \bar{k} + \beta_1^1 \bar{i} + \hat{\beta}_1^1 \bar{j} \quad [32a]$$

and

$$[\nabla p_\infty]_{r=0} = 10\, a^{-2} \mu (\alpha_1^0 \bar{k} + \alpha_1^1 \bar{i} + \hat{\alpha}_1^1 \bar{j}). \quad [32b]$$

Then the expression for the terminal settling velocity is merely:

$$\bar{U} = \frac{2(\varrho' - \varrho) a^2 \bar{g}}{3\mu} \frac{(1+\lambda)}{2+3\lambda} + [\bar{V}_\infty]_{r=0} + \frac{a^2}{2\mu} \frac{\lambda}{2+3\lambda} [\nabla p_\infty]_{r=0}. \quad [33]$$

The drag force on a spherical droplet suspended in an unbounded fluid for the case when the velocity distribution far from the droplet is $\bar{V}_\infty$, is obtained by substitution of eq. [33] into the expression for the drag:

$$\bar{F}_D = 2\pi \mu a \frac{2+3\lambda}{1+\lambda} [\bar{V}_\infty]_{r=0} + \pi a^3 \frac{\lambda}{1+\lambda} [\nabla p_\infty]_{r=0}. \quad [34]$$

This result is a generalization of *Faxen*'s law for a spherical particle. This law is a particular case of eq. [34] when $\lambda = \infty$.

3.3. *The equation of the interface*

Thus far only six of the boundary conditions have been utilized for solving the flow fields in and around a spherical droplet. The seventh boundary condition, i. e.

$$\bar{\tau}^*_{(r)} \cdot \bar{i}_r + \bar{\tau}^*_{\infty(r)} \cdot \bar{i}_r - \bar{\tau}'^*_{(r)} \cdot \bar{i}_r = \sigma \left[\frac{1}{R_1} + \frac{1}{R_2}\right] \quad [35]$$

is now used for determining the equation of the interface. This new interface may then be used for calculating the velocity fields of the second iteration.

The radius of nearly spherical droplets may be represented by:

$$r = a[1 + \xi(\theta, \varphi)], \quad [36]$$

where $|\xi(\theta,\varphi)| \ll 1$.

Landau and *Lifshitz* (6) derived the following expression:

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{2}{a} - \frac{2\xi}{a} - \frac{1}{a}\left[\frac{1}{\sin^2\theta}\frac{\partial^2 \xi}{\partial \varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \xi}{\partial \theta}\right)\right]. \quad [37]$$

Let the deviation function $\xi(\theta, \varphi)$ be described by a sum of surface harmonics as follows:

$$\xi(\theta,\varphi) = \sum_{n=1}^{\infty} L_n S_n(\theta,\varphi), \quad [38]$$

where the notation is analogue to eq. [16].

Substituting eq. [38] in eq. [37] and recalling that

$$\frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_n}{\partial\theta}\right) + n(n+1) S_n = 0$$

we readily obtain:

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{a}\left\{2 + \sum_{n=1}^{\infty}[n(n+1) - 2] L_n S_n\right\}. \quad [39]$$

Substitution of eq. [39] into boundary condition [35], we obtain (Appendix B):

$$\sum_{n=1}^{\infty}\left\{\frac{2n(n-1)\lambda}{a}\Phi_n^* + \frac{(n^2-n-3)\lambda a}{(2n+3)\mu'} p_n^* - \frac{2(n+1)(n+2)}{a}\Phi_{-n-1}^* + \frac{(n^2+3n-1)}{(2n-1)\mu} p_{-n-1}^* - \frac{2n(n-1)}{a}\Phi_n^{\infty *} - \frac{(n^2-n-3)a}{(2n+3)\mu} p_n^{\infty *} - \frac{2(n+1)(n+2)}{a}\Phi_{-n-1}^{\infty *} + \frac{(n^2+3n-1)}{(2n-1)\mu} p_{-n-1}^{\infty *} - \frac{a}{\mu}(p_0^{\infty *} + p_{-1}^{\infty *} + p_0^* - p_{-1}^*) - \frac{(\varrho'-\varrho) g a}{\mu} P_1(\cos\theta)\right\} = -\frac{\sigma}{\mu}\left\{2 + \sum_{n=1}^{\infty}[n(n+1)-2] L_n S_n\right\}. \quad [40]$$

