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Sensitive Dependence to Initial Conditions for One Dimensional Maps*

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Abstract. This paper studies the iteration of maps of the interval which have negative Schwarzian derivative and one critical point. The maps in this class are classified up to topological equivalence. The equivalence classes of maps which display sensitivity to initial conditions for large sets of initial conditions are characterized.

There has been recent interest in the relationship between the "chaotic" asymptotic behavior of complicated solutions to ordinary differential equations and physically unstable phenomena such as those encountered in fluid flow [21]. This has led to numerical studies of a variety of systems of differential and difference equations which appear to have large sets of initial conditions yielding complicated asymptotic behavior. The mathematical theory of Axiom A "strange attractors" provides a satisfying description of some systems which do have complicated asymptotic behavior, but there is little overlap between the numerical studies referred to above and the class of systems with Axiom A attractors. Perhaps the "Lorenz system" [16], is the only example of an explicit system of equations used in a physical problem for which there is a convincing argument that a large set of its solutions have complicated asymptotic behavior. Nonetheless, the numerical studies of such examples as the "Henon" map [9], the mechanical systems studied by Holmes and Moon [11], the "strange attractor" of Spiegel [26], and the density dependent population models of [8] all provide evidence for the prevalence of complicated solutions in systems near those with homoclinic tangencies. From a practical point of view the distinction between trajectories tending to complicated periodic orbits with very long periods and trajectories with aperiodic asymptotic behavior may be slight, but we would like to understand the extent to which numerical computations of strange attractors reflect

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the mathematical properties of the systems being studied. This paper aims to study some of these mathematical questions in the simplest situation in which they arise: the iteration of a single real valued function of "quadratic" type. In this context we give a reasonably complete topological picture of what is involved in chaotic behavior and sensitive dependence to initial conditions while failing to answer the outstanding question about their prevalence.

1. Introduction

There have been many numerical studies of the asymptotic properties of the iterates of a real function $f:I\to I$, $I=[0,1]$. The quadratic functions $f_a(x) = ax(1-x)$, $0\leqq a\leqq 4$, have been studied intensively. These numerical investigations indicate that there are many functions f which have typical solutions with complicated asymptotic behavior. For example, the calculations of Shaw [24] indicate that for the quadratic family, there is a set of $a\in[3,4]$ of large measure for which the typical trajectories have positive Liapounov exponents. This leads one to speculate that in some suitable class of one parameter families of maps $f_a:I\to I$, each family has a set of parameter values of positive measure such that the corresponding maps have trajectories which depend sensitively to initial conditions. We give our definition of sensitive dependence.

Definition. Let X be a metric space with a measure μ and $f:X\to X$ a continuous map. Then f has *sensitive dependence to initial conditions* if there is a set $Y\subset X$ of positive measure and an $\varepsilon>0$ such that for any $x\in Y$ and neighborhood U of x, there is $y\in U$ and $n\geqq 0$ with $d(f^n(x), f^n(y))>\varepsilon$.

Certainly other definitions of sensitive dependence are possible, but this one is suitable for our purposes. We shall specify a class of maps $f:I\to I$ for which we can give a precise topological characterization of when they have sensitivity to initial conditions.

Two maps $f,g:X\to X$ are *topologically equivalent* if there is a homeomorphism $h:X\to X$ such that $hf=gh$. A topological equivalence h maps f-trajectories to g-trajectories. For the maps which we study, sensitivity to initial conditions will depend only on the topological equivalence class of the map. We shall give a complete enumeration of the topological equivalence classes having sensitivity to initial conditions. Thus we are able to focus the question of the prevalence of sensitivity to initial conditions in rather sharp terms. Let us proceed to describe these results precisely.

The Schwarzian derivative [4] of a function $f:I\to I$ is defined by $Sf(x) = \dfrac{f'''(x)}{f'(x)} - \dfrac{3}{2}\left(\dfrac{f''(x)}{f'(x)}\right)^2$. *We shall work with the class of functions C defined by*

1) $f:I\to I$ with $f(0)=f(1)=0$ and $f\in C^3(I)$.

2) f has a single local maximum $c=c_f$. The function f is strictly increasing on $[0,c]$ and strictly decreasing on $[c,1]$. $f''(c)<0$.

3) The Schwarzian derivative of f is negative: for all $x\in I-\{c\}$, $Sf(x)<0$. We explain below the significance of 3), which we call the *Schwarzian condition.*

Throughout the paper the symbol "c" means the critical point of whichever map is under discussion. If there is ambiguity we write c_f for the critical point of f. It is clear that many of our results hold for larger classes of function than those in $\mathscr{C}$ and we occasional indicate results which hold in greater generality, but for the most part we focus attention on $\mathscr{C}$. Here the theory we describe is most complete. Note that quadratic functions have negative Schwarzian derivative since their third derivatives are zero.

A primary reason for working with negative Schwarzian derivative is *Singer's theorem.* To state this theorem recall a couple of definition. A point $x \in I$ is *periodic* with period n if $f^n(x)=x$ but $f^i(x) \neq x$ for $i<n$. A periodic point x of period n is (one-sided) *stable* if there is a non-trivial interval U with $f^n(y) \to x$ for all $y \in U$. An interval is non-trivial if it has positive length; a trivial interval is a point. A necessary condition for x to be stable is that $|Df^n(x)| \leqq 1$. A sufficient condition for x to be stable is that $|Df^n(x)| < 1$. We denote the derivative of f by either f' or Df as convenient.

Theorem (Singer [25]). *Let $f: I \to I$ have negative Schwarzian derivative. For every stable periodic point x of period n, there is an $i<n$ and a critical point c or endpoint of I such that $y \in [c, f^i(x)]$ implies $f^{kn}(y) \to f^i(x)$ as $k \to \infty$.*

The proof of Singer's theorem is based upon several facts about Schwarzian derivatives which we list here and use later:

1) $S(f \circ g)(x) = Sf(g(x)) \cdot (g'(x))^2 + Sg(x)$. If Sf is negative, then Sf^n is negative for all $n>0$.

2) If Sf is negative, then $|f'|$ has no positive local minimum. If $J=[a,b]$ is an interval on which f is monotone and $x \in J$, then $|f'(x)| \geqq \min(|f'(a)|, |f'(b)|)$.

3) Denote $(x_1, x_2, x_3, x_4) = \dfrac{(x_4-x_1)(x_3-x_2)}{(x_4-x_3)(x_2-x_1)}$. Sf is negative if and only if for each $x_1<x_2<x_3<x_4$ contained in an interval on which f is monotone, $(x_1, x_2, x_3, x_4) < (f(x_1), f(x_2), f(x_3), f(x_4))$.

4) If f is a polynomial such that all zeros of f' are real, then Sf is negative.

The proof of Singer's theorem rests upon the fact that an increasing function with negative Schwarzian derivative cannot have a stable fixed point between two unstable fixed points. Each stable periodic orbit has associated with it a critical point.

Corollary. *If $f \in \mathscr{C}$, then f has at most one stable periodic orbit*

In Sect. 3, we prove that if f has a stable periodic orbit, then it is not sensitive to initial conditions. Thus the primary interest in sensitivity to initial conditions forces us to devote our attention to those $f \in \mathscr{C}$ which do not have stable periodic orbits. The structure of these maps can be quite complicated. They are far from being hyperbolic or structurally stable, so the techniques used to study Axiom A dynamical systems are not sufficient for a topological characterization of these. It is here that the one-dimensionality is exploited in a rigorous way. Just as order of points on the circle leads to the theory of rotation numbers for homeomorphisms of S^1, the order on the interval leads to invariants of a map $f: I \to I$ with respect to topological equivalence. Milnor and Thurston [17] have introduced a language, the *kneading theory*, which systematically exploits the order of the line in studying topological properties of a map under iteration. We shall use this language in Sect. 2 to construct the topological classification of maps $f \in \mathscr{C}$, so we introduce the necessary background from the kneading theory here.

Milnor and Thurston study piecewise monotone maps. The map $f: I \to I$ is piecewise monotone if it is continuous and there are $0 = c_0 < c_1 < \ldots < c_l = 1$ such that $f|[c_{i-1}, c_i]$ is strictly increasing or strictly decreasing. Assume that the set $\{c_i\}_{i=0}^{l}$ is chosen as small as possible. Then the c_i's are called *turning points* and the intervals $I_i = (c_{i-1}, c_i)$ are called *laps*. Associated to each $x \in I$ is a sequence $\underline{A}(x) = \{A_n(x)\}_{n=0}^{\infty}$, called the *itinerary* of x. The n-th term $A_n(x)$ is called the n-th *address* of x and is defined by $A_n(x) = I_i$ or C_i as $f^n(x) \in I_i$ or $f^n(x) = c_i$. The itineraries of the turning points are called *kneading sequences*. It is easily seen that if h is an orientation preserving topological equivalence from f to g, then the kneading sequences of f and g must be the same.

By introducing signs to the addresses, one can order these sequences in a way which is consistent with order on the interval. Define the sign $\varepsilon(I_i)$ of the symbol I_i to be $+1$ or -1 as $f|I_i$ is increasing or decreasing. The sign $\varepsilon(C_i)$ is defined to be 0. Given a sequence $\underline{a} = \{a_n\}_{n=0}^{\infty}$ of the symbols I_i, C_i, define the invariant coordinate $\theta(\underline{a}) = \{\theta_n(\underline{a})\}_{n=0}^{\infty}$ by $\theta_n(a) = \varepsilon(a_0)\, \varepsilon(a_1) \ldots \varepsilon(a_{n-1}) a_n$. If $x \in I$, we write $\underline{\theta}(x) = \theta(\underline{A}(x))$. We introduce an order of symbols by

$$-C_l < -I_l < -C_{l-1} < \ldots < -I_1 < -C_0 < {}^{0 \cdot C_i}_{0 \cdot I_i} < C_0 < I_1 < \ldots < I_l < C_l\,.$$

With this order of symbols, we order the sequences $\theta(\underline{a})$ lexicographically: $\theta(\underline{a}) < \theta(\underline{b})$ if $\underline{a} \neq \underline{b}$ and $\theta_n(\underline{a}) < \theta_n(\underline{b})$ for the smallest n with $a_n \neq b_n$. The fundamental observation of Milnor-Thurston is the *monotonicity of invariant coordinates:*

Theorem. *Let $f: I \to I$ be piecewise monotone. If $x < y$, then $\theta(x) \leqq \theta(y)$.*

The proof of this theorem follows easily from the fact that the sign of $\theta_n(x)$ is $+1$ or -1 according to whether f^n is increasing or decreasing at x. If $x < y$ and n is the smallest integer with $A_n(x) \neq A_n(y)$, then f^n is monotone on the interval $[x, y]$. One has $f^n(x) < f^n(y)$ or $f^n(x) > f^n(y)$ depending upon whether the signs of $\theta_n(x)$ and $\theta_n(y)$ are $+1$ or -1.

For a map $f \in \mathscr{C}$, the itineraries of the endpoints are fixed, and there is one interior turning point c. The itinerary $\underline{\gamma}$ of c we call the *kneading sequence* of f. It plays a role for f analogous to the rotation number of a homeomorphism of the circle. In the first part of Sect. 2, we discuss how $\underline{\gamma}$ determines the itineraries of other points $x \in I$. We also examine which symbol sequences occur as the kneading sequence of some map $f \in \mathscr{C}$. An important role in these considerations is played by the *shift map* σ on sequences. If $\underline{a} = \{a_n\}_{n=0}^{\infty}$ is a sequence, then $\sigma(\underline{a}) = \underline{b}$ is the sequence obtained by dropping a_0 and renumbering: $\underline{b} = \{b_n\}_{n=0}^{\infty}$ and $b_n = a_{n+1}$. Using the monotonicity of the invariant coordinate and the shift map, we determine recursive conditions for a sequence $\underline{a}$ of I_0, C, and I_1 to occur as the kneading sequence of $f \in \mathscr{C}$. (We number the two laps of f as I_0 and I_1 rather than I_1 and I_2.)

The kneading sequence of $f \in \mathscr{C}$ comes very close to being a complete invariant of its topological equivalence class. In the second half of Sect. 2, we show that the kneading sequence of f determines whether or not f has a stable periodic orbit. If

f and g in $\mathscr{C}$ have the same kneading sequence and do not have stable periodic orbits, then we prove that they are topologically equivalent. The earlier topological discussion determines that f and g have the same sets of itineraries, and the monotonicity of invariant coordinates implies that points with given itineraries occur in the same order on the interval. The analytic part of the argument which uses the Schwarzian condition is that each itinerary is assumed by just one point. There are no non-trivial intervals J for which $f^n|J$ is monotone for all n. (Misiurewicz has suggested the name *homterval* for intervals with this property.) A topological equivalence from f to g is then constructed by associating points with the same itinerary.

The kneading sequences of $f \in \mathscr{C}$ which do have stable periodic orbits are of three types. If $\underline{\gamma}$ is the kneading sequence of such an f, then $\sigma(\underline{\gamma})$ is periodic of period n. If γ_n is C and g has the same kneading sequence as f, then f and g are topologically equivalent. If $\gamma_n \neq C$, then the possibilities depend upon the number k of I_1's among $\gamma_1, \ldots, \gamma_n$. If k is even, then the stable periodic orbit of f has period n and is stable either from one side or from both sides. This additional bit of information determines the topological equivalence class of f. If k is odd, then the stable periodic orbit of f has period n or $2n$, and this bit of information determines its topological equivalence class.

