f_A-SPACES

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UDC 517.11

Let $\mathbb{A} = \langle A, \in, \ldots \rangle$ be an admissible set [4], let $\langle X, X_0, \leq \rangle$ be an f-space [1], let $B \subseteq A$ be a Σ-set, and let $\nu : B \to X_0$ be an enumeration (mapping onto) of the basis X_0.

The quadruple $\mathfrak{X} = \langle X, X_0, \leq, \nu : B \to X_0 \rangle$ is said to be an f_A-space if the following three conditions hold:

1. If $c \in B^* (\rightleftharpoons \{c \mid c \in A, c \subseteq B\})$ is such that

$$\exists \xi \in X \, \forall b \in c (\nu b \leq \xi), \tag{1}$$

then the set $\nu c \rightleftharpoons \{\nu b \mid b \in c\}$ has in $\langle X, \leq \rangle$ a least upper bound $\sqcup \nu c$ and $\sqcup \nu c \in X_0$.

Let $Con_{X,\nu} \rightleftharpoons \{c \mid c \in B^*$ and (1) is valid for c$\}$.

2. The set $L \rightleftharpoons \{\langle c, b \rangle \mid c \in Con_{X,\nu}, b \in B, \nu b = \sqcup \nu c\}$ is a Σ-set.

3. For any $\xi \in X$ the set $B_\xi \rightleftharpoons \{b \mid b \in B, \nu b \leq \xi\}$ is a Σ-set.

<u>Remark.</u> Every f_A-space is a f_0-space [1]; indeed, $\emptyset \in Con_{X,\nu}$ and $\perp \rightleftharpoons \sqcup \emptyset$ is the least element in $\langle X, \leq \rangle$.

<u>LEMMA 1.</u> The set $L_0 \rightleftharpoons \{\langle b_0, b_1 \rangle \mid b_0, b_1 \in B, \nu b_0 \leq \nu b_1\}$ is a Σ-set.

Indeed, $\langle b_0, b_1 \rangle \in L_0 \leftrightarrow \langle \{b_0, b_1\}, b_1 \rangle \in L$, and L is a Σ-set.

We mention the following two properties of the sets of the form B_ξ, $\xi \in X$:

1. $B_\xi \neq \emptyset$; if $\langle b_0, b_1 \rangle \in L_0$ and $b_1 \in B_\xi$ then $b_0 \in B_\xi$.

2. If $c \in B_\xi^*$, then $c \in Con_{X,\nu}$ and $\langle c, b \rangle \in L$ implies $b \in B_\xi$.

We denote by $m_\nu(B)$ the collection of all Σ-sets $B' \subseteq B$, satisfying conditions 1, 2 for B_ξ.

The correspondence $\xi \longmapsto B_\xi$ defines a mapping $\beta : X \to m_\nu(B)$; an f_A-space $\mathfrak{X}$ is said to be complete if β is an onto mapping.

<u>Remark.</u> From the general properties of f-spaces [1] if follows easily that β is a differently valued mapping.

For any f_A-space $\mathfrak{X}$ one can construct its completion $\mathfrak{X}^*$ in the following manner: we set

$$X^* \rightleftharpoons m_\nu(B); \quad X_0^* \rightleftharpoons \beta(X_0) \subseteq X^*; \quad \nu^*(b) \rightleftharpoons \beta(\nu b), \; b \in B,$$
$$\text{and} \quad \mathfrak{X}^* \rightleftharpoons \langle X^*, X_0^*, \subseteq, \nu^* : B \to X_0^* \rangle.$$

Translated from Algebra i Logika, Vol. 25, No. 5, pp. 533-543, September-October, 1986. Original article submitted April 16, 1986.

 0002-5232/86/2505-0336$12.50

It is easy to verify that $\mathfrak{X}^*$ is a complete f_A-space; β is a homeomorphic imbedding of X into X^*, whiile $\beta \upharpoonright X_0$ is a homeomorphism of X_0 and X_0^*.