Substitution of eqs. [15], [17] and [18] in eq. [40], and using the orthogonality properties of the surface harmonics, we obtain an infinite set of linearly independent equations. The first of these equations, namely for $n = 0$, yields:

$$p_0^{\infty *} + p_{-1}^{\infty *} + p_0^* - p_{-1}^* = \frac{2\sigma}{a}.$$

The second equation, for $n = 1$, yields an identity. This implies that the assumption of a spherical droplet is correct up to second term in eq. [36].

Solving the equations with $n = 2, 3, \ldots, \infty$, for L_n^m, we finally obtain:

$$L_n^m = \frac{1}{(n^2+n-2)\, n(n+1)} \frac{1}{(1+\lambda)} \times \left\{\frac{\alpha_n^m \mu}{\sigma}[(4n^3+6n^2+2n+3)\lambda + (4n^3+6n^2-4n-6)] + \frac{\beta_n^m \mu}{\sigma}[(4n^3+6n^2+2n-3)\lambda + (4n^3+6n^2-4n)]\right\} \quad [41]$$

and similar expressions for $\hat{L}_n^m$. Notice that L_n^m is at least of order $0(1/n^2)$.

The deviation from sphericity function is thus obtained by substitution of eq. [41] into eq. [38].

4. Examples

The usefulness of the solution presented herein is now demonstrated by solving the flow fields, settling velocity and the deviation from sphericity for two simple flows which were solved by *G. I. Taylor* (3a, 3b), and for *Poiseuill*ian flow, which was not solved previously.

4.1. Couette flow

The unperturbed velocity distribution is given by:

$$\bar{v}_\infty = G(y + l)\,\bar{i} - U\,\bar{i}, \qquad [42]$$

where l is the distance of the center of the droplet from the point of zero velocity and G is the shear.

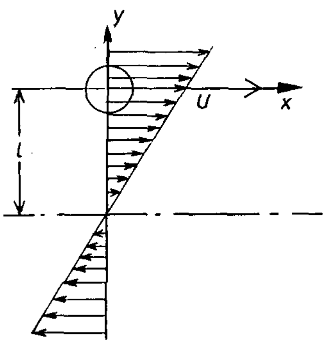

Fig. 1. *Couette* flow

Multiplying eq. [42] by a unit vector in the r direction $\bar{i}_r$:

$$\bar{v}_\infty \cdot \bar{i}_r = (G\,l - U)\sin\theta\cos\varphi + G\,r\sin^2\theta\sin\varphi\cos\varphi$$

$$= (G\,l - U)\cos\varphi\,P_1^1(\cos\theta) + \frac{G\,a}{6}\,\frac{r}{a}\sin 2\varphi\,P_2^2(\cos\theta)\,.$$

Hence

$$\beta_1^1 = G\,l - U \quad \text{and} \quad \hat{\beta}_2^2 = \frac{G\,a}{6}\,. \qquad [43]$$

Similarly,

$$\bar{r}\cdot\nabla x\,\bar{v}_\infty = -\,G\,a\,\frac{r}{a}\cos\theta\,.$$

Hence

$$\gamma_1^0 = -\,G\,a \qquad [44]$$

Substitution of eqs. [43] and [44] in eqs. [18], one obtains the coefficients A_1^1, B_1^1, A_{-2}^1, B_{-2}^1, $\hat{A}_2^2$, $\hat{B}_2^2$, $\hat{A}_{-3}^2$, $\hat{B}_{-3}^2$ and C_1^0. All other coefficients are identically zero. Substitution of the coefficients into eqs. [7] through [11], using eqs. [15], the velocity fields are obtained.