Throughout Sect. 2, for $x \in I - \{c\}$ we have occasion to look at the point $y \neq x$ such that $f(y) = f(x)$. The point y is well defined since f is 2 to 1 on $I - \{c\}$. We denote this point y by x'. If $S \subset I - \{c\}$, then we write S' for the set $\{y \in I |$ there is $x \in S$ with $y \neq x$ and $f(y) = f(x)\}$. One uses this map "$'$" in the following way. If $y \in (x, x')$, one says that y is *closer to c* than x. We study the iterates of x, looking for ones which are closer to c. The *induced map F* of f is defined by $F(x) = f^n(x)$ with n the smallest integer such that $f^n(x) \in (x, x')$. The induced map is not defined at all points and it is discontinuous, but it has good expanding properties if $f \in \mathscr{C}$. This is the basis of the analytic arguments in Sect. 2.

In Sect. 3, we determine which $f \in \mathscr{C}$ have sensitivity to initial conditions. As we have already mentioned, if $f \in \mathscr{C}$ without stable periodic orbits, some have sensitivity to initial conditions and some do not. To understand this, we examine periodic points of f. If p is a periodic point of period n, then we say p is *central* if f^n is monotone on the interval $[p, c]$ and $Df^n(p) > 0$. The central point p is *restrictive* if $f^n(c) \in (p, p')$. This means that f^n maps the interval (p, p') into itself. The condition for $f \in \mathscr{C}$ without a stable periodic orbit to have sensitivity to initial conditions is that there is an N such that f has no restrictive central points of period larger than N. This requirement can be expressed in terms of the kneading sequence $\underline{\gamma}$ of f. This condition is closely related to the decomposition of the non-wandering set of f into "basic" sets by Jonker and Rand [15]. Those $f \in \mathscr{C}$ without stable periodic orbits and a non wandering set with a finite number of basic sets have sensitivity to initial conditions. The $f \in \mathscr{C}$ which have an infinite number of basic sets have neither stable periodic orbits nor sensitivity to initial conditions. If $f \in \mathscr{C}$ has no stable periodic orbit and is not sensitive to initial conditions, then there is a Cantor set Λ such that almost all points $x \in I$ have orbits which are asymptotic to Λ and $f|\Lambda$ is a homeomorphism. The set Λ has partitions $\Lambda = \bigcup_{i=1}^{n} \Lambda_i$ such that f permutes the sets Λ_i and Λ_i have diameter as small as one pleases. A well studied example of such maps are f with topological entropy 0 and infinitely many periodic orbits [18].

The final section studies the relationship between sensitivity to initial conditions, topological entropy, and conjugacy to piecewise-linear maps of constant slope. The maps $g_\mu(x)=\mu/2-\mu|x-1/2|$ have the properties that $g(0)=g(1)=0$ and $|g'_\mu(x)|=\mu$ for $\mu \neq 1/2$. These maps are said to have constant slope. Piecewise linear maps with constant slope larger than 1 have a special property in terms of ergodic theory, namely, they have invariant measures of maximal entropy which are absolutely continuous with respect to Lebesgue measure. Not all maps $f\in\mathscr{C}$ are topologically equivalent to a piecewise linear map g_μ. We determine those which are.

The topological entropy and growth numbers of the maps we study are equivalent [19]. The growth number λ of $f\in\mathscr{C}$ is $\limsup (N_k)^{1/k}$ where N_k is the number of fixed points of f^k. If $h(f)$ is the topological entropy of f, then $h(f)=\log\lambda$. So we can work with growth numbers as easily as topological entropy. If $1<\mu\leqq 2$, the growth number of g_μ is μ. If $\sqrt{2}<\mu\leqq 2$, then g_μ has no restrictive central points in the interior of I. Thus if f has growth number larger than $\sqrt{2}$ and it does have restrictive central points, then f is not topologically equivalent to any g_μ. We prove a converse of this statement. If $f\in\mathscr{C}$ has no restrictive central points then the growth number of f is larger than $\sqrt{2}$ and f is topologically equivalent to g_μ with μ the growth number of f.

The next piece of this story has to do with the topological entropy of f and restrictive central points. We prove that if f has growth number larger than $\sqrt{2}$ and a restrictive central point p which is not stable from just one side, then the measure of maximal entropy for f is supported in $I-\bigcup_{i=0}^{n-1} f^i(J)$ where $J=(p,p')$ and p has period n. This implies that if $g\in\mathscr{C}$ is close to f, then g and f have the same topological entropy and the same growth numbers. Thus, most of the maps $f\in\mathscr{C}$ for which the function $\lambda(f)$ is not locally constant and $\lambda(f)>\sqrt{2}$ are topologically equivalent to piecewise linear g_μ. These f have sensitivity to initial conditions. Indeed, if f has sensitivity to initial conditions, then there is an $n>0$ and a subinterval $J=[p,p']$, p the closest restrictive central point to c, such that $f^n(J)\subset J$ and $f^n|J$ is topologically equivalent to a piecewise linear g_μ. Thus sensitivity to initial conditions for $f\in\mathscr{C}$ is related to both topological entropy and topological equivalence to piecewise linear maps with constant slope.

Consider a one parameter family $f_a\in\mathscr{C}$ in which the growth number varies, say from 1 to 2. We are interested in studying the set S of a for which f_a has sensitivity to initial conditions. We can write $S=\bigcup_{n=0}^{\infty} S_n$ where S_n is the set of a for which the closest restrictive central point p_a of f_a has period n. Then for $a\in S_n$, $f_a^n|(p_a,p'_a)$ is topologically equivalent to a piecewise linear g_μ of constant slope. If the kneading sequence of $f_a^n|(p_a,p'_a)$ does not remain constant for $a\in S_n$, then the growth number of $f_a^n|(p_a,p'_a)$ is not locally constant at a. This suggests that the measure of the set S will be positive for all families f_a if and only if the measure of the set $\tilde{S}=\{a|\lambda(f_a)$ not locally constant at $a\}$ is always positive.

2. The Topological Classification

In this section we give a topological classification of one dimensional maps with negative Schwarzian derivative and one critical point. There are two aspects to this classification, one topological and one analytic. The topological part of the theory can be applied to all continuous maps with one critical point while the analytical part relies strongly upon the Schwarzian condition to prove statements about the size of certain derivatives of f. While much of this theory can be generalized to a class of maps with more than one critical point, we focus here on the one critical point case.

The language we use in constructing our topological classification is that of the kneading sequences of Milnor and Thurston [17]. This terminology has been introduced in the previous section. Given two maps f and g, we want to determine whether f and g are topologically equivalent. A topological equivalence h with $hf = gh$ must map the critical point of f to the critical point of g. Therefore, h maps the orbit of the critical point of f to the orbit of the critical point of g in an order preserving way. This implies immediately that f and g have the same kneading sequences if they are topologically equivalent. The first part of our classification will determine the set of kneading sequences which do occur for maps $f \in \mathscr{C}$.

The strategy of our proof will be to try to determine the extent to which the kneading sequence of $f \in \mathscr{C}$ determines its topological equivalence class. We find that this sometimes occurs, but sometimes the kneading sequence does not quite determine the topological equivalence class of a map. The dichotomy here is roughly between maps which have stable periodic orbits and those which do not. A nonsingular "periodic" kneading sequence occurs for two topological equivalence classes, while all the maps $f \in \mathscr{C}$ with a given "aperiodic" kneading sequence are topologically equivalent to one another. The proofs of both these statements rely upon a determination of the itineraries which occur for a given map from its kneading sequence. This is done simultaneously with the determination of the possible kneading sequences in the topological part of the argument. In the periodic case, the points whose orbits tend to the stable period orbit form an open and dense set whose complement is topologically equivalent to a subshift of finite type. This can be analyzed quite explicitly. In the aperiodic case, we prove that the set $\{x | f^n(x) = c \text{ for some } n > 0\}$ is dense. Using the monotonicity of the invariant coordinate, any topological equivalence defined on this set extends to a topological equivalence on all of I.

Let us then begin our argument with an easy lemma.

Lemma 2.1. *Let $f: I \to I$ have the single critical point c and assume that $f(0) = f(1) = 0$. Then $x \in I$ implies that $\theta(x) \leqq \theta(f(c))$. (Here $\theta(x)$ is the invariant coordinate of x and the order is the lexicographic order of invariant coordinates as explained in Sect. 1.)*

Proof. The point $f(c)$ is the maximum value of f. Thus $x \leqq f(c)$ and the lemma follows immediately from the monotonicity of the invariant coordinate.

Corollary 2.2. *If $\underline{a}$ is the kneading sequence of a map f and σ is the shift map on sequences, then $\theta(\sigma^i(\underline{a})) \leqq \theta(\sigma(\underline{a}))$ for all $i \geqq 0$. If $\underline{b}$ is the itinerary of any point $x \in I$ for f, then $\theta(\sigma^i(\underline{b})) \leqq \theta(\sigma(\underline{a}))$ for all $i > 0$.*

The topological part of our discussion centers on the extent to which converses to this corollary are true. The first statement almost characterizes the f-itineraries of points $x \in I$.

Proposition 2.3. *Let $f: I \to I$ satisfy $f(0)=f(1)=0$, f is C^1, and f has a single turning point c. Denote the itinerary of $f(c)$ by $\underline{\gamma}$. If $\underline{a}$ is a sequence of I_0, I_1 with the property that $\theta(\sigma^i(\underline{a})) < \theta(\underline{\gamma})$ for all $i > 0$, then there is a point $x \in I$ such that $\underline{a}$ is the itinerary $\underline{A}(x)$.*

Proof. We deal separately with the cases in which c is periodic and c is not periodic. Assume first that there is no $n > 0$ with $f^n(c) = c$. Then the sequence $\underline{\gamma}$ consists entirely of the symbols I_0, and I_1. We shall examine the sets $L = \{x | \theta(x) < \theta(\underline{a})\}$ and $U = \{x | \theta(x) > \theta(\underline{a})\}$. We assert that each of these sets is non-empty and open unless $\underline{a}$ is the itinerary of an endpoint. Then the connectedness of I implies that there is $y \in I - \{L \cup U\}$. We must have $\underline{A}(y) = \underline{a}$. If $\underline{a}$ is one of the sequences $\{I_0, I_0, \ldots, I_0, \ldots\}$ or $\{I_1, I_0, I_0, \ldots, I_0, \ldots\}$, then $A(0) = \underline{a}$ or $A(1) = \underline{a}$. Otherwise $0 \in L$ and $1 \in U$.

Assume $x \in L$. Then there is a smallest $n > 0$ with $A_n(x) \neq a_n$ since $\theta(x) < \theta(\underline{a})$. If $a_n = I_1$, then $A_n(x) = I_0$ or C while if $a_n = I_0$, then $A_n(x) = C$ or I_1. If $A_n(x) \neq C$, then there is a neighborhood V of x with $A_i(y) = A_i(x)$ for $i \leq n$ and $y \in V$. This implies that $V \subset L$ and x is an interior point. Let $m \geq 0$ be the smallest integer with $\sigma^{n+1}(a)_m \neq \gamma_m$. If $A_n(x) = C$, then $f^n(x) = c$ and $m+n+1 = j$ is the smallest integer larger than n with $A_i(x) \neq a_i$. Since $f^{m+1}(c) \neq c$, we have $A_j(x) \neq C$. There is a neighborhood V of x such that $y \in V$ implies $A_i(y) = A_i(x)$ for all $i \leq j$ except $i = n$. We then have $V \subset L$. We conclude that L is open. A similar argument establishes that U is open, proving the proposition when c is not periodic.

Assume now that c is periodic with period n. Since f is C^1 and $f'(c) = 0$, the orbit of c is stable. Therefore, there is a neighborhood V of c such that $f^n(V) \subset V$. If V is small enough, then $y \in V$ implies that $A_i(y) = \gamma_{i-1}$ for all $i > 0$ with $i \not\equiv 0 \pmod{n}$. One also has $A_{in}(y) = A_n(y)$ for all $i > 0$. Thus points $y \in V - \{c\}$ have one of two itineraries $\underline{\alpha}$ or $\underline{\beta}$ depending upon $A_0(y)$. It is easily checked that these two itineraries $\underline{\alpha}$ and $\underline{\beta}$ satisfy the hypotheses for $\underline{a}$ in the proposition. We also assert that there are no itineraries $\underline{a}$ between $\underline{\alpha}$ and $\underline{\beta}$ which satisfy the hypotheses of the theorem. Indeed, the only itineraries between $\underline{\alpha}$ and $\underline{\beta}$ begin with the symbol C.

Suppose now that $\underline{a}$ is a sequence of I_0 and I_1 such that $\theta(\sigma^i(\underline{a})) < \theta(\underline{\gamma})$ for all $i > 0$. Once again denote by $L = \{x | \theta(x) < \theta(\underline{a})\}$ and $U = \{x | \theta(x) > \theta(\underline{a})\}$. Let $x \in L$, and let n be the smallest integer such that $A_n(x) \neq a_n$. If $A_n(x) \neq C$, then x has a neighborhood V such that $y \in V$ implies $A_i(y) = A_i(x)$ for all $i \leq n$. The set V is in L. If $A_n(x) = C$, then it may happen that $\sigma^n(\underline{a})$ is one of the itineraries $\underline{\alpha}$ or $\underline{\beta}$. In this case, x is an endpoint of L, but there is an interval V with left endpoint x such that $y \in V$ implies $A_i(y) = A_i(x) = a_i$ for $i < n$, and $f^n(x)$ has itinerary $\underline{\alpha}$ or $\underline{\beta}: A(f^n(x)) = \sigma^n(\underline{a})$. Then points of V have $\underline{a}$ as itinerary and the proposition is proved. If $\sigma^n(\underline{a})$ is not one of the itineraries $\underline{\alpha}$ or $\underline{\beta}$, then x has a neighborhood V such that $y \in V$ implies $A_i(y) = A_i(x)$ for $i < n$, and $f^n(y)$ has itinerary $\underline{\alpha}$ or $\underline{\beta}$ or $f^n(y) = c$. Since $\theta(\sigma^n(\underline{a}))$ cannot lie between $\theta(\underline{\alpha})$ and $\theta(\underline{\beta})$, all points of V must belong to L. A similar argument implies that either U is open or immediately to the left of U there are points whose itinerary is $\underline{a}$. Thus we have one of three possibilities, but in each we find a $y \in I$ whose itinerary is $\underline{a}$. This finishes the proof of the proposition.