Proposition 1. If $\mathfrak{X}$ is a complete f_A-space, $C \subseteq B$ is a Σ-set such that $c \in Con_{X,\nu}$ for any $c \in C^*$, then in $\langle X, \leq \rangle$ there exists a least upper bound $\sqcup \nu C$ for set $\nu C \rightleftharpoons \{\nu b \mid b \in C\}$.

We consider the set $\mathfrak{D} \rightleftharpoons \{b \mid b \in B, \exists c \in C^* \exists b_0(\langle c, b_0 \rangle \in L \wedge \langle b, b_0 \rangle \in L_0)\}$. We show that $\mathfrak{D} \in m_\nu(B)$ the facts that $\mathfrak{D}$ is a Σ-set and that $\mathfrak{D}$ satisfies condition 1 follow at once from the definition of $\mathfrak{D}$. We verify whether property 2 for $\mathfrak{D}$ holds: if $c \in \mathfrak{D}^*$, then we have

$$\forall b \in c \exists c_0 \exists b_0 (c_0 \subseteq C \wedge \langle c_0, b_0 \rangle \in L \wedge (b, b_0) \in L_0).$$

By the Σ-sample principle [4], there exists $d \in A$ such that

$$\forall b \in c \exists c_0 \in d \exists b_0 (c_0 \subseteq C \wedge \langle c_0, b_0 \rangle \in L \wedge \langle b, b_0 \rangle \in L_0) \wedge$$

$$\wedge \forall c_0 \in d \exists b \in c \exists b_0 (c_0 \subseteq C \wedge \langle c_0, b_0 \rangle \in L \wedge \langle b, b_0 \rangle \in L_0).$$

We set $c_1 \rightleftharpoons \cup d$, then from $\forall c_0 \in d\ (c_0 \subseteq C)$ there follows that $c_1 \subseteq C$. From $c_1 \in C^*$ and from the assumptions of the proposition there folows that there exists $b_1 \in B$ such that $\langle c_1, b_1 \rangle \in L$. From the validity of

$$\forall b \in c \exists c_0 \subseteq c_1 \exists b_0 (c_0 \subseteq C \wedge \langle c_0, b_0 \rangle \in L \wedge \langle b, b_0 \rangle \in L_0)$$

there follows that $\nu b \leq \nu b_0 = \sqcup \nu c_0 \leq \sqcup \nu c_1 = \nu b_1$ for any $b \in c$; consequently, $c \in Con_{X,\nu}$ and $\sqcup \nu c \leq \nu b_1$, and for $b \in B$ such that $\langle c, b \rangle \in L$ we have $\nu b = \sqcup \nu c \leq \nu b_1$; $b_1 \in \mathfrak{D}$, consequently, $b \in \mathfrak{D}$, and property 2 is verified.

Since $\langle \{b\}, b \rangle \in L$ for any $b \in B$, then $C \subseteq \mathfrak{D}$, and, as one can easily see, if $C \subseteq \mathfrak{D}_0$, $\mathfrak{D}_0 \in m_\nu(B)$, then $\mathfrak{D} \subseteq \mathfrak{D}_0$. Since $\mathfrak{X}$ is complete, there exists a (unique) element $\xi \in X$ such that $\mathfrak{D} = B_\xi$. From what has been said above there follows that $\xi = \sqcup \nu C$.