The settling velocity is readily obtained from eqs. [28]:

$$U_x = \beta_1^1 = G\,l, \quad U_y = U_z = 0\,. \qquad [45]$$

The deviation from sphericity is obtained from eq. [41] as follows:

$$\hat{L}_2^2 = \frac{G\,a\,\mu}{3\sigma}\cdot\frac{16 + 19\lambda}{16(1+\lambda)}\,.$$

Finally, the radius is given by:

$$r = a[1 + \hat{L}_2^2\sin 2\varphi\,P_2^2(\cos\theta)]$$

$$= a\left[1 + \frac{G\,a\,\mu}{\sigma}\,\frac{16 + 19\lambda}{16(1+\lambda)}\,\frac{x\,y}{r^2}\right]. \qquad [46]$$

Eqs. [45] and [46] are the familiar results of *Taylor* (3) for a neutrally buoyant droplet in *Couette* flow.

4.2. Hyperbolic flow

The unperturbed velocity distribution is given by

$$\bar{v}_\infty = G\,x\,\bar{i} - G\,y\,\bar{j} - U_x\,\bar{i} - U_y\,\bar{j}\,. \qquad [47]$$

Using a procedure entirely analogue to that in section 4.1, it is easy to obtain the required coefficients, i. e.

$$\beta_2^2 = \frac{G\,a}{3}\,; \quad \beta_1^1 = -\,U_x\,; \quad \hat{\beta}_1^1 = -\,U_y \qquad [48]$$

from which the settling velocity and the deviation from sphericity are calculated, namely:

$$U_x = U_y = 0$$

$$r = a\left[1 + \frac{2\,G\,a\,\mu}{\sigma}\,\frac{16 + 19\lambda}{16(1+\lambda)}\sin^2\theta\cos 2\varphi\right] \qquad [49]$$

which are the familiar results of *Taylor* (3).

4.3. Poiseuille flow

Use a spherical coordinate system which is centered at the center of mass of the droplet (fig. 2). In this system the unperturbed velocity distribution is:

$$\bar{v}_\infty = \left\{U_0\left[1 - \left(\frac{r}{R_0}\right)^2\sin^2\theta - \left(\frac{b}{R_0}\right)^2 - \frac{2\,r\,b}{R_0{}^2}\cos\varphi\sin\theta\right] - U\right\}\bar{k}\,, \qquad [50]$$

where U_0 is the maximum velocity which is at a distance b from the droplet and where R_0 is the distance to the point of zero velocity.

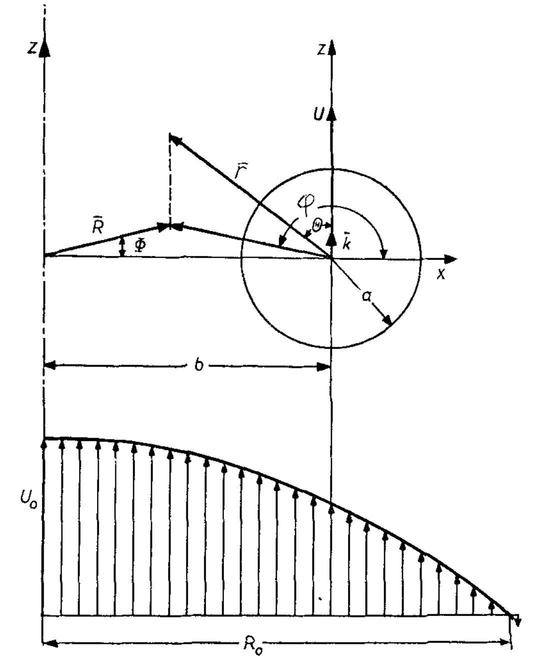

Fig. 2. The geometry and coordinate systems used in the analysis

Multiplying eq. [50] by a unit vector in the r direction $\bar{i}_r$, one obtains:

$$\bar{v}_\infty \cdot \bar{i}_r = U_0\left[\cos\theta - \left(\frac{r}{R_0}\right)^2 \sin^2\theta\cos\theta - \left(\frac{b}{R_0}\right)^2\cos\theta - \frac{2rb}{R_0^2}\cos\theta\sin\theta\cos\varphi\right] - U\cos\theta\,.$$