Example. The hypothesis that f is C^1 is not used in the case when the critical point does not lie in a periodic orbit. If c is periodic, then some hypothesis is necessary. Consider the function $g_\mu(x)=\mu/2-\mu|x-\frac{1}{2}|$. If we take $\mu=(1+\sqrt{5})/2$, then $g_\mu(\frac{1}{2})=(1+\sqrt{5})/4$, $g((1+\sqrt{5})/4)=(-1+\sqrt{5})/4$, and $g((-1+\sqrt{5})/4)=\frac{1}{2}$. Thus $\frac{1}{2}$ is periodic with period 3. If x is near $\frac{1}{2}$ then $g^3(x)\geqq\frac{1}{2}$. For any n, the points immediately to the right of $\frac{1}{2}$ have itineraries with $A_i(x)=I_0$ or I_1 as $i\equiv 2\,(\mathrm{mod}\,3)$ or $i\equiv 0,1\,(\mathrm{mod}\,3)$ and $i\leqq n$. Since the slope of g^n is $\pm\left(\frac{1+\sqrt{5}}{2}\right)^n$ except at the turning points, any two distinct points eventually lie in different laps. In particular, for any $x\neq\frac{1}{2}$, there will be an n such that g^n has a turning point between $\frac{1}{2}$ and x. Thus x will not have the itinerary $\underline{a}=I_1I_1I_0I_1I_1I_0\ldots$.

The sequence a is an itinerary which Proposition 2.3 says should exist, as we now show. The itinerary of $\frac{1}{2}=c$ is $CI_1I_0CI_1I_0\ldots$. A check of the hypothesis of Proposition 2.3 shows that for $i\equiv 1,2,0\,(\mathrm{mod}\,3)$ we do have $\theta(\sigma^i(\underline{a}))<\theta(\underline{\gamma})$ where $\gamma=I_1I_0CI_1I_0C\ldots$. Thus Proposition 2.3 is not valid for the map g. It follows that g is a map with the property that it is not topologically equivalent to any smooth function $f:I\to I$. In Sect. 4 we shall find smooth maps which are not topologically equivalent to any of the piecewise linear maps $g_\mu(x)=\mu/2-\mu|x-\frac{1}{2}|$. This question of topological equivalence to g is closely related to the question of sensitive dependence to initial conditions.

Proposition 2.3 does not yet characterize all of the itineraries of a map $f\in\mathscr{C}$. There are two types of itineraries which must be described: those which contain the symbol C, and those itineraries $\underline{a}$ for which there is an n with $\sigma^n(\underline{a})=\underline{\gamma}$. The itineraries $\underline{a}$ containing C must have $\sigma^{n+1}(\underline{a})=\underline{\gamma}$ if $a_n=C$. The hypotheses of Proposition 2.3 give sufficient criteria for $\underline{a}$ to exist: if $a_n=C$, $\theta(\sigma^i(\underline{a}))<\theta(\underline{\gamma})$ for $i<n$, and $\sigma^{n+1}(\underline{a})=\gamma$, then there is an $x\in I$ with $\underline{A}(x)=\underline{a}$. The proof follows the argument of Proposition 2.3. The question of whether there are itineraries for which $\sigma(\underline{a})=\underline{\gamma}$ but $a_0\neq C$ is more delicate. For functions $f\in\mathscr{C}$ we shall prove that such itineraries exist if and only if f has a stable periodic orbit. Some analytic condition is necessary to decide whether or not such orbits exist from the kneading sequence of a map. Before turning to these analytic questions, we want to state a proposition similar to 2.3 which determines the set of kneading sequences which do occur for $f\in\mathscr{C}$.

Theorem 2.4. *Let $\underline{\gamma}$ be a sequence with the properties that*
1) $\theta(\sigma^i(\underline{\gamma}))\leqq\theta(\underline{\gamma})$ *for all* $i>0$,
2) *If* $\gamma_j=C$, *then* $\gamma_{j+k+1}=\gamma_k$ *for all* $k\geqq 0$.

Then there is a map $f\in\mathscr{C}$ such that $\underline{\gamma}$ is the itinerary of $f(c)$.

Proof. The proof of Theorem 2.4 is similar in spirit to that of Proposition 2.3. Consider a continuous one parameter family $f_\mu\in\mathscr{C}$, $\mu\in[0,1]$, such that $f_0(c)<c$ and $f_1(c)=1$. Denote by $\underline{\beta}^\mu$ the itinerary of $f_\mu(c)$. We seek μ with $\underline{\beta}^\mu=\underline{\gamma}$. For $\mu=0$, we have $\beta_i^\mu=0$, and for $\mu=1$, we have $\beta_0^\mu=1$, $\beta_i^\mu=0$ when $i>0$. Denote by $L=\{\mu|\theta(\underline{\beta}^\mu)\leqq\underline{\gamma}\}$. The set L is non-empty. Let $\nu=\sup_{\mu\in L}\mu$. We want to see if $\underline{\beta}^\nu=\underline{\gamma}$. If $\underline{\beta}^\nu\neq\underline{\gamma}$, there is a smallest $n>0$ such that $\beta_n^\nu\neq\gamma_n$. If $\beta_i\neq C$ for any $i\leqq n$, there is a neighborhood V of $\nu\in[0,1]$ such that $\mu\in V$ implies that $\beta_i^\mu=\beta_i^\nu$ for all $i\neq n$. If $\theta(\underline{\beta}^\nu)$

$>\theta(\underline{\gamma})$, then $V \cap L = \emptyset$. If $\theta(\underline{\beta}^\nu) < \theta(\underline{\gamma})$, then $V \subset L$. In either case, ν is not in the boundary of L contrary to assumption. Therefore, if $\beta_n^\nu \neq \gamma_n$, there is an $j \leqq n$ with $\beta_j^\nu = C$.

Now $\underline{\beta}^\nu$ is the itinerary of $f_\nu(c)$ for f_ν, so $f_\nu^{j+1}(c) = c$. This implies that $\beta_{k+j+1}^\nu = \beta_k^\nu$ for all $k \geqq 0$. Comparison with hypothesis 2) of the theorem reveals that $j < n$ implies that $\underline{\gamma} = \underline{\beta}^\nu$, contrary to assumption. Therefore $j = n$. This allows us to determine $\underline{\gamma}$. The itinerary $\underline{\beta}^\nu$ is periodic with period $n+1$ and $\beta_i^\nu = C$ whenever $i \equiv n \pmod{n+1}$. Now f_ν is differentiable, so the orbit of c is a stable periodic orbit. For μ sufficiently close to ν, f_μ has a stable periodic orbit of period $n+1$ containing a point p_μ near c. For μ close enough to c, we have $|f_\mu^{n+1}(c) - p_\mu| < |c - p_\mu|$ and $|f_\mu^{2(n+1)}(c) - p_\mu| < |c - p_\mu|$. These inequalities imply that the itinerary $\underline{\beta}^\mu$ is still periodic of period $(n+1)$, and it satisfies $\beta_i^\mu = \beta_i^\nu$ for $i \equiv n \pmod{n+1}$. Now one of the itineraries $\underline{\beta}^\mu$ is larger than $\underline{\beta}^\nu$ and one is smaller. Therefore, one of the $\underline{\beta}^\mu$ lies in L and one does not. The $\underline{\beta}^{\mu_1}$ which lies in L satisfies $\theta(\underline{\beta}^{\mu_1}) \leqq \theta(\underline{\gamma})$ and the $\underline{\beta}^{\mu_2}$ which does not lie in L satisfies $\theta(\underline{\gamma}) < \theta(\underline{\beta}^{\mu_2})$. Now $\underline{\beta}^\nu$ is the only sequence $\underline{\delta}$ which satisfies 1) and 2) and $\theta(\underline{\beta}^{\mu_1}) < \theta(\underline{\delta}) < \theta(\underline{\beta}^{\mu_2})$. Therefore, we must have $\underline{\gamma} = \underline{\beta}^{\mu_1}$ since $\underline{\gamma} \neq \underline{\beta}^\nu$ and $\underline{\gamma}$ satisfies 1) and 2). But then there is $\mu_1 = \mu$ with $\underline{\beta}^{\mu_1} = \underline{\gamma}$ as was to be proved.

With Lemma 2.1, the proof of Theorem 2.4 clearly implies the following result which is stronger than the statement of 2.4:

Corollary 2.5. *Let $f_\mu \in \mathscr{C}$, $\mu \in [0,1]$ be a continuous one parameter family with $f_0(c) < c$ and $f_1(c) = 1$. If $g: I \to I$ is a continuous map with a single turning point, and $g(0) = g(1) = 0$, then there is a $\mu \in [0,1]$ such that f_μ and g have the same kneading sequences. For every sequency $\underline{\gamma}$ satisfying hypotheses 1) and 2) of Theorem 2.4, there is a μ such that f_μ has kneading sequence $\underline{\delta}$ and $\underline{\gamma} = \sigma(\underline{\delta})$.*

Let us turn now to the analytic investigation of maps $f \in \mathscr{C}$. Our goal is to prove that there at most two topological equivalence classes among the maps with a given kneading sequence. If the kneading sequence is "periodic", then there are two topological equivalence classes; otherwise, there is just one. The negative Schwarzian condition plays a dual role in that it restricts f to have at most one periodic orbit, and it prevents non-trivial intervals from existing on which f^n is a homeomorphism for all n. It is possible that a larger class of maps than $\mathscr{C}$ satisfies this last property (for instance, the negative Schwarzian condition might be replaced merely by $f \in C^2$).

Theorem 2.6. *Let $f \in \mathscr{C}$. If there is a non-trivial interval J such that $f^n|J$ is a homeomorphism for all n, then f has a stable periodic orbit γ.*

The proof of this theorem is quite long, so we isolate several steps as lemmas. Assume that J is a maximal interval such that $f^n|J$ is a homeomorphism and that f does not have a stable periodic orbit.

Lemma A. *For all $m \neq n$, $f^m(J) \cap f^n(J)$ has no interior.*

Proof. If $f^m(J) \cap f^n(J)$ has non-empty interior, then for each $k > 0$, f^k is a strictly monotone function on the interval $f^m(J) \cup f^n(J)$. If $m > n$ and $x \in f^m(J) \cap f^n(J)$, then $f^{m-n}(x) \in f^{m+(m-n)}(J) \cap f^m(J)$. Inductively, $f^{l(m-n)+n}(J) \cap f^{(l-1)(m-n)+n}(J)$ has non-empty interior. Denote by K the set $\bigcup_{l \geqq 0} f^{l(m-n)+n}(J)$. Then K is an interval since it is connected. Moreover, $f^k|K$ is strictly monotone for each $k > 0$ and $f^{m-n}(K) \subset K$.

Therefore f^{m-n} is a homeomorphism of K into itself, and f^{m-n} must have a stable fixed point in the closure of K. This proves Lemma A.

Now denote by $K_n = L_n \cup J \cup R_n$ the largest interval containing J on which f^n is a homeomorphism. We denote the length of an interval K by $l(K)$.

Lemma B. *If* $k \leqq n$, *then* $\dfrac{l(f^k(J))}{l(f^k(L_n))} \geqq \dfrac{l(J)}{l(L_n)}$ *or* $\dfrac{l(f^k(J))}{l(f^k(R_n))} \geqq \dfrac{l(J)}{l(R_n)}$.

Proof. This is a direct consequence of the negative Schwarzian derivative of f; the map f^k is a homeomorphism of K_n with negative Schwarzian derivative. Therefore $|Df^k|$ has no local minimum in the interior of K_n. If $J=(a,b)$ and $|Df^k(a)| \lesseqgtr |Df^k(b)|$, then $|Df^k(x)| > |Df^k(y)|$ for all $x \in J$ and $y \in \{^{L_n}_{R_n}$. Integrating this inequality over J and either L_n or R_n gives the conclusion of Lemma B.

The strategy of the remaining part of the proof will be to find bounds on the lengths of some of the images of J. We shall find a constant $\alpha > 0$ and a sequence $k_i \to \infty$ with $l(f^{k_i}(J)) > \alpha$. Since Lemma A implies that the $f^{k_i}(J)$ have disjoint interiors, this contradicts the fact that I has finite length and suffices to prove the theorem. We make one final reduction before proceeding further. If $f^n(J)$ has c as an endpoint for some $n \geqq 0$, then we replace J by $f^{n+m}(J)$ for any $m > 0$. Thus we may assume that c is not an endpoint of any $f^n(J)$. With this reduction, we split the remainder of the argument into two cases. In the first there is an $n \geqq 0$, such that $f^n(J)$ is closer to c than all other $f^i(J)$. In the second, for each n, there is an $i > n$ such that $f^i(J)$ is closer to c than $f^n(J)$.