Let $\mathfrak{X} = \langle X, X_0, \leq, \nu_0 : B_0 \to X_0 \rangle$ and $\mathfrak{Y} = \langle Y, Y_0, \leq, \nu_1 : B_1 \to Y_0 \rangle$ be two f_A-spaces; let $\mu : X \to Y$ be a continuous mapping and

$$L_\mu \rightleftharpoons \{\langle b_0, b_1 \rangle \mid b_0 \in B_0,\ b_1 \in B_1,\ \nu_1 b_1 \leq \mu \nu_0 b_0\} \subseteq B_0 \times B_1.$$

Remark. The continuous mapping μ can be restored from set L_μ: for $\xi \in X$ we have

$$\mu(\xi) = \sqcup \{\nu_1 b_1 \mid \exists b_0 (\langle b_0, b_1 \rangle \in L_\mu \wedge \nu_0 b_0 \leq \xi)\}.$$

A continuous mapping $\mu : X \to Y$ is said to be a computable mapping from $\mathfrak{X}$ into $\mathfrak{Y}$ if L_μ is a Σ-set. By $C(\mathfrak{X}, \mathfrak{Y})$ we denote the family of all computable mappings from $\mathfrak{X}$ into $\mathfrak{Y}$.

We assume that $\mathfrak{Y}$ is a complete f_A-space and that $B \subseteq B_0 \times B_1$ is a Σ-set such that we have the condition

$$\forall c \in B^* (\delta c \in Con_{X,\nu_0} \Rightarrow \rho c \in Con_{Y,\nu_1}). \tag{2}$$

From such a B we construct a mapping $\mu_B : X \to Y$ in the following manner:

Let $\xi \in X$ and $C_\xi \rightleftharpoons \{b_1 \mid \exists b_0 (\langle b_0, b_1\rangle \in B \wedge b_0 \in B^X_\xi)\}$; C_ξ is a Σ-set. We show that $c \in C^*_\xi \Rightarrow c \in Con_{Y,\nu_1}$. Since $c \in C^*_\xi$, we have

$$\forall b_1 \in c\, \exists b\, \exists b_0 (b \in B \wedge b = \langle b_0, b_1\rangle \wedge b_0 \in B^X_\xi);$$

by the Σ-sample principle there exists $d \in A$ such that

$$\forall b_1 \in c\, \exists b \in d\, \exists b_0\ (b \in B \wedge b = \langle b_0, b_1\rangle \wedge b_0 \in B^X_\xi) \wedge$$
$$\wedge\, \forall b \in d\, \exists b_1 \in c\, \exists b_0 (b \in B \wedge b = \langle b_0, b_1\rangle \wedge b_0 \in B^X_\xi).$$

Then $d \subseteq B$, $\rho d = c$, $\delta d \subseteq B^X_\xi$; since $\delta d \in B^X_\xi$, we have $\delta d \in Con_{X,\nu_0}$, and, consequently, by (2) we have $c = \rho d \in Con_{Y,\nu_1}$.

By Proposition 1 in $\mathcal{Y}$ there exists $\sqcup \nu_1 C_\xi$. We set $\mu_B(\xi) = \sqcup \nu_1 C_\xi$, $\xi \in X$.

Proposition 2. The mapping $\mu_B : X \to Y$ is a computable mapping from $\mathfrak{X}$ into $\mathcal{Y}$.

First we verify the continuity of the mapping μ_B. Let $\xi \in X$, $b_1 \in B_1$ and $\nu_1 b_1 \leq \mu_B(\xi)$; then $b_1 \in B^Y_{\mu_B(\xi)}$. The set $B^Y_{\mu_B(\xi)}$ is obtained from C_ξ as $\mathfrak{D}$ from C in the proof of Proposition 1. Consequently, there exist $c \in C^X_\xi$ and $\bar{b}_1 \in B_1$ such that $\langle c, \bar{b}_1\rangle \in L^Y \wedge \langle b_1, \bar{b}_1\rangle \in L^Y_0$. Since $c \subseteq C_\xi$, we have

$$\forall b_1 \in c\, \exists b_0 (\langle b_0, b_1\rangle \in B \wedge b_0 \in B^X_\xi);$$

by the Σ-sample principle there exists $d \in A$ such that

$$\forall b_1 \in c\, \exists b_0 \in d\, (\langle b_0, b_1\rangle \in B \wedge b_0 \in B^X_\xi) \wedge \forall b_0 \in d\, \exists b_1 \in c\, (\langle b_0, b_1\rangle \in B \wedge b_0 \in B^X_\xi).$$