Similarly, one obtains the following product:

$$\bar{r}\cdot \nabla x\,\bar{v}_\infty = U_0\frac{2br}{R_0^2}\sin\theta\sin\varphi\,.$$

Recalling that:

$$\cos\theta = P_1(\cos\theta)$$

$$\sin\theta = P_1^1(\cos\theta)$$

$$3\sin\theta\cos\theta = P_2^1(\cos\theta)$$

and

$$\sin^2\theta\cos\theta = \frac{2}{5}[P_1(\cos\theta) - P_3(\cos\theta)]$$

one obtains:

$$\bar{v}_\infty\cdot\bar{i}_r = U_0\left\{\left[1-\left(\frac{b}{R_0}\right)^2 - \frac{2}{5}\frac{a^2}{R_0^2}\left(\frac{r}{a}\right)^2\right]P_1(\cos\theta) - \frac{2}{3}\frac{ab}{R_0^2}\left(\frac{r}{a}\right)\cos\varphi\,P_2^1(\cos\theta) + \frac{2}{5}\left(\frac{a}{R_0}\right)^2\left(\frac{r}{a}\right)^2 P_3(\cos\theta) - U\,P_1(\cos\theta)\right\} \quad [51]$$

$$\bar{r}\cdot\nabla x\,\bar{v}_\infty = U_0\frac{2ab}{R_0}\left(\frac{r}{a}\right)\sin\varphi\,P_1^1(\cos\theta)\,. \quad [52]$$

Hence:

$$\alpha_1^0 = -\frac{2}{5}U_0\left(\frac{a}{R_0}\right)^2$$

$$\beta_1^0 = U_0\left(1-\frac{b^2}{R_0^2}\right) - U$$

$$\beta_2^1 = -\frac{2}{3}U_0\frac{ab}{R_0^2}$$

$$\beta_3^0 = \frac{2}{5}U_0\left(\frac{a}{R_0}\right)^2 \quad [53]$$

and

$$\hat{\gamma}_1^1 = U_0\frac{2ab}{R_0^2}\,.$$

Substitution of eqs. [53] in eq. [18] one obtains the coefficients

$A_1^0, B_1^0, A_2^1, B_2^1, A_3^0, B_3^0, A_{-2}^0, B_{-2}^0, A_{-3}^1, B_{-3}^1, A_{-4}^0, B_{-4}^0$ and $\hat{C}_1$.

All other coefficients are identically zero. Substitution of the coefficients into eqs. [7] through [11] and using eqs. [15], the velocity fields are obtained.

The settling velocity is obtained from eqs. [28],

$$U_x = U_y = 0$$

$$U = U_Z = \frac{2}{3}\frac{(\varrho'-\varrho)a^2 g}{\mu}\frac{1+\lambda}{2+3\lambda} + U_0\left(1-\frac{b^2}{R_0^2} - \frac{2\lambda}{2+3\lambda}\frac{a^2}{R_0^2}\right). \quad [54]$$

The deviation from sphericity is obtained from eq. [41]:

$$L_2^1 = -\frac{\mu U_0}{\sigma}\frac{ab}{R_0^2}\frac{19\lambda+16}{12\lambda+12}$$

$$L_3^0 = \frac{\mu U_0}{\sigma}\left(\frac{a}{R_0}\right)^2\frac{11\lambda+10}{20\lambda+20}\,.$$