Lemma C. *Suppose that* $n > 0$ *has the property that* $i \neq n$ *and* $x \in f^i(J)$ *implies* $f^n(J) \subset (x, x')$. *Assume that* c *is not in the closure of* $f^n(J)$. *Then there is an* $\alpha > 0$ *such that* $l(f^i(J)) > \alpha$ *for all* $i \geqq 0$.

Proof. We may assume that $n=0$ by replacing J with $f^n(J)$. This may change α, but if the lemma is true with $f^n(J)$ in place of J, then it is true for J. Now we have assumed that c is not an endpoint of J, so there is a point ξ between J and c and $k > 0$ with $f^k(\xi) = c$. Denote by J' the interval on the opposite side of c from J with $f(J) = f(J')$ and by M and M' the intervals joining J and J' to c. If $f^n(L_n)$ or $f^n(R_n)$ has c as an endpoint, then it contains M or M' and hence ξ or ξ'. Therefore $c \in f^{n+k}(L_n)$ if $f^n(L_n)$ has c as an endpoint, and similarly for R_n. There is an N such that K_N does not have 0 or 1 as an endpoint because the images of J are all distinct. For $n \geqq N$, both endpoints of K_{n+k+1} are in the interior of K_n. It follows that the endpoints of $f^n(L_n)$, $f^n(R_n)$ must be one of the points $f^i(c)$, $1 \leqq i \leqq k$ when $n \geqq N$. When n is chosen so that c is contained in $f^n(L_n)$ or $f^n(R_n)$ then this interval contains M or M'. Therefore, when $n \geqq N$, $f^n(L_n)$ and $f^n(R_n)$ each contain one of the intervals $f^i(M)$, $1 \leqq i \leqq k$. Define $\beta = \min l(S)$ with S one of the intervals $f^i(M)$, $1 \leqq i \leqq k$, $f^i(L_i)$, $i \leqq N$, or $f^i(R_i)$, $i \leqq N$. Then β is a lower bound for $l(f^n(L_n))$ and $l(f^n(R_n))$. Using Lemma B and noting that $l(L_{n+1}) \leqq l(L_n)$, $l(R_{n+1}) \leqq l(R_n)$, we find a positive lower bound α for $l(f^n(J))$.

Assume now that for each N, there is an $n > N$ such that $f^n(J)$ is closer to c than $f(J)$ for $l < n$. We want to find a sequence k_n such that the lengths of the $f^{k_n}(J)$ are bounded away from zero. As we noted above, this will finish the proof of the theorem. We shall find the sequence k_n inductively, starting with $k_0 = 0$. Suppose that $k_1, \ldots, k_{n-1}$ have been chosen so that k_i is closer to c than $f^l(J)$ for $l < k_i$.

Denote by $0<a=\min_{f(x)=f(y)} \frac{|f'(x)|}{|f'(y)|}$ and $\beta=\min_{0\leq i\leq n} l(f^{k_i}(J))$. Suppose further that if $K_m=L_m\cup J\cup R_m$ is the maximal interval containing J on which f^m is monotone and $m>k_{n-1}$, then $\frac{l(L_m)}{l(J)}<a^2$ and $\frac{l(R_m)}{l(J)}<a^2$. With this notation, we have the final lemma:

Lemma D. *If k_n is the smallest integer l such that $f^l(J)$ is closer to c than $f^{k_{n-1}}(J)$, then $(f^{k_n}(J))>\beta$.*

The first assertion is that if k_{n+1} is the smallest integer such that $f^{k_{n+1}}(J)$ is closer to c than $f^{k_n}(J)$, then $l(f^{k_{n+1}}(J))>al(f^{k_n}(J))$. To prove this assertion, denote by S the set $\{x|f^i(x)\notin(x,x')$ for $i<k_{n+1}-k_n\}$. Then $f^{k_n}(J)\subset S$, but the ends of the maximal interval containing $f^{k_n}(J)$ on which $f^{k_{n+1}-k_n}$ is monotone are not in S. The component of S containing J has endpoints which each satisfy one of the equations $f^{k_{n+1}-k_n}(x)=x$ or x'. At a solution of $f^{k_{n+1}-k_n}(x)=x$ we have $|Df^{k_{n+1}-k_n}(x)|>1$ and at a solution of $f^{k_{n+1}-k_n}(x)=x'$ we have $|Df^{k_{n+1}-k_n}(x)|>a$ since f has no stable periodic orbits. Now S contains no critical points of $f^{k_{n+1}-k_n}$, so the negative Schwarzian condition implies $|Df^{k_{n+1}-k_n}(x)|>a$ for all $x\in f^{k_n}(J)$. Hence $l(f^{k_{n+1}}(J))>al(f^{k_n}(J))$.

Denote $m=k_n$ and $f^m(K_m)=(\xi,\eta)$. Our next assertion is that $c\in f^m(K_m)$. If $c\notin f^m(K_m)$, then either $f^m(R_m)$ or $f^m(L_m)$ is between $f^m(J)$ and c, say $f^m(R_m)$. There is an $i<m$ with $f^i(\eta)=c$. Since $f^i(R_m)$ joins c to $f^i(J)$, and $f^i(J)$ is farther from c than $f^m(J)$, we have $f^m(R_m)\subset f^i(R_m)$ or $f^m(R_m)\subset(f^i(R_m))'$. Now $f^{m-i}f^i(R_m)$ is monotone, so f^{m-i} has a stable fixed point in $f^i(R_m)$ or $(f^i(R_m))'$. This contradicts the assumption that f has no stable periodic orbit, so $c\in f^m(R_m)$ or $c\in f^m(L_m)$.

Assume now that $c\in f^m(R_m)$. Then we assert that $\frac{l(f^m(R_m))}{l(f^m(J))}>\frac{l(R_m)}{l(J)}$. Since $c\in f^m(R_m)$, $f^m(R_m)$ contains $f^{k_{n+1}}(J)$ or $(f^{k_{n+1}}(J))'$ where k_{n+1} is the smallest integer with $f^{k_{n+1}}(J)$ closer to c than $f^{k_n}(J)$. The estimate of the first paragraph gives $l(f^{k_{n+1}}(J))>al(f^m(J))$. Therefore $l(f^m(R_m))>a^2l(f^m(J))$. Since we assumed that $a^2>\frac{l(R_m)}{l(J)}$, we have $\frac{l(f^m(R_m))}{l(f^m(J))}>\frac{l(R_m)}{l(J)}$[1].

Still assuming that $c\in f^m(R_m)$, the last paragraph and Lemma B imply that $\frac{l(f^m(L_m))}{l(f^m(J))}<\frac{l(L_m)}{l(J)}$. We assert that $f^m(L_m)$ contains $f^{k_{n-1}}(J)$ or $(f^{k_{n-1}}(J))'$. This assertion will prove the lemma, since $l(f^{k_{n+1}}(J))\geqq r$ then implies that $l(f^m(L_m))>ra$. Together with $\frac{l(L_m)}{l(J)}<a^2$, we obtain $l(f^m(J))>a^{-1}r\geqq r$, as was to be proved. So we finish the proof of Lemma D by showing that $f^m(L_m)$ contains $f^{k_{n-1}}(J)$ or $(f^{k_{n-1}}(J))'$. With ξ the left endpoint of L_m, there is an $i<m$ with $f^i(\xi)=c$. If M is the interval joining $f^{k_{n-1}}(J)$ to c, then $f^i(L_m)$ contains M or M' because $f^i(J)$ is at least as far from c as $f^{k_{n-1}}(J)$. Now $f^{m-i}|f^i(L_m)$ is monotone and $f^m(J)\subset M$ or M'. Therefore, if $f^m(\xi)\in M$, then f^{m-i} has a stable fixed point. Moreover, if $f^m(\xi)\in f^{k_{n-1}}(J)$, then f^{m-i} maps the interval $f^{k_{n-1}}(J)\cup M$ monotonely into itself.

1 This step in the argument is essentially due to Misiurewicz.

Similarly, $f^m(\xi)\notin(f^{k_{n-1}}(J))'$. Thus $f^m(\xi)$ is farther from c than $f^{k_{n-1}}(J)$ and $f^m(L_m)$ contains either $f^{k_{n-1}}(J)$ or $(f^{k_{n-1}}(J))'$. This completes the proof of Theorem 2.6.

With Theorem 2.6, we take up the question of topological equivalence of maps in $\mathscr{C}$. Assume that $f\in\mathscr{C}$. Denote by $\underline{\gamma}$ the itinerary of $f(c)$. If f has a stable periodic of period n, and p is the closest point in this stable periodic orbit to c, then f^{2n} is monotone on the interval (p,c) and $f^{2n}(p,c)\subset(p,c)$. It follows that the sequence $\underline{\gamma}$ is periodic with period n. Conversely, suppose that $\underline{\gamma}$ is periodic of period n. Then f^k is monotone on the interval $(f^{n+1}(c),f(c))$ for all $k>0$ since the endpoints of this interval have the same itineraries. If $f^{n+1}(c)\neq f(c)$, then Theorem 2.6 implies that f has a stable periodic orbit. If $f^{n+1}(c)=f(c)$, then c lies in a periodic orbit which is stable, or $f(c)$ lies in an unstable periodic and Singer's theorem implies that f has no stable periodic orbit. We conclude that if $\underline{\gamma}$ is periodic, then f has a stable periodic orbit unless $f(c)$ lies in an unstable periodic orbit. This last possibility occurs if and only if there are itineraries allowed by Proposition 2.3 whose invariant coordinates lie between those of c and $f^n(c)$. Thus, one can determine from the kneading sequence of f whether or not $f\in\mathscr{C}$ has a stable periodic orbit.

Proposition 2.7. *Suppose that $f,g\in\mathscr{C}$ have the same kneading sequence and that f does not have a stable periodic orbit. Then f and g are topologically equivalent.*

Proof. The discussion preceding the proposition implies that g does not have a stable periodic orbit. Theorem 2.6 implies that the sets $\{x|f^n(x)=c \text{ for some } n\geqq 0\}$ and $\{y|g^n(y)=c \text{ for some } n\geqq 0\}$ are each dense in I. Proposition 2.3 and the discussion following its proof imply that the sets of f-itineraries and g-itineraries are the same. [Note that Theorem 2.6 implies that if $x\neq c$, then the f-itineraries of $f(x)$ and $f(c)$ are different. So $x\neq c$ implies that $\theta(x)<\theta(c)$ in Lemma 2.1.] Theorem 2.6 also implies that each f-itinerary and each g-itinerary is assumed by just one point. Thus we can define a bijection $h:I\to I$ by the requirement that the f-itinerary of x is the same as the g-itinerary of $h(x)$. The monotonicity of invariant coordinates implies that h is monotone. Since h is $1-1$ and onto, it is a homeomorphism of I.

We end this section with a discussion of the topological equivalence classes of $f\in\mathscr{C}$ which do have stable periodic orbits. These results were discovered independently by Singer and Wolfe. To begin, we shall state a proposition related to Theorem 2.6 for the wider class of C^2 maps.

Proposition 2.8. *Suppose $f\in C^2(I)$ and that $U\subset I$ is a finite union of open intervals which contains all the critical points of f. If $f|I-U$ has no stable periodic orbits then the set $E_U=\{x\in I|f^n(x)\in I-U\}$ is totally disconnected (i.e., contains no non-trivial intervals).*

Remark. The proposition allows a stable periodic orbit in $I-U$ provided all nearby points asymptotic to the stable orbit have trajectories which contain points of U. Such a periodic orbit is stable from only one side.

Proof. Let δ be the distance from $I-U$ to the set of critical points. Then $\delta>0$ since it is the distance between two disjoint compact sets. Suppose that $J\subset E_U$ is a non-trivial interval which is a component of E_U. We shall denote $K_n\supset J$ a maximal interval with the properties that (1) $f^i(K_n)\cap f^j(K_n)$ has empty interior if $i\neq j$ and $i,j\leqq n$, and (2) $f^i(K_n)\subset I-U$ for $i<n$. Write $K_n=L_n\cup J\cup R_n$ and $K_n=[\xi_n,\eta_n]$. The

points ξ_n, η_n each satisfy one of the equations $f^i(x)=f^j(\xi_n)$, $f^i(x)=f^i(\eta_n)$ or $f^i(x)\in\partial U$ for some $j<i<n$. Note that Lemma A of Theorem 2.6 implies that $f^n(J)\cap f^m(J)$ has no interior for $m\neq n$.

We shall establish an estimate of the "non-linearity" of f^n on the interval K_n similar to that used in the proof of the Denjoy theorem about diffeomorphisms of S^1 with irrational rotation numbers. A similar estimate also appears in Sect. 3. Denote by β the Lipschitz constant of $\log|Df(x)|$ on the set $I-U$. For any $x, y\in K_n$, we have

$$\begin{aligned}\log\left|\frac{Df^n(x)}{Df^n(y)}\right| &= \sum_{i=0}^{n-1}\log|Df(f^i(x))|-\log|Df(f^i(y)| \\ &\leqq \beta\sum_{i=0}^{n-1}|f^i(x)-f^i(y)| \\ &\leqq \beta l(I-U)\leqq\beta\end{aligned}$$

since the intervals $(f^i(x), f^i(y))$ have disjoint interiors and are contained in $I-U$. This estimate implies that there is a constant $\alpha>0$ such that

$$\frac{l(f^i(L_n))}{l(f^i(J))}<\alpha\frac{l(L_n)}{l(J)} \quad\text{and}\quad \frac{l(f^i(R_n))}{l(f^i(J))}<\alpha\frac{l(R_n)}{l(J)} \quad\text{for}\quad i\leqq n\,.$$

We now want to prove that J and U can be chosen so that $f^i(K_n)\cap U\neq\emptyset$ for some large n and $i<n$. First, observe that $E_U=E_V$ if $V=\{x\,|\,\text{there is } y\in U \text{ with } [x,y]\cap E_U=\emptyset\}$. If we enlarge U to V, then all points of the boundary of V lie in E_U. Now each component of V contains a component of U, so V still has finitely many components. It is open since E_U is closed. Thus we may replace U by V without changing E_U or invalidating any of the hypotheses for U. Since U has a finite number of boundary points, there is an $n\geqq 0$ such that $i\geqq n$ implies $f^i(J)$ has no points in the boundary of U. Replacing J by $f^n(J)$ we may assume $n=0$.