Then $d \subseteq B^X_\xi$, $d \in Con_{X,\nu_0}$; let b_0 be such that $\langle d, b_0\rangle \in L^X$ (i.e., $\nu_0 b_0 = \sqcup \nu_0 d$). Then $\nu_0 b_0 \leq \xi$ and if $\nu_0 b_0 \leq \xi' \in X$, then $d \subseteq B^X_{\xi'}$, $c \subseteq C_{\xi'}$, $\mu_B(\xi') = \sqcup \nu_1 C_{\xi'} \geq \nu_1 \bar{b}_1 \geq \nu_1 b_1$. Consequently, $\xi \in \check{\nu_0 b_0} \subseteq \mu_B^{-1}(\check{\nu_1 b_1})$ and the continuity of μ_B is proved.

The computability of μ_B follows now from the following easily verifiable equality:

$$L_{\mu_B} = \{\langle b_0, b_1\rangle \mid \exists d\, \exists \bar{b}_1 (d \subseteq B \wedge \forall b \in \delta d\ (\langle b, b_0\rangle \in L^X_0) \wedge \langle \rho d, \bar{b}_1\rangle \in L^Y \wedge \langle b_1, \bar{b}_1\rangle \in L^Y_0)\}.$$

Remark. a) $B \subseteq L_{\mu_B}$; b) if μ is a computable mapping from $\mathfrak{X}$ into $\mathcal{Y}$, then L_μ satisfies condition (2).

We consider now the question when on the set $C(\mathfrak{X}, \mathcal{Y})$ of all computable mappings from $\mathfrak{X}$ into $\mathcal{Y}$ one can define a "natural" structure of f_A-space. It is reasonable to restrict ourselves to the case when $\mathcal{Y}$ is a complete f_A-space.

Proposition 3. If $C \rightleftharpoons \{c \mid c \in (B_0 \times B_1)^*$, c satisfies condition (2)$\}$ is a Σ-set, then $C(\mathfrak{X}, \mathcal{Y})$ has a "natural" structure of an f_A-set.

For $c \in C$ we set $\nu c \rightleftharpoons \mu_c$; then ν is an enumeration $C \to C_0(\mathfrak{X}, \mathcal{Y})$, where $C_0(\mathfrak{X}, \mathcal{Y}) \rightleftharpoons \{\nu c \setminus c \in C\}$. A partial order on $C(\mathfrak{X}, \mathcal{Y})$ is defined in the following manner: for $\mu_0, \mu_1 \in C(\mathfrak{X}, \mathcal{Y})$

$$\mu_0 \leq \mu_1 \rightleftharpoons \forall \xi \in X \ (\mu_0(\xi) \leq \mu_1(\xi)).$$

We show that the quadruple

$$\langle \mathcal{C}(\mathfrak{X},\mathfrak{Y}), \mathcal{C}_0(\mathfrak{X},\mathfrak{Y}), \leq, \ \nu : \mathcal{C} \longrightarrow \mathcal{C}_0(\mathfrak{X},\mathfrak{Y}) \rangle$$

is a complete f_A-space.

First one has to prove that $\langle \mathcal{C}(\mathfrak{X},\mathfrak{Y}), \ \mathcal{C}_0(\mathfrak{X},\mathfrak{Y}), \leq \rangle$ is an f-space. We establish at once condition 1 of the definition of an f_A-space, from which there will follow this statement too.