Finally, the radius is given by:

$$r = a[1 + L_2^1\cos\varphi\,P_2^1(\cos\theta) + L_3^0\,P_3(\cos\theta)]$$

$$= a\left[1 - \frac{\mu U_0}{\sigma}\frac{ab}{R_0^2}\frac{19\lambda+16}{12\lambda+12}\cos\varphi\,P_2^1(\cos\theta) + \frac{\mu U_0}{\sigma}\left(\frac{a}{R_0}\right)^2\frac{11\lambda+10}{20\lambda+20}P_3(\cos\theta)\right]. \quad [55]$$

Eqs. [54] and [55] have not been derived previously.

To illustrate the deviation from sphericity, eq. [55] was evaluated for the following conditions:

$$\frac{\mu U_0}{\sigma} = 2; \quad \frac{a}{R_0} = 0.316; \quad \frac{b}{R_0} = 0.08$$

$$\lambda \to 0$$

and the resulting shape is depicted in fig. 3.

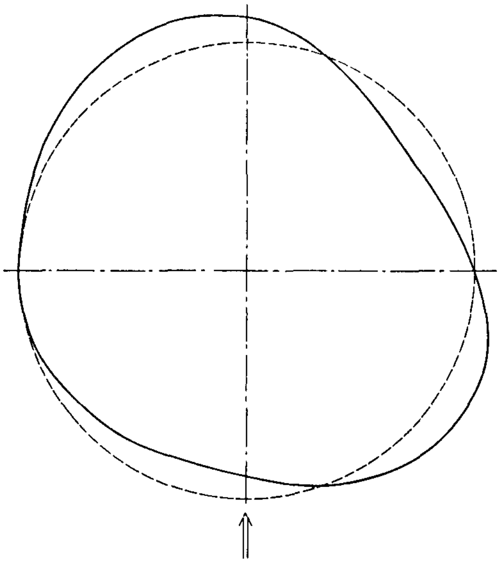

Fig. 3. The deviation from sphericity of a droplet submerged in an unbounded *Poiseuille* flow, with $\mu U_0/\sigma = 2$, $a/R_0 = 0.316$, $b/R_0 = 0.08$, $\lambda_1 \to 0$

One should note that these conditions are not physically meaningful. The value of $\mu\, U_0/\sigma$ has been deliberately exaggerated in order to make the deviation from sphericity visible. Under conditions which are physically more realistic, the deviation from sphericity will have a much smaller amplitude.

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Summary

A solution is presented for the flow fields interior and exterior to a single spherical droplet submerged in an unbounded fluid, for the general case when the unperturbed velocity is *Stokes*ian but otherwise quite arbitrary.

A general equation for the terminal settling velocity is derived which contains as special cases the solutions of *Hadamard-Rybczynski*, *Taylor* and others. The drag force on a spherical droplet is also formulated. This equation contains as a special case the law of *Faxen.*

The function describing the interface and its deviation from sphericity is derived. This may be used for determining more accurate flow fields in an iterative procedure.

Zusammenfassung

Das Problem der Strömungsfelder in einer unbegrenzten Flüssigkeit wird sowohl innerhalb wie außerhalb eines einzelnen kugelförmigen Tropfens für den allgemeinen Fall gelöst, wenn die ungestörte Geschwindigkeit durch die *Stokes*sche Gleichung angegeben werden kann, aber sonst willkürlich ist.

Es wird eine allgemeine Formel für die Geschwindigkeit, die nach langer Zeit erreicht wird, angegeben. Sie enthält als Spezialfall die Lösungen von *Hadamard-Rybczynski*, *Taylor* u. a. Die schleppende Kraft, die auf den Tropfen einwirkt, wird formuliert. Diese Formel enthält das Gesetz von *Faxen* als Spezialfall.