Let K be an open interval containing J. Then we assert that $(K-J)\cap E_U\neq\emptyset$. There is $x\in K-E_U$ since J is a component of E_U. So there is an n such that $f^n(x)\in U$. Assume that n is chosen as small as possible. Then there is $y\in K$ such that $f^n(y)$ is in the boundary of U, hence in E_U. Now $f^i(y)$ cannot be in U for $i<n$ since n was chosen to be minimal, so $y\in E_U$. This argument shows that each component of $(K-J)$ intersects E_U. It follows that $l(L_n)\to 0$ and $l(R_n)\to 0$.

Consider now $f^n(K_n)$. Since $f^i(K_n)\cap U=\emptyset$ for $i<n$, and U contains the critical points of f, f^n is monotone on K_n. If $f^n(L_n)\cap U\neq\emptyset$, then we assert that $f^n(L_n)$ contains a component of U. Look at $f^n(\xi_n)$. If ξ_n is periodic of period $m<n$, then ξ_n is in E_U. If $f^i(\xi_n)=f^j(\eta_n)$ with $j<i$, then $f^n(\xi_n)=f^{n-i+j}(\eta_n)\in I-U$. If $f^i(\xi_n)$ is in the boundary of U for $i<n$, then ξ_n is in E_U. In all cases, $f^n(\xi_n)\in I-U$. Since f^n is monotone on L_n, it must contain a whole component of U. Now if n is sufficiently large, the estimate $l(f^n(J))>\alpha^{-1}l(f^n(L_n))\dfrac{l(J)}{l(L_n)}$ gives $l(f^n(J))>1=l(I)$ since $l(L_n)\to 0$ as $n\to\infty$ and $l(f^n(L_n))>\min l(U_i)=\gamma$, U_i a component of U. This is absurd, so $f^n(L_n)\cap U=\emptyset$ for n sufficiently large. Similarly $f^n(R_n)\cap U=\emptyset$ for n sufficiently large. Indeed, we may pick n large enough that $l(f^{n+1}(L_n))<\gamma$ and $l(f^{n+1}(R_n))<\gamma$.

Choose N large enough that $f^N(K_N)\cap U=\emptyset$, $(f^{N+1}(L_N))<\gamma$, and $l(f^{N+1}(R_N))<\gamma$ when $n\geqq N$. If n is large enough, there are $N<i<j\leqq n$ such that $f^i(K_n)\cap f^j(K_n)\neq\emptyset$. Denote $j-i$ by m. We assert that, for all $k\geqq i$, $f^{m+k}(J)$ and $f^k(J)$

lie in the same component of $I-U$. If $m+k \leqq n$ and $i \leqq k$, then $f^{(k-j}(f^i(K_n) \cap f^j(K_n)) \neq \emptyset$ implies that $f^{m+k}(J)$ and $f^k(J)$ lie in the same component of U. Suppose that the assertion is true for all $l<k$ with $m+k>n$. Then $f^{m+k-1}(K_{m+k-1}) \cap U = \emptyset$ and $f^{m+k-1}(K_{m+k-1})$ and $f^{k-1}(K_{m+k-1})$ lie in the same component of U. There is $l<m+k-1$ such that $f^{m+k-1}(K_{m+k-1}) \cap f^l(K_{m+k-1}) \neq \emptyset$. Therefore $f^{m+k}(K_{m+k-1}) \cap f^{l+1}(K_{m+k-1}) \neq \emptyset$. Since $l(f^{m+k}(L_{m+k-1}))<\gamma$ and $l(f^{m+k}(R_{m+k-1}))<\gamma$, all points of $f^{m+k}(K_{m+k-1}) \cap I-U$ lie in the same component of $I-U$. This must be the component of $I-U$ which contains $f^{l+1}(K_{m+k-1})$, which by the inductive assumption is the component of $I-U$ which contains $f^k(K_{m+k-1})$. Therefore $f^{m+k}(J)$ and $f^k(J)$ lie in the same component of $I-U$.

Now $K=\{x \mid f^k(x)$ lies in the same component of U as $f^{i+k}(J)$ for $0 \leqq k \leqq m = j-i\}$ is an interval containing J since f is monotone on each component of $I-U$. Moreover $f^m|K$ is monotone. In the last paragraph, we proved that $f^{i+km}(J) \subset K$ for all $k>0$. If $f^m|K$ is increasing, then $f^{i+km+m}(J)$ lies to the right of $f^{i+km}(J)$. If $f^m|K$ is decreasing, $f^{i+km+m}(J)$ lies to the left of $f^{i+km}(J)$. In either case, there is a point $x \in K$ which is a limit point of $f^{i+km}(J)$ as $k \to \infty$. Then $f^m(x)=x$ and x is a stable fixed point of f^m. Moreover $f^k(x) \in I-U$ for all $k \geqq 0$, contradicting the fact that f has no stable periodic orbits approached by points in $I-U$. This proves the proposition.

Let us reconsider $f \in \mathscr{C}$ with a stable periodic orbit. Singer's theorem implies that there is an interval U containing c such that all $x \in U$ have orbits which tend to the stable periodic orbit, and U contains a point in the periodic orbit. Note that if the periodic orbit attracts from only one side, then we must take U closed, a situation which we discuss in more detail below. Proposition 2.8 implies that if J is a non-trivial interval on which f^n is monotone for all n, then there is an n with $f^n(J) \subset U$. This implies that points of J tend to the stable periodic orbit of f. If $E_f = \{x \mid f^n(x)$ does not tend to the stable periodic orbit of $f\}$, then E_f is totally disconnected.

Let $U \subset I$ be the maximal interval containing c and consisting of points whose orbits tend to the stable periodic orbit of f. If this periodic orbit is stable from both sides, then U is an open interval (a, a') and one of the end points of U is periodic. Let n be the smallest integer with $f^n(U) \cap U \neq \emptyset$. Then $f^n(U) \subset U$, $f^n(a)=a$ or a', $f^n|U$ is a map with one critical point [$x \in U$ implies $f^i(x) \neq c$ for $i<n$] and all orbits of $f^n|U$ tend to a stable periodic orbit. Since f^n has a fixed point p in U, this must be the stable periodic orbit. There are different kneading sequences in the cases when $Df^n(p)$ is positive, zero, and negative.

Lemma 2.9. *Let $f, g \in \mathscr{C}$ have stable fixed points p_f and p_g. Assume that if x is in the interior of I, then $f^n(x) \to p_f$ and $g^n(x) \to p_g$. Assume further that $f'(p_f)$ and $g'(p_g)$ have the same sign or are both zero and that $p_f = p_g = 0$ if p_f or p_g is in ∂I. Then f and g are topologically equivalent.*

Proof. We shall consider one case and leave the others to the reader. If $f'(p_f)>0$, then the interval $[p_f, c_f]$ has the property that $f[p_f, c_f] \subset [p_f, c_f]$. Define h to be any strictly increasing function of $[f(c_f), c_f]$ onto $[g(c_g), c_g]$. Then h extends to a homeomorphism of $[p_f, c_f]$ onto $[p_g, c_g]$ by the formula $h(f^n(x)) = g^n(h(x))$ and $h(p_f) = h(p_g)$. Extend h to $[p_f, p'_f]$ by $h(x') = (h(x))'$. To define h on $[0, p_f)$ and $(p'_f, 1]$, pick points $x \in (0, p_f)$ and $y \in (0, p_g)$ and define h to be any strictly increasing function of $[x, f(x)]$ onto $[y, g(y)]$. If $z \in (0, p_f)$, there is a unique $n \in \mathbb{Z}$ with $f^n(z) \in [x, f(x))$. Define $h(z)$ by $h(f^n(z)) = g^n(h(z))$. Since $f|[0, p_f]$ and $g|[0, p_g]$ are homeomorphisms, this formula is well defined. Finally set $h(0)=0$ and $h(x') = (h(x))'$ for $x \in (p'_f, 1]$. One easily checks that h so defined is a topological equivalence of f to g.

The other cases are similar. If $p_f = c_f$, then one defines h first from an interval $[x, f(x)]$ onto an interval $[y, g(y)]$ for arbitrary choices of $x \in (0, c_f)$ and $y \in (0, c_g)$. If $f'(p_f) < 0$, then one begins by defining h from the interval $[c_f, f^2(c_f)]$ onto $[c_g, g^2(c_g)]$. The extensions of h to topological equivalences are then uniquely determined. We leave the details to the reader.

Return now to consideration of $f \in \mathscr{C}$ with a stable periodic orbit, U the maximal interval containing c consisting of points tending to the stable periodic orbit, and n the least integer with $f^n(U) \subset U$. We next ask the extent to which the kneading sequence of f determines the integer n. There are two different possibilities if n is even. The itinerary of $f(c)$ could be periodic with period $n/2$. This case occurs if $f^{n/2}(U)$ and U share a common endpoint. If f has kneading sequence $\underline{\gamma}$, and $\gamma_i = I_0$ for $i > 0$, then f has a stable fixed point since f maps the interval $[0, c]$ into itself in an orientation preserving way. If f has kneading sequence $\underline{\gamma}$ and $\gamma_i = I_1$ for $i > 1$, then f maps the interval $(c, f(c))$ into itself in an orientation reversing way. Either there is a stable fixed point or there is a stable periodic orbit of period 2. Thus if f has a kneading sequence $\underline{\gamma}$ such that $\sigma(\underline{\gamma})$ is periodic of period n, then f has a stable periodic orbit of period n or $2n$. Period $2n$ can occur only if the number of I_1's among $\gamma_1, \ldots, \gamma_n$ is odd.

With these observations, we can determine the topological equivalence classes associated to a "periodic" kneading sequence. Suppose that $\underline{\gamma}$ is a periodic sequence of period n and that $\underline{\gamma}$ is the itinerary of $f(c)$. We shall say $\underline{\gamma}$ is of *positive* or *negative type* if the number of I_1's among $\gamma_0, \ldots, \gamma_{n-1}$ is even or odd and of *critical type* if $\gamma_{n-1} = C$. If $\underline{\gamma}$ is of positive type, then there is a closed interval U containing c such that $f^n(U) \subset U$, $f^n(\partial U) \subset \partial U$ and f^n has a stable fixed point $p \in U$ with $Df^n(p) > 0$. There are two possibilities: the stable fixed point is an interior point of U or it is an endpoint of U. Each of these possibilities represents a different topological equivalence class, but if f and g have kneading sequence $\underline{\gamma}$ and the same possibility occurs for f and g, then we assert that they are topologically equivalent. This follows easily from 2.3, 2.8, and 2.9.

Define a topological equivalence in stages. Proposition 2.3 implies that the set of f-itineraries and g-itineraries are the same. Proposition 2.8 implies that if $x \in I$ satisfies $f^n(x) \in I - U$ for all $n > 0$; i.e. $x \in E_f$, then there are no other points of I with the same itinerary. But $f^n(x) \in U$ if and only if $\sigma^{n+1}(\underline{A}(x)) = \underline{\gamma}$, so we can determine whether $f^n(x) \in E_f$ from its itinerary. The same considerations apply to g and we can define a homeomorphism $h: E_f \to E_g$ by the property that the f-itinerary of x and the g-itinerary of $h(x)$ are the same. We extend h to the set $\bigcup_{i \geq 0} f^{-i}(U_f)$ by first defining h to be a topological equivalence from $f^n|U_f$ to $g^n|U_g$ using Lemma 2.9. For each component of $f^{-i}(U)$ other than U, there is a $j \leq i$ with f^j mapping $f^{-i}(U)$ homeomorphically onto U. We can then define h on this component of $f^{-i}(U)$ by $h(f^j(x)) = g^j(h(x))$ and the requirement that the f-itinerary of x and the g-itinerary of $h(x)$ are the same. We leave it to the reader to check that h so defined is indeed a topological equivalence from f to g.

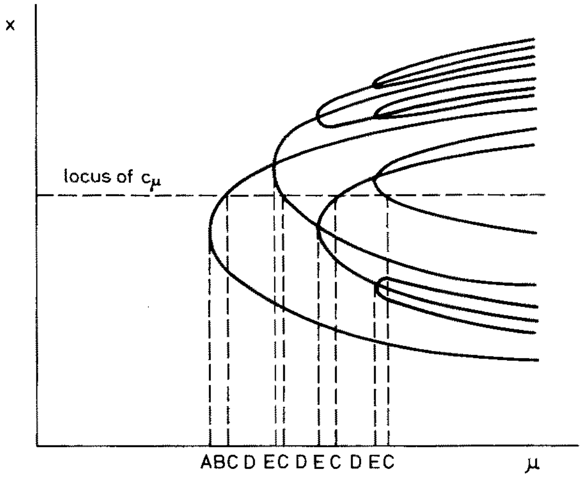


Fig. 1. A: $\underline{\gamma}$ positive type, one sided stable orbit; B: $\underline{\gamma}$ positive type, two sided stable orbit; C: $\underline{\gamma}$ critical type; D: $\underline{\gamma}$ negative type, period $\underline{\gamma}$ = period stable orbit; E: $\underline{\gamma}$ negative type, period $\underline{\gamma} = \frac{1}{2}$ (period stable orbit). Intervals D are closed on the right, intervals E are open

If $\underline{\gamma}$ is of critical type, then the period of its stable periodic orbit is n and the fixed point of $f^n|U$ is the critical point of f, an interior point of U. The construction of the preceding paragraph can be applied to give a topological equivalence from f to g if g has the same kneading sequence as f. Finally, if $\underline{\gamma}$ has negative type, there are two possibilities. Either the stable periodic point of $\underline{\gamma}$ has period n or period $2n$. In either case, the stable fixed point of f^n or f^{2n} is an interior point of U, and the construction of topological equivalences can be applied. If g has the same kneading sequence as f, and if the stable periodic orbits f and g have the same period, then f and g are topologically equivalent. This discussion is summarized by the following theorem.