Let $d \in \mathcal{C}^+$ and assume that there exists $\mu \in \mathcal{C}(\mathfrak{X},\mathfrak{Y})$ such that $\nu c \leq \mu$ for all $c \in d$. We set $c_0 \rightleftharpoons \cup d$ and we verify that for c_0 we have (2). Let $c_1 \subseteq c_0$ and assume that $\delta c_1 \in Con_{X,\nu_0}$; then there exists $\xi_0 = \sqcup \nu_0(\delta c_1)$; if $b_1 \in \rho c_1$ then $\langle b_0, b_1 \rangle \in c_1 \subseteq \cup d$ for some $b_0 \in B_0$ and $\langle b_0, b_1 \rangle \in c$ for some $c \in d$. Then $\nu_1 b_1 \leq [\nu c](\xi_0) \leq \mu(\xi_0)$; consequently, $\nu_1 b_1 \leq \mu(\xi_0)$ for any $b_1 \in \rho c_1$; from here $\rho c_1 \in Con_{Y,\nu_1}$ and c_0 satisfies condition (2). We show that $\nu c_0 = \sqcup \nu d$; for this we establish the following fact, needed also later: for $c, c' \in \mathcal{C}$ we have

$$\nu c \leq \nu c' \Longleftrightarrow \forall \langle b_0, b_1 \rangle \in c \, \exists c'' \subseteq c' (\forall \langle b'_0, b'_1 \rangle \in c'' (\nu_0 b'_0 \leq \nu_0 b_0) \wedge \nu_1 b_1 \leq \sqcup \nu_1 \rho c''). \qquad (3)$$

We note that from the definition there follows easily that $c \subseteq c' \in \mathcal{C}$ implies $\nu c \subseteq \nu c'$; further, $\langle b_0, b_1 \rangle \in c$ implies $\nu_1 b_1 \leq [\nu c](\nu_0 b_0)$; the condition in the brackets means that $\nu_1 b_1 \leq [\nu c''](\nu_0 b_0) (\leq [\nu c'](\nu_0 b_0))$. From these remarks there follows the implication from left to right in (3).

Assume that the right-hand side of the equivalence (3) is true and let $\xi \in X$; then $[\nu c](\xi) = \sqcup \{\nu_1 b_1 | \exists b_0 (\langle b_0, b_1 \rangle \in c \wedge \nu_0 b_0 \leq \xi)\}$; let $\langle b_0, b_1 \rangle \in c$ and let $\nu_0 b_0 \leq \xi$; by virtue of the right-hand side in (3), there exists $c'' \subseteq c'$ such that $\nu_1 b_1 \leq [\nu c''](\nu_0 b_0) \leq [\nu c''](\xi) \leq [\nu c'](\xi)$, from where $[\nu c](\xi) \leq [\nu c'](\xi)$ and $\nu c \leq \nu c'$.

From the equivalence (3) there follows at once that $\nu c = \sqcup \{\nu(\{\langle b_0, b_1 \rangle\}) | \langle b_0, b_1 \rangle \in c\}$ for any $c \in \mathcal{C}$; thus, returning to the proof of the proposition, we obtain

$$\nu c_0 = \sqcup \{\nu(\{\langle b_0, b_1 \rangle\}) | \langle b_0, b_1 \rangle \in c_0\} = \sqcup \{\sqcup \{\nu(\{\langle b_0, b_1 \rangle\}) | \langle b_0, b_1 \rangle \in c\} | c \in d\} = \sqcup \nu d.$$

Property 1 is verified.

From the fact that the right-hand side of the equivalence (3) is a Σ-relation, there follows that

$$\angle_0 \rightleftharpoons \{\langle c_0, c_1 \rangle | c_0, c_1 \in \mathcal{C}, \nu c_0 \leq \nu c_1\}$$

is a Σ-set. If $d \in Con_{\mathcal{C}(\mathfrak{X},\mathfrak{Y}),\nu}$, $c \in \mathcal{C}$, then $\nu c = \sqcup \nu d \Longleftrightarrow \nu c \leq \nu(\cup d) \wedge \nu(\cup d) \leq \nu c$; from here there follows that

$$\angle = \{\langle d, c \rangle | \ d \in Con_{\mathcal{C}(\mathfrak{X},\mathfrak{Y}),\nu}, \ c \in \mathcal{C}, \nu c = \sqcup \nu d\}$$

is a Σ-set. Condition 2 is also verified.