Die Funktion der Grenzfläche und ihre Abweichung von der Kugelform wird bestimmt. Sie kann zur Bestimmung genauerer Strömungsfelder in einem Iterationsverfahren angewendet werden.

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Appendix A

The boundary conditions for the first iteration are as follows:

$$v_r^* = v_r'^* = 0 \qquad \text{[A-1]}$$

$$\bar{v}^* = \bar{v}'^* \qquad \text{[A-2]}$$

$$\bar{\pi}_{(r)}^* - \bar{\tau}_{(r)}'^* = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\bar{i}_r, \qquad \text{[A-3]}$$

where $\bar{\pi}_{(r)}$ and $\bar{\tau}'_{(r)}$ are based on $\bar{v}$ and $\bar{v}'$, respectively.

From these boundary conditions the following vectorial equations can be obtained:

$$\bar{v}^* \cdot \bar{i}_r = \bar{v}'^* \cdot \bar{i}_r$$

$$-r\,\nabla \cdot \bar{v}^* = -r\,\nabla \cdot \bar{v}'^*$$

$$\bar{r} \cdot \nabla x\, \bar{v}^* = \bar{r} \cdot \nabla x\, \bar{v}'^*$$

$$\bar{r} \cdot \nabla x\,(\bar{r}\; x\; \bar{\pi}_{(r)}^*) = \bar{r} \cdot \nabla x\,(\bar{r}\; x\; \bar{\tau}_{(r)}'^*)$$

$$\bar{r} \cdot \nabla x\, \bar{\pi}_{(r)}^* = \bar{r} \cdot \nabla x\, \bar{\tau}_{(r)}'^*$$

$$\bar{\pi}_{(r)}^* \cdot \bar{i}_r - \bar{\tau}_{(r)}'^* \cdot \bar{i}_r = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

But since

$$\nabla \cdot \bar{v}' = 0$$

$$-r\,\nabla \cdot \bar{v}'^* = \left(r \frac{\partial v_r'}{\partial r}\right)^*$$

Also,

$$\bar{r} \cdot \nabla x\, \bar{v}'^* = [\bar{r} \cdot \nabla x\, \bar{v}']^*$$

$$\bar{r} \cdot \nabla x\,(\bar{r}\; x\; \bar{\tau}_{(r)}'^*) = [\bar{r} \cdot \nabla x\,(r\; x\; \bar{\tau}_{(r)}')]^*$$

$$\bar{r} \cdot \nabla x\, \bar{\tau}_{(r)}'^* = [\bar{r} \cdot \nabla x\, \bar{\tau}_{(r)}']^*$$

and

$$\bar{\tau}_{(r)}'^* \cdot \bar{i}_r = \tau_{rr}'^*.$$

Similar expressions can be written for $\bar{v}$ and $\bar{\pi}_{(r)}$. Equating the two sets, together with eqs. [A-1] through [A-3], one obtains:

$$v_r^* = v_r'^*$$

$$\left[r\frac{\partial v_r}{\partial r}\right]^* = \left[r\frac{\partial v_r'}{\partial r}\right]^*$$

$$[\bar{r}\cdot \nabla x\,\bar{v}]^* = [\bar{r}\cdot \nabla x\,\bar{v}']^*$$

$$[\bar{\tau}_r\cdot \nabla x(\bar{r}\,x\,\bar{\pi}_{(r)})]^* = [\bar{\tau}_r\cdot \nabla x(\bar{r}\,x\,\bar{\tau}_{(r)}']^*$$

$$[\bar{\tau}_r\cdot \nabla x\,\bar{\pi}_{(r)}]^* = [\bar{\tau}_r\cdot \nabla x\,\bar{\tau}_{(r)}']^*$$

$$\pi_{rr}^* - \tau_{rr}'^* = \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