Theorem 2.10. *Let $f, g \in \mathscr{C}$ have the same kneading sequence $\underline{\gamma}$. Either both f and g have stable periodic orbits or neither does. If f and g do not have stable periodic orbits they are topologically equivalent. If f and g do have stable periodic orbits, then $\sigma(\underline{\gamma})$ is periodic with period n. If $\underline{\gamma}$ has positive type, then the stable orbits of f and g have period n. Then f and g are topologically equivalent if their stable orbits are both stable from one side or both stable from both sides. If $\underline{\gamma}$ is of critical type, then f and g are topologically equivalent. If $\underline{\gamma}$ is of negative type, then the stable periodic orbits of f and g have period n or $2n$. The maps f and g are topologically equivalent if and only if these periods are the same.*

Remark. This theorem can be best understood in terms of the bifurcation diagram of periodic orbits in a one parameter family (see [6]). This diagram is illustrated above for the points in the stable periodic orbits of period 2^n. The curves show the locus of periodic points in a one parameter family f_μ.

3. On Sensitive Dependence

In this section we want to establish conditions as to when a particular map has sensitivity to initial conditions.

Definition. A map $f:X \to X$ of a compact metric space with measure μ has sensitivity to initial conditions if there is a set $K \subset X$ and an $\varepsilon > 0$ such that (1) $\mu(K) > 0$, and (2) if $x \in K$ and U is a neighborhood of x, then there is $y \in U$ and $n > 0$ with $d(f^n(x), f^n(y)) > \varepsilon$.

In all of our considerations, we will have $X = I$ with the usual metric and Lebesque measure. Maps in $\mathscr{C}$ which give stable periodic orbits will not have sensitivity to initial conditions. This is a corollary of the following theorem:

Theorem 3.1. *Let $f \in \mathscr{C}$ and let U be a neighborhood of c. If $E_U = \{x | f^n(x) \in I - U$ for all $n \geqq 0\}$ contains no non-trivial intervals, then the Lebesque measure of E_U is 0.*

Reduction. We shall first reduce to the case in which one of the endpoints of U is periodic. Note that $U \supset V$ implies that $E_U \subset E_V$. Therefore, if the theorem is true for E_V, $V \subset U$, then it is true for E_U. We seek now $(a, b) = V \subset U$ such that (1) $f(a) = f(b)$, (2) a or b is periodic, and (3) E_V contains no intervals.

We obtain V in stages. First, assume $U = (x, y)$ with $f(x) \leqq f(y)$. Then pick $z \in [y, c)$ with $f(z) = f(x)$ and replace U by (x, z). Then the first hypothesis for V is satisfied. Now assume that ξ and ξ' are the points of E_U closest to c. Then $f(\xi) = f(\xi')$ since $\xi \in E_U$ and $f(\xi) = f(\xi')$ imply $f^n(\xi') \in I - U$ for all $n > 0$ and since U satisfies (1). Assume that there is no stable fixed point in U. Then there is an n such that f^n fails to be a homeomorphism on the interval (ξ, c). There is an $\eta \in (\xi, c)$ and an $n > 0$ such that $f^n(\eta) = c$ and f^n is a homeomorphism on the interval $(\xi, \eta) = K$. If $f^n(\xi)$ and ξ fall on the same side of c, then the interval $f^n(K) = (f^n(\xi), c)$ contains K in its image because $f^n(\xi)$ is as far from c as ξ. Thus f^n has a fixed point a in (ξ, η). If $f^n(\xi)$ and ξ lie on opposite sides of c, then $f^n(\xi')$ and ξ' lie on the same side of c and f^n has a fixed point a in the interval (ξ', η'). Provided f has no stable periodic orbit, we take $V = (a, a')$ as the set we seek.

If there is a stable periodic orbit of f, then at least one of its points p must lie in $\bar{U}$ since E_U has no intervals. If p is in the boundary of U and U is symmetric with p' its other endpoint, then (1)–(3) are satisfied. So assume $p \in U$ is of period n and that no other points in the orbit of p lie in the interval (p, p'). Denote by V the component of p in $\{x | f^{kn}(x) \to p$ as $k \to \infty\}$. Then Singer's theorem implies that $c \in V$. This implies that if $x \in V$, then $x' \in V$. If $x \in V$, then $f^{kn}(x) \in U$ for some $k > 0$, so $E_U \subset E_V$ and we may replace U by V. One of the endpoints of V is easily seen to be a fixed point of f^n since f^n must map the boundary of V to itself. If f has a periodic point p of period n such that f^n is monotone on (p, c), $Df^n(p) = 1$, and p is stable from one side, then p is the limit of periodic points q_m. If $S = \{x | f^i(x) = p$ for some $i\}$, $U_m = (q_m, q'_m)$ and $U = (p, p')$, then $E_U = \bigcup_{m \geqq 0} E_{U_m} \cup S$.

Proof of Theorem. We assume now that the neighborhood U of c in the theorem satisfies the following properties: U is an interval (a, b) with (1) $f(a) = f(b)$, (2) a is periodic with $f^n(a) = a$ and $Df^n(a) > 1$, (3) E_U has no intervals. The proof of the theorem has two further steps. The first proves that f is "uniformly expanding" on E_U in the sense that there is a k such that $|Df^k(x)| > 1$ for all $x \in E_U$. The second part of the argument then estimates the "thickness" of the Cantor set E_U. This is seen to be finite, and the theorem is an easy consequence of this fact.

We seek a value of k such that $|Df^k(x)| > 1$ for all $x \in E_U$. Consider the sets $E_k = I - \bigcup_{i=0}^{k} f^{-i}(U)$. Each E_k is a finite union of closed intervals, $E_U = \bigcap_{k=0}^{\infty} E_k$, and f^k

is a local homeomorphism on E_k. The negative Schwarzian derivative of f implies that $\inf_{X \in E_k} |Df^k(x)|$ is assumed at a point of the boundary of E_k. Thus we seek k for which $|Df^k(x)| > 1$ for all x in the boundary of E_k.

Let $\alpha \leqq \min\{l(f^i(U)), 0 < i \leqq n; |a-c|, |b-c|\}$. If there is a stable periodic point $p \in U$, take $\alpha < |a-p|$. Since there are non non-trivial intervals in E_U, there is a K such that each component of E_K has length smaller than α. Now all points of the boundary of E_K have orbits which eventually lie in the unstable periodic orbit containing a. Therefore, there is $k \geqq K$ such that $x \in \partial E_K$ implies $|Df^k(x)| > 1$. We assert that also $|Df^k(x)| > 1$ for all $x \in \partial E_k$. Suppose that $x \in \partial E_k - \partial E_K$ and that J is the component of E_k containing x. Assume that $L \cup J \cup R$ is the maximal interval containing x on which $f^k(x)$ is a homeomorphism. We assert that either $L \subset E_K$ or $L \cap E_K = \emptyset$ since f^l maps each component $E_l - E_{(l-1)}$ onto U. Suppose $L \subset E_K$ and consider $f^k(L)$. This set has the form $f^i(U)$ for some $i > 0$. If f has a stable periodic orbit containing $p \in U$, then $l(f^i(U)) > |a-p| \geqq \alpha$ for all i. If f has no stable periodic orbit, then there is a $\xi \in (a, c)$ such that $f^n(\xi) = c$ since $f^n(a) = a$ and f^n does not map the interval (a, c) into itself. It follows that in this case that $f^k(L)$ has the form $f^i(U)$ with $i \leqq n$. Thus, we have the estimate $l(f^k(L)) \geqq \alpha$ in this case as well. Since $l(f^k(L)) \geqq \alpha > l(L)$, there is a point $y \in L$ with $|Df^k(y)| > 1$. If $L \cap E_K = \emptyset$, then we have assumed that if y is the common endpoint of L and J, then $|Df^k(y)| > 1$. The same argument we have used for L applies also to R. There are points $y \in L$, $z \in R$ with $|Df^k(y)| > 1$ and $|Df^k(z)| > 1$. Therefore, the negative Schwarzian condition implies $|Df^k(x)| > 1$ if $x \in J$. Thus $|Df^k(x)| > 1$ for all $x \in E_U$.

We now come to the final part of the proof that E_U has Lebesque measure zero. We have found an integer k such that $|Df^k(x)| > 1$ for all $x \in E_k = \{y \mid f^i(x) \in I - U$ for $0 \leqq i \leqq k\}$. Since E_k is compact, there is $\lambda > 1$ with $|Df^k(x)| > \lambda$ for all $x \in E_k$. We want to estimate the sizes of the "gaps" in $E_l - E_{l+1}$ which appear in the construction of E_U as $\bigcap_{l=0}^{\infty} E_l$. Suppose that J is a component of $E_l - E_{l+1}$, and that K is the component of E_l containg J. We assert that there is a constant $\gamma > 0$ (depending only on f) such that $l(J)/l(K) > \gamma$.

The constant γ is obtained from "nonlinearity" estimates of the sort used in Proposition 2.8. If J and K are as above, then f^{l+1} is a homeomorphism on K and $f^{l+1}(J) = U$. For any $x, y \in K$, we have

$$\log \left| \frac{Df^{l+1}(x)}{Df^{l+1}(y)} \right| = \sum_{i=0}^{l} \log |Df(f^i(x))| - \sum_{i=0}^{l} |\log Df(f^i(y))|$$
$$\leqq \Lambda \sum_{i=0}^{l} |f^i(x) - f^i(y)| ,$$

where Λ is a Lipschitz constant for the function $\log |Df(x)|$ on $I - U$. We use here the assumption that $f^i(K) \subset I - U$ for $i \leqq l$. Since $|Df^k(z)| > \lambda > 1$ for all $z \in E_k$ and $l(f^i(K)) < 1$ for all $i \leqq l$, we have $\sum_{i=0}^{l} |f^i(x) - f^i(y)| \leqq \frac{1}{1-\lambda^{-1}} = \frac{\lambda}{\lambda - 1}$. This gives the final estimate that

$$\log \left| \frac{Df^{l+1}(x)}{Df^{l+1}(y)} \right| \leqq \frac{\lambda \Lambda}{\lambda - 1} \quad \text{for all} \quad x, y \in K .$$

This produces the estimate for $l(J)/l(K)$:

$$\frac{l(U)}{l(I)} \leqq \frac{l(f^{l+1}(J))}{l(f^{l+1}(K))} \leqq \exp\left(\frac{\lambda A}{\lambda-1}\right)\frac{l(J)}{l(K)}.$$

Thus, if $\gamma = \exp\left(-\frac{\lambda A}{\lambda-1}\right) l(U)$, we have $l(J)/l(K) > \gamma$.

The theorem now follows quite easily: If $m > l$ is chosen such that every component of E_l contains points which are not in E_m, then $l(E_m) \leqq (1-\gamma) l(E_l)$. Iterating this estimate and noting that $(1-\gamma)^i \to 0$ as $i \to \infty$, we find that $l(E_m) \to 0$ as $m \to \infty$. Thus E_U has Lebesque measure zero.

Theorem 3.1 implies that the sensitivity of $f \in \mathscr{C}$ to initial conditions is determined by the properties of the set $\bigcap_{U \text{ nbhd of } c} \bigcup_{i \geqq 0} f^i(U)$, which we shall denote by Λ. We remark that Λ contains the support of any invariant measure which is absolutely continuous with respect to Lebesque measure. In particular, if $f \in \mathscr{C}$ has a (strongly) stable periodic orbit, then f does not have sensitivity to initial conditions. We want to examine f without stable periodic orbits and give topological criteria for sensitivity to initial conditions.

Definition. The fixed point p of f^n, $n > 1$, is *central* if (1) $Df^n(p) > 1$, and (2) f^n is a homeomorphism on the interval $J = (p, c)$. The central point p is *restrictive* if $f^n(J) \subset (p, p')$. As usual, the point p' is determined by $f(p) = f(p')$. The point 0 is not considered a central point.

We say a few words about the motivation for these definitions before proceeding further. If p is a restrictive central point of period n and if $U = (p, p')$, the $f^n(U) \subset U$. This implies that the set $\bigcup_{i=0}^{n-1} f^i(U)$ is forward invariant for f and that a point x which maps into this set cannot escape. This is the basis of the "spectral decomposition" of non-wandering sets given by Jonker and Rand [15]. Here we shall interpret the restrictive central points as establishing barriers which prevent separation of points in orbits with nearby initial conditions.

Theorem 3.2. *Suppose $f \in \mathscr{C}$ has no stable periodic orbit. Then f has sensitivity to initial conditions if and only if there is an integer N such that $n \geqq N$ implies f^n does not have a restrictive central point.*

The conditions for p to be a central point and a restrictive central point can be determined directly from invariant coordinates. Thus this theorem does yield strictly topological criteria for determining whether $f \in \mathscr{C}$ has sensitivity to initial conditions. Before embarking upon the proof of this theorem, we shall discuss briefly the existence of central points.