Condition 3 also holds since for $\mu \in C(\mathfrak{X},\mathfrak{Y})$ we have

$$B_\mu = \angle_\mu^* .$$

It remains to verify the completeness of the f_A-space

$$\langle C(\mathfrak{X},\mathfrak{Y}),\ C_0(\mathfrak{X},\mathfrak{Y}),\ \leqslant,\ \nu : C \to C_0(\mathfrak{X},\mathfrak{Y})\rangle .$$

(which will be denoted simply by $C(\mathfrak{X},\mathfrak{Y})$). Assume that the Σ-set $B \subseteq C$ satisfies conditions 1 and 2 for the sets B_ξ. In particular, from condition 2 there follows that for any $d \in B^*$ $d \in Con_{C(\mathfrak{X},\mathfrak{Y}),\nu}$ and, consequently, $\cup B$ satisfies condition (2). By Proposition 2, the mapping $\mu_{\cup B}$ belongs to $C(\mathfrak{X},\mathfrak{Y})$. A simple verification shows that $B_{\mu_{\cup B}} = \angle^*_{\mu_{\cup B}} = B$.

Remark. If we consider the category $\mathbb{F}_A$ of f_A-spaces (the objects of this category are the f_A-spaces and the morphisms are the computable mappings), then it is easy to verify that in this category there exists a direct product $\mathfrak{X} \times \mathfrak{Y}$ for any two f_A-spaces $\mathfrak{X}$ and $\mathfrak{Y}$. The nat natural character of the structure (when it exists) of the f_A-space $C(\mathfrak{X},\mathfrak{Y})$ is confirmed by the following two easily verifiable facts:

1) the f_A-spaces $C(\mathfrak{X}\times\mathfrak{Y},\mathfrak{Z})$ and $C(\mathfrak{X},C(\mathfrak{Y},\mathfrak{Z}))$ (when they exist) are isomorphic (in the category $\mathbb{F}_A$) ;

2) the signification mapping $\upsilon : \mathfrak{X} \times C(\mathfrak{X},\mathfrak{Y}) \to \mathfrak{Y}; (\upsilon(\xi,\mu) \rightleftharpoons \mu(\xi),\ \xi \in X,\ \mu \in C(\mathfrak{X},\mathfrak{Y}),\ \mu(\xi) \in Y)$ is a morphism of $\mathbb{F}_A$.

We indicate a series of sufficient conditions in order that the requirements of Proposition 3 should hold.

1. If $Con_{Y,\nu_1} = B_1^*$, then C is a Σ-set.

Then, obviously, $C = (B_0 \times B_1)^*$.

In the formulation of the subsequent conditions we shall assume that the following condition holds:

$B_0^* \setminus Con_{X,\nu_0}$ is a Σ-set (Δ-Con).

II. If in the admissible set $\mathbb{A}$ there exists a Σ-function P such that $P(a) = \{b \mid b \in A,\ b \subseteq a\}$ for all $a \in A^*$, then under the condition Δ-Con on $\mathfrak{X}$ the set C is a Σ-set.

Indeed,

$$c \in C \Leftrightarrow \forall c' \in P(c)(\delta c \in B_0^* \setminus Con_{X,\nu_0} \vee \rho c \in Con_{Y,\nu_1}).$$

We consider two more conditions: for a natural number N

$$B_0^N \setminus \{\langle b_1,\dots,b_N\rangle \mid \{b_1,\dots,b_N\} \in Con_{X,\nu_0}\} \text{ is a } \Sigma\text{-set} \quad (\Delta(N)\text{-}Con);$$

$$\forall c \in B_0^* \,(c \in Con_{X,\nu_0} \Leftrightarrow \forall b_1,\dots,b_N \in c(\{b_1,\dots,b_N\} \in Con_{X,\nu_0})) \quad (Con(N)).$$

III. If there exists N such that $\mathfrak{X}$ satisfies the condition Δ(N)-Con, while $\mathfrak{Y}$ satisfies condition Con(N), then C is a Σ-set.