Recalling that $\bar{v} = \bar{v}_\infty + \bar{V}$, we obtain the following set of transformed boundary conditions:

$$v_r'^* = 0$$

$$V_r^* = -v_{\infty r}^*$$

$$\left[r\frac{\partial v_r'}{\partial r}\right]^* - \left[r\frac{\partial V_r}{\partial r}\right]^* = \left[r\frac{\partial v_{\infty r}}{\partial r}\right]^*$$

$$[\bar{r}\cdot \nabla x\,\bar{v}']^* - [\bar{r}\cdot \nabla x\,\bar{V}]^* = [\bar{r}\cdot \nabla x\,\bar{v}_\infty]^*$$

$$[\bar{r}\cdot \nabla x(\bar{r}\,x\,\bar{\tau}_{(r)}')]^* - [\bar{r}\cdot \nabla x(\bar{r}\,x\,\bar{\tau}_{(r)})]^* = [\bar{r}\cdot \nabla x(\bar{r}\,x\,\bar{\tau}_{(r)\infty})]^*$$

$$[\bar{r}\cdot \nabla x\,\bar{\tau}_{(r)}'] - [\bar{r}\cdot \nabla x\,\bar{\tau}_{(r)}]^* = [\bar{r}\cdot \nabla x\,\bar{\tau}_{(r)\infty}]^*$$

$$\tau_{rr}^* + \tau_{\infty rr}^* - \tau_{rr}' = \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right),$$

where $\bar{\tau}_{(r)}$ and $\bar{\tau}_{\infty(r)}$ are based on $\bar{V}$ and $\bar{v}_\infty$, respectively.

Appendix B

Lamb's solution for *Stokes*ian flow is:

$$\bar{v} = \sum_{n=-\infty}^{\infty}\left[\nabla x(\bar{r}\,\chi_n) + \nabla\Phi_n + \frac{n+3}{2(n+1)(2n+3)\mu} r^2 \nabla p_n - \frac{n}{(n+1)(2n+3)\mu}\bar{r}\,p_n\right].$$

Happel and *Brenner* (5, p. 63) showed that:

$$\bar{v}\cdot\bar{i}_r = \sum_{n=-\infty}^{\infty}\left[\frac{n}{2\mu(2n+3)} r\,p_n + \frac{n}{r}\Phi_n\right]$$

$$r\frac{\partial v_r}{\partial r} = \sum_{n=-\infty}^{\infty}\left[\frac{n(n+1)}{2\mu(2n+3)} r\,2\,p_n + \frac{n(n-1)}{r}\Phi_n\right]$$

$$\bar{r}\cdot\nabla x\,\bar{v} = \sum_{n=-\infty}^{\infty} n(n+1)\,\chi_n$$

and

$$\bar{\tau}_{(r)} = \frac{\mu}{r}\sum_{n=-\infty}^{\infty}\left[(n-1)\nabla x(\bar{r}\,\chi_n) + 2(n-1)\nabla\Phi_n - \frac{2n^2+4n+3}{\mu(n+1)(2n+3)}\bar{r}\,p_n + \frac{n(n+2)}{\mu(n+1)(2n+3)} r^2\nabla p_n\right].$$

The following relations are easy to prove:

$$\bar{r}\cdot\nabla x\,\bar{\tau}_{(r)} = \frac{\mu}{r}\sum_{n=-\infty}^{\infty}(n-1)\,n(n+1)\,\chi_n$$

$$\bar{r}\cdot\nabla x(\bar{r}\,x\,\bar{\tau}_{(r)}) = -\mu\sum_{n=-\infty}^{\infty}\left[\frac{2(n-1)\,n(n+1)}{r}\Phi_n + \frac{n^2(n+2)}{(2n+3)} r\,p_n\right]$$

$$\bar{\tau}_r\cdot\bar{i}_r = \frac{\mu}{r}\sum_{n=-\infty}^{\infty}\left[\frac{2n(n-1)}{r}\Phi_n + \frac{(n^2-n-3)}{\mu(2n+3)} r\,p_n\right]$$

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