Let $f \in \mathscr{C}$ have no stable periodic orbit. Then the theory of Sect. 2 implies that $\{x \mid f^n(x) = c \text{ for some } n \geqq 0\}$ is dense in I. Therefore given $N > 0$, we can find $n > N$ such that the critical point $x \neq c$ of f^{n+1} closest to c satisfies $f^n(x) = c$. Then f^n has a fixed point q in one of the intervals (x, c) or (x', c) because $f^n(x) - x$ and $f^n(x') - x'$ have opposite signs. We do not expect, however, that $Df^n(q) > 0$. Let S be the set of points in the lap K of x for f^n such that $f^n(y) \in (y, y')$, but $f^i(y) \notin (y, y')$ for $i \leqq n$. Then the ends of the lap K are not in S, so points of the boundary of S satisfy one of the equations $f^n(y) = y$ or $f^n(y) = y'$. The points at opposite ends of interval in S do not both satisfy the same equation since f has no stable periodic orbits. This implies that each of S and S' has one end fixed by f^n. At one of them, Df^n is positive and this point is a central fixed point for f^n. The results of this discussion will be used below. They are summarized in the following lemma:

Lemma 3.3. *Let $f \in \mathscr{C}$ have no stable periodic orbit. If U is a neighborhood of c, there is an n such that f^n has a central fixed point in U.*

Proof of Theorem. Suppose that $f \in \mathscr{C}$ has no stable periodic orbit, but does have a sequence $n_k \to \infty$ such that f^{n_k} has a restrictive central point p_k. Set $U_k = (p_k, p'_k)$. For almost all $x \in I$, there is an N such that $f^n(x) \in \bigcup_{i=0}^{n_k-1} f^i(U_k)$ for $n \geqq N$. It follows, that there is a set $K \subset I$ of full measure such that $x \in K$ implies that $f^n(x)$ has ω-limit set in $\bigcap_{k=1}^{\infty} \bigcup_{i=0}^{n_k-1} f^i(U_k)$, which we denote Λ. Now Λ contains no non-trivial intervals, $\bigcup_{i=0}^{n_{k+1}-1} f^i(U_{k+1}) \subset \bigcup_{i=0}^{n_k-1} f^i(U_k)$, so $\mu_k = \max_{0 \leqq i < n_k} l(f^i(U_k)) \to 0$ as $k \to \infty$. If $\varepsilon > 0$, choose k such that $\mu_k < \varepsilon$. Then $x \in K$ implies that there is $N > 0$ such that $f^N(x) \in U_k$. We can find a neighboorhood V of x such that $f^N(V) \in U_k$ and $\max_{0 \leqq i < N} (f^i(V)) < \varepsilon$. Then for all n, we have $l(f^n(V)) < \varepsilon$ since $l(f^i(U_k)) \leqq \mu_k < \varepsilon$. This proves that f does not have sensitivity to initial conditions.

Consider now $f \in \mathscr{C}$ with the properties that (1) f has no stable periodic orbit, and (2) there is an $N > 0$ such that f^n has no restrictive central fixed point when $n > N$. We shall prove that f does have sensitivity to initial conditions.

The actual statement which we prove is that there is an $\varepsilon > 0$ such that if J is any non-trivial interval, then there is an n with $l(f^n(J)) > \varepsilon$. This easily implies sensitivity to initial conditions because $x \in J$; $l(f^n(J)) > \varepsilon$ implies that there is $y \in J$ with $d(f^n(x), f^n(y) > \varepsilon/2$. Moreover, since the set $\{y | f^n(y) = c \text{ for some } n \geqq 0\}$ is dense in I, we may assume that the interval J contains c. The lemma above implies then that J contains an interval (p_k, p'_k) with p_k a central point for some iterate of f.

The next step in the proof is the following lemma:

Lemma 3.4. *Suppose f^k has no restrictive central point for $k \geqq n$ and that p is central for f^n. Then there is a central point q and a k such that q is closer to c than p, but $f^k(q, q') \supset (p, p')$.*

Before proving this lemma, let us see how it implies that the theorem is true. Beginning with $p = q_0$, we can find a sequence $\{q_k\}$ of central points and integers n_k such that $f^{n_k}(q_k, q'_k) \supset (q_{k-1}, q'_{k-1})$ and q_k is closer to c than q_{k-1}. Then $f^{n_k + n_k + \dots + n_1}(q_k, q'_k) \supset (p, p')$ and $\{q_k\} \to c$. Given any neighborhood V of c, there is a q_k such that $(q_k, q'_k) \subset V$. Then we find n such that $f^n(V) \supset (p, p')$. Using $\varepsilon \leqq l(p, p')$, for any interval J, there is a k with $l(f^k(J)) > \varepsilon$. As we noted above, this suffices to prove the theorem.

We now prove the lemma by the construction of a "return map". Assume that p is a central fixed point of f^n. Define the discontinuous map $g: (p, p') \to (p, p')$ by $g(x) = f^k(x)$ where k is the smallest integer such that $f^k(x) \in (p, p')$. Since p is non-restrictive, f^n does not map (p, p') into itself and g is discontinuous. Now g is defined almost everywhere on (p, p') and it is monotone on each interval not containing c on which it is continuous. At the two ends of such an interval, the values of g must approach p and p'. Consider now the two cases in which g has only two points of discontinuity, and the case in which g has more than two points of discontinuity.

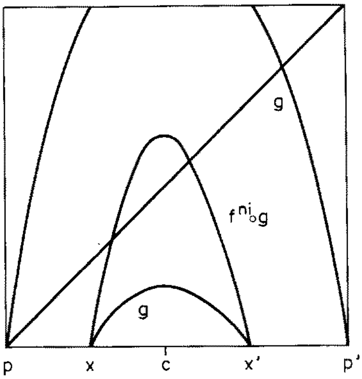


Fig. 2

In the first case, there are points x and x' in (p, p') such that g is continuous on the intervals (p, x), (x, x'), and (x', p). On (p, x) and (p', x'), the value of g goes from p to p'. On (x, x'), g has a single critical point at c and $g(y) \to p$ as $y \to x$ or x'. Consider now $f^{ni} \circ g$ on the interval (x, x'). For some i, this map has a fixed point since $f^{ni}(g(x)) = f^{ni}(g(x')) = f^{ni}(p) = p$ and for each $y \in (p, c)$, and there is an i with $c \in (f^{ni}(y), p)$. In particular, take i as small as possible with $g(c) = y$. Then we locate a fixed point q of $f^{ni} \circ g$ which is a central fixed point of f^k (see Fig. 2). By assumption q is not restrictive, so $f^k(q, q')$ contains a neighborhood of q. It is then evident that, for some l, $f^l(q, q')$ contains the interval (p, q) and $f^n(p, q) \supset (p, p')$. Therefore the lemma holds in this case.

Now consider the case in which g has more than two points of discontinuity in (p, p'). Then there is a fixed point q of g in (p, p') at which $Dg(q) > 0$. Choose q to be the closest fixed point of g to c with $Dg(q) > 0$. If $g(x) = f^k(x)$ in a neighborhood of q, then we assert that q is a central fixed point of f^k. If g is continuous on the interval (q, c), this is clear. If g is not continuous on (q, c), it has just one point of discontinuity y, and $g(x) = f^{k+n}(x)$ on (y, c). On (q, y), f^k gives the first return to (p, p'), while on (y, c) f^k has not yet returned to (p, p'). In neither case can there be $x \in (q, c)$ which is a critical point of f^k.

Now the argument proceeds as in the previous case. The point q is not restrictive, so $f^k(q, q')$ contains a neighborhood of q. If U is a neighborhood of q, then there is an i such that $f^{ik}(U)$ contains (p, q) because the domain of g contains an interval with endpoint q on which g is increasing and takes the value p at the other end. But $f^n(p, q)$ contains (p, p') since (p, q) contains a point of discontinuity for g. Thus the lemma and the theorem are proved.

4. Topological Entropy and Piecewise Linear Maps

Having established a topological criterion for the sensitivity to initial conditions of a map $f \in \mathscr{C}$, we want to explore further the relationship of this criterion with other topological properties of a map. Here we shall focus upon two issues which are seen to be closely connected with the sensitivity of a map. The first of these has to do with topological entropy, which is equivalent to the growth numbers of maps in the case we deal with.

Definition. Let $f: X \to X$ be a map of a set such that for each k, f^k has a finite number, N_k, of fixed points. Then the *growth number* of f is $\limsup (N_k)^{1/k}$.

The second issue we consider concerns those maps $f \in \mathscr{C}$ which are topologically equivalent to piecewise linear maps g with $|g'| = \mu$ constant. The interest in these maps g rests with the facts that their growth numbers are easily computed and that they posses invariant measures absolutely continuous with respect to lebesgue measure which have the largest possible entropy [20].

The first result we prove is a theorem whose proof is largely a series of computations.

Theorem 4.1. *Suppose $f \in \mathscr{C}$ has a restrictive central point p fixed by f^k, k odd. If $U = (p, p')$ and $E_U = \{x \mid f^n(x) \in I - U$ for all $n \geqq 0\}$, then the growth number of $f|E_U$ is larger than the growth number of $f \left| \bigcup_{i=0}^{k-1} f^i(U) \right.$.*

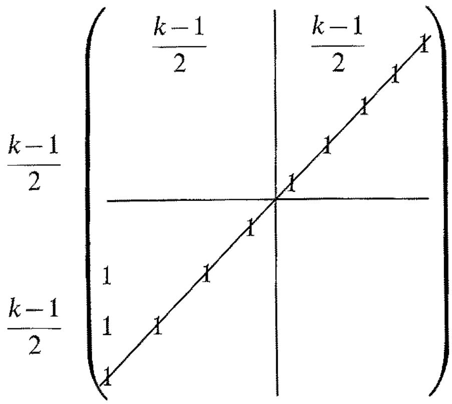
Corollary 4.2. *If $f \in \mathscr{C}$ has a restrictive central point p fixed by f^k, k odd, then f has a neighborhood $\mathscr{U}$ in $\mathscr{C}$ such that all maps in $\mathscr{U}$ have the same growth number.*

The proof of the theorem relies upon the characterization of the smallest itineraries which correspond to periodic orbits of each odd period. These were calculated in [6] and the entropy which we calculate here has also been calculated by Misiurewicz and Jonker-Rand [14]. From each periodic orbit of odd period k, we pick the largest point and then seek the one among these having the smallest invariant coordinate. The itinerary $\underline{a}^k$ of this periodic orbit is a periodic repetition of $I_1 \underbrace{I_0 I_1 I_1 \dots I_1}_{k-2}$. Any periodic orbit of period k contains a point whose invariant coordinate is at least as large as $\theta(\underline{a}^k)$. This is easily deduced from the fact that any cyclic sequence of I_0, I_1 of length k contains a block of I_1's of even length. For a sequence which begins $I_1 I_0 \underbrace{I_1 I_1 \dots I_1}_{2l} I_0$, the larger l, the smaller the invariant coordinate.

Using the above fact, we can explicitly describe a subshift of finite type which must be contained in E_U. It is the subshift of finite type present when f has a stable periodic orbit with the itinerary described above. Partitioning I along this orbit, we obtain a Markov partition $A_1, \dots, A_{k-1}$ for E_U. If these sets are labelled in increasing order along I, then

$$\begin{aligned} f(A_{(k+1)/2}) &= A_{(k+1)/2} \cup A_{(k-1)/2} \\ f(A_{(k+1)/2+l}) &= A_{(k-1)/2-l}, \qquad 0 \leqq l \leqq (k-3)/2 \\ f(A_{(k-1)/2-l}) &= A_{(k+1)/2+l+1}, \qquad 0 \leqq l \leqq (k-3)/2 \\ f(A_1) &= A_{(k+1)/2} \cup A_{(k+3)/2} \cup \dots \cup A_{k-1}. \end{aligned}$$

The transition matrix of A has 1's in the indicated positions:



If we renumber the A_i's starting with $A_{(k+1)/2}$ and then alternatively taking the next set to the left and to the right, the transition matrix is $\tilde{A}$ shown below

$$\tilde{A}=\begin{pmatrix} 1 & & & & & 1\\ & 1 & & & & 0\\ & & 1 & & & 0\\ & & & 1 & & \\ & & & & 1 & 1\\ & & & & & 0\end{pmatrix}$$

Direct calculation shows that the characteristic polynomial of $\tilde{A}$ is $P(t)=t^{2l}-t^{2l-1}-t^{2l-2}+t^{2l-3}-\ldots+t-1$ where we write $l=(k-1)/2$. Then $P(t)(t+1)=t^{2l+1}-2t^{2l-1}-1$. The growth number of the corresponding subshift of finite type is the largest root λ of $P(t)$. Evaluating $P(t)$ at $t=2^{1/2}$, we find $P(t)(t+1)=-1$.

Thus $\lambda>2^{1/2}$, which is the crucial estimate necessary to prove the theorem.

The map f cyclically permutes the sets $f^i(U)$, $0\leqq i<k$, $U=(p,p')$. Therefore, all periodic orbits in $\cup f^i(U)$ have periods which are divisible by k. On each $f^i(U)$, f^k has exactly one critical point. Therefore, the number of fixed points of f^{mk} in $\bigcup_{i=0}^{k-1} f^i(U)$ is at most $k\cdot 2^m$. This implies that the growth rate of $f|\cup f^i(U)$ is at most $2^{1/k}$. Comparing this estimate with the one for λ proves the theorem.

In the theory of rotation numbers of diffeomorphisms of the circle, the rotations represent a distinguished set which one might regard as "normal forms". Given a diffeomorphism, one would like to change coordinates so that it becomes a rotation if possible. In the theory we are studying, perhaps the closest analog to a rotation is a piecewise linear map g_μ defined by

$$g_\mu(x)=\begin{cases}\mu x & \text{if } 0\leqq x\leqq \frac{1}{2}\\ \mu(1-x) & \text{if } \frac{1}{2}\leqq x\leqq 1\,.\end{cases}$$

Clearly, $|g'_\mu(x)|=\mu$ for all $x\neq\frac{1}{2}$. It is known that g_μ has an invariant measure ν, absolutely continuous with respect to Lebesgue measure, whose entropy is $\log\mu$. Here $\log\mu$ is the topological entropy of g and μ is the growth number of g.

Not every diffeomorphism of S^1 is conjugate to a rotation, but within the class of C^2 diffeomorphisms, those with irrational rotation numbers are. An analogous fact is true here – not every map $f\in\mathscr{C}$ is topologically equivalent to a g_μ. Topological conditions can be used to specify which equivalence classes are represented by a g_μ. The fundamental observation is the next simple proposition.

Proposition. 4.3. *If $\sqrt{2}<\mu<2$, then the map $g(x)=\mu/2-\mu|\frac{1}{2}-x|$ has no restrictive central points.*

Proof. Let p be a central fixed point of g^k, $k>1$. Then $p'=1-p$ and g^k is monotone on $(p,\frac{1}{2})$. But $|Dg^k|>2$ by assumption, so $|g^k(\frac{1}{2})-g^k(p)|>2|\frac{1}{2}-p|=|1-2p|$. Since $|p'-p|=|1-2p|$, this implies that $p'\in(p,g^k(\frac{1}{2}))$ and p is not restrictive.

Corollary 4.4. *If $f\in\mathscr{C}$ has a restrictive central point and growth number larger than $\sqrt{2}$, then there is no μ such that f is topologically equivalent to g_μ.*

We next want to establish a converse to these last statements which gives a positive criterion for a map $f \in \mathscr{C}$ to be topologically equivalent to one which is piecewise linear of constant slope.

Theorem 4.5. *Let $f \in \mathscr{C}$ have no restrictive central point. Then there is a $\mu \in (\sqrt{2}, 2)$ such that f is topologically equivalent to the map g defined by $g_\mu(x) = \mu/2 + \mu \cdot |\frac{1}{2} - x|$.*

Proof. Since f has no stable periodic orbit, the set $\{x | f^n(x) = c$ for some $n \geqq 0\}$ is dense and the theory of Sect. 2 can be applied. We need only prove that there is a μ such that f and g_μ have the same kneading sequence. Then f and g will be topologically equivalent. What we shall prove is that there is a unique kneading sequence with the growth number of f. This argument depends on the following lemma.

Lemma 4.6. *If $\Sigma_1 \subsetneqq \Sigma_2$ are topologically mixing subshifts of finite type, then the growth number of Σ_1 is smaller than the growth number Σ_2.*

Proof. Consider a common Markov partition for Σ_1 and Σ_2 with transition matrices A and B. Now $\Sigma_2 - \Sigma_1$ is open in Σ_2 and periodic points of Σ_2 are dense. Therefore $\Sigma_2 - \Sigma_1$ contains periodic points, and there is an n such that $\operatorname{Tr} A^n < \operatorname{Tr} B^n$. Now $\Sigma_1 \subset \Sigma_2$ implies $(A^n)_{i_j} \leqq (B^n)_{i_j}$ for all n, i, j. The strict inclusion implies that $(A^n)_{i_j} < (B^n)_{i_j}$ for some i, j and each n. We claim that there is an n for which $(A^n)_{i_j} < (B^n)_{i_j}$ for all i, j. The number $(B^n)_{i_j}$ is the number of sequences $b_{i_0}, b_{i_1 i_2} \ldots b_{i_{n-1} i_n}$ with $b_{i\,ji\,j+1} = 1$ for each $0 \leqq j \leqq n$ and $i = i_0$, $j = i_n$. Topological mixing and $\Sigma_1 \neq \Sigma_2$ gurantees that for n large there will be a chain of this sort with some $a_{i\,ji\,j+1} = 0$ for each $i = i_0$, $j = i_n$. (This can be used as the definition of topological mixing in this case.) Now if v is any vector with positive components and $(A^n)_{i_j} < (B^n)_{i_j}$ for all i, j, then each component of $A^n v$ is smaller than the corresponding component of $B^n v$. Taking v to be the eigenvector of A corresponding to its largest eigenvalue, we find that B has an eigenvalue larger than all eigenvalues of A. But the growth rates of Σ_1 and Σ_2 are the largest eigenvalues of A and B.

Using this lemma we now prove that there is only one kneading sequence with the growth rate of $f \in \mathscr{C}$. Let $\underline{a}$ be the kneading sequence of f and let $\underline{b}$ be another kneading sequence, say larger than $\underline{a}$. Between $\underline{a}$ and $\underline{b}$ is a periodic kneading sequence $\underline{d}$. Let g be a map with kneading sequence $\underline{d}$ and let p be the restrictive central point of g farthest from c. The map g has a stable periodic orbit and hence a restrictive central point. Then the growth number of g is the growth number of $g|E_U$; $U = (p, p')$, $E_U = \{x | g^n(x) \in I - U$ for all $n \geqq 0\}$. The invariant coordinate of p must be larger than $\underline{a}$ because $\theta(c)$ lies between $\theta(p)$ and $\theta(p')$ for g and $\theta(\underline{a}) \leqq \theta(\underline{d}) = \theta(c_g)$. If the invariant coordinate of $\underline{a}$ were larger than that of p, then f would have a periodic orbit with the same itinerary as p, and this point would be a restrictive central point. Now $g|E_U$ is topologically equivalent to a subshift of finite type which is topologically mixing. We can find a proper subset of E_U which is also a subshift of finite type, topologically transitive, and with growth rate at least as large as f. Then the Lemma implies that g has larger growth number than f. Therefore, any map with kneading sequence $\underline{b}$ has larger growth number than f.

Assume now that $\underline{b}$ has a smaller invariant coordinate than $\underline{a}$. Then an argument similar to the one above implies that if g has kneading sequence $\underline{b}$, then g

has growth number smaller than that of f. There is no other kneading sequence than the one of f with the same growth number. Let g_μ be the piecewise linear map $g_\mu(x)=\mu/2-\mu\cdot|\frac{1}{2}-x|$ where μ is the growth number of f. The uniqueness argument then implies that g has the same kneading sequence as f since μ is also its growth number. Finally, the results of Sect. 2 imply that f and g are topologically equivalent.

Thus far, we have considered piecewise-linear maps with growth numbers in $(\sqrt{2},2]$ and smooth maps with odd periodic orbits. Let us briefly describe the general situation, starting with the piecewise linear maps. Suppose $g_\mu(x)=+\mu/2-\mu\cdot|\frac{1}{2}-x|$ with $\mu\in(2^{1/2^m},2^{1/2^{m-1}}]$. Then all of the periodic orbits of g_μ except a finite number (one for each 2^i, $i<m$) will have periods which are divisible by 2^m. There is a subinterval J of I, with an endpoint at the closest restrictive central point of g to 1/2 such that $g^{2^m}|J$ is a piecewise linear map with slope in $(\sqrt{2},2]$. The preceding theorems then apply to $g^{2^m}|J$. Similarly, if the map f has periodic points of the form $2^m\cdot k$, $k>1$ odd but not of the form $2^{m-1}\cdot k$, then the growth number of f lies in the interval $(2^{1/2^{m+1}},2^{1/2^m}]$. The map f has a restrictive central point p of period 2^m, and if q is a restrictive central point closer to c than p, then f is not topologically equivalent to a piecewise linear map g_μ. If no such q exists, then f is topologically equivalent to the g_μ with μ the growth rate of f. As a final corollary of the theory we have developed thus far, we have the following:

Theorem 4.7. *Let $f\in\mathscr{C}$. Then f has sensitivity to initial conditions if and only if there is a subinterval $J\subset I$ and an $n>0$* such that *$f^n(J)\subset J$ and $f^n|J$ is topologically equivalent to a piecewise linear map $g_\mu(x)=\mu/2-\mu\cdot|x-\frac{1}{2}|$. Here μ is the growth rate of $f^n|J$ and $\mu\in(\sqrt{2},2]$.*

We turn now to one parameter families f_ν of maps in $\mathscr{C}$ for some final remarks. An outstanding question about such families is the prevalence of parameter values ν for which f_ν is "chaotic". If one interprets "chaotic" as "having sensitive dependence on ititial conditions", then the theory we have developed can be applied to yield some new perspective on this problem. To cast the problem into the terms we desire, we make a "genericity" hypothesis for the family f_ν:

Hypothesis. If J is a nontrivial interval in the parameter space of the family f_ν such that $\nu_1,\nu_2\in J$ imply that f_{ν_1} and f_{ν_2} have the same kneading sequence, then f_{ν_1} has a stable periodic orbit.

While it is not known that any family satisfies this hypothesis, it seems likely that the set of families in $C^k(I,\mathscr{C})$ which satisfy the hypothesis is generic set; i.e., a countable intersection of open dense sets. In any case we shall assume that all families we discuss do satisfy the hypothesis.

Let f_ν be such a family. If there is a set B of positive measure in the parameter space such that $\nu\in B$ implies that f_ν has sensitive dependence to initial conditions, then there is a subset $\tilde{B}$ and an n such that $\nu\in\tilde{B}$ implies that f_ν^n restricted to a suitable subinterval is topologically equivalent to a piecewise linear map of constant slope. With this n, we rescale the maps so that their domain of definition is I. We then have a family for which there is a set of positive measure in parameter space for which the members of the family are topologically equivalent to piecewise linear maps.

Interpret this via Theorem 4.5 in terms of growth rates. Using the hypothesis, the places where the growth rate function of a family f_v is not constant are the parameter values v at which f_v is topologically equivalent to a piecewise linear function g_μ. This suggests that one should study the growth rate function associated to a family: $\gamma(v)$ = growth rate of f_v. An alternative question to the one raised earlier is whether the function γ is absolutely continuous. The discussion above suggest that the typical family f_v has a set of positive measure in parameter space for which f_v has sensitivity to initial conditions if and only if the growth rate function $g(v)$ has a distributional derivative $\frac{dg}{dv}$ whose support has positive measure in the parameter space. Since we prove nothing, we refrain from a formal statement of this principle. It does motivate discussion of the growth rate function, however.

One cannot expect a simple argument to prove that the growth rate function is absolutely continuous. The following example gives an upper bound of 1/2 for the Hölder exponent of the growth rate function of the quadratic family $f_v(x) = vx(1-x)$. When $v=4$, then f_v has growth number 2. All roots of the polynomial $f_v^k(x)-x$ of degree 2^k are real. There is a sequence of values $v_n \to 4$ such that when $v=v_n$, 1/2 is periodic with period n and with $0<f^2(\frac{1}{2})<f^3(\frac{1}{2})<\ldots<f^{n-1}(\frac{1}{2}) <f^n(\frac{1}{2})=\frac{1}{2}$. These stable periodic orbits are the "largest of each period" in the sense of invariant coordinates. Since the derivative of $4x(1-x)$ at 0 is 4, one can estimate the way in which $\varepsilon_n = 4-v_n \to 0$ as $n\to\infty$. For any $\delta>0$, $(4-\delta)^n\varepsilon_n \to 0$. For n large, we must have $f^2_{v_{n+1}}(\frac{1}{2}) \approx \frac{1}{4} f^2_{v_n}(\frac{1}{2})$.

We can also calculate the growth rate of f_{v_n}. Partitioning I along the orbit of $\frac{1}{2}$, we find that the rest of the nonwandering set of f_{v_m} is topologically equivalent to a subshift of finite type with $(n-1)\times(n-1)$ transition matrix A_n:

$$A_n = \begin{pmatrix} 0 & & & & 1 \\ 1 & & & & 1 \\ & 1 & & & 1 \\ & & \ddots & & 1 \\ & & & 1 & 1 \end{pmatrix}.$$

The growth rate of f_n is the largest root of the characteristic polynomial $P_n(t)$ of A_n. Now $P(t) = t^{n-1} - t^{n-2} - \ldots - 1 = t^{n-1} - \frac{t^{n-1}-1}{t-1}$. The largest root of $P(t)$ is approximately $2-2^{1-n}$. Thus for the v_n's we have the rough estimate for the growth rate function that $\gamma(4)-\gamma(v_n) \sim 2^{1-n}$ while $4-v_n \sim \beta 4^{-n}$ for some constant β. It follows that γ will not be in the Hölder class C^α if α is larger than 1/2. In particular the growth rate function is not Lipschitz. More careful estimates will be necessary to determine whether or not the support of $\frac{d\gamma}{dv}$ typically has positive measure.

There are two other "measure theoretic" questions which we ask concerning the theory developed in this paper. The first question is whether the set Λ for $f \in \mathscr{C}$ having an infinite number of restrictive central points always has Lebesgue measure zero. A positive answer would imply that if $f \in \mathscr{C}$ is not sensitive to initial conditions, then f has no invariant measure absolutely continuous with respect to Lebesgue measure. The second question is whether $f \in \mathscr{C}$ sensitive to initial conditions implies that f does have an invariant measure absolutely continuous with respect to Lebesgue masure. Is sensitivity equivalent to absolutely continuous invariant measure?

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