Indeed,

$$c \in C \Leftrightarrow \forall \langle b_0^1, b_1^1\rangle, \ldots, \langle b_0^N, b_1^N\rangle \in c\,(\langle b_0^1, \ldots, b_0^N\rangle \in$$
$$\in B_0^N \setminus \{\langle b_1, \ldots, b_N\rangle \mid \{b_1, \ldots, b_N\} \in Con_{X,v_0}\} \vee \{b_1^1, \ldots, b_1^N\} \in Con_{Y,v_1}).$$

Some of the derived properties can be carried over to the space of computable mappings $C(\mathfrak{X}, \mathfrak{Y})$. For example, we have the following

<u>THEOREM.</u> a) For any admissible set $\mathbb{A}$ and for any natural number N the category $\mathbb{F}^*_{\mathbb{A},N}$ of complete f_A-spaces, satisfying the properties $\Delta(N)$-Con and Con(N), is Cartesian closed.

b) for an admissible set $\mathbb{A}$, having a Σ-function P such that $\forall a(P(a)=\{b \mid b \in A, b \subseteq a\})$, the category $\mathbb{F}^*_{\mathbb{A},\Delta\text{-}Con}$ of complete f_A-spaces, satisfying the property Δ-Con, is Cartesian closed.

Cartesian closedness means the existence of a direct product of objects and (in the considered case) closedness with respect to the formation of the space $C(\mathfrak{X}, \mathfrak{Y})$.

a) The following fact can be easily verified:

If $\mathfrak{X}$ satisfies property $\Delta(N)$-Con while (the complete) $\mathfrak{Y}$ satisfies property Con(N), then $C(\mathfrak{X}, \mathfrak{Y})$ (which exists according to III) satisfies the condition Con(N).

It remains to verify that if $\mathfrak{Y}$ satisfies also the condition $\Delta(N)$-Con, then also $C(\mathfrak{X}, \mathfrak{Y})$ satisfies condition $\Delta(N)$-Con.

We show that $C(\mathfrak{X}, \mathfrak{Y})$ satisfies even condition Δ-Con. If $d \in C^*$ then $d \notin Con_{C(\mathfrak{X},\mathfrak{Y}),v} \Leftrightarrow$

$$\cup d \notin C \Leftrightarrow \exists \langle b_0^1, b_1^1\rangle, \ldots, \langle b_0^N, b_1^N\rangle \in \cup d \;\; (\{b_0^1, \ldots, b_0^N\} \in Con_{X,v_0} \wedge \{b_1^1, \ldots, b_1^N\} \notin Con_{Y,v_1});$$

the latter is the Σ-condition.

b) If the assumption of part b) of the theorem holds for $\mathbb{A}$, and $\mathfrak{X}$ satisfies the condition Δ-Con, and (the complete) $\mathfrak{Y}$ satisfies condition Δ-Con, then for $d \in C^*$ we have

$$d \notin Con_{C(\mathfrak{X},\mathfrak{Y}),v} \Leftrightarrow \cup d \notin C \Leftrightarrow \exists c' \in P(\cup d)(\delta c \in Con_{X,v_0} \wedge \rho c \in B_1^* \setminus Con_{Y,v_1});$$

the latter is the Σ-condition.

The proved theorem allows us to define for any admissible $\mathbb{A}$ the concept of a partially computable (or Σ-) functional of any finite type. Unfortunately, the family F_σ of all such Σ-functionals of any fixed type σ need not have "good" (computable) enumeration, as it has been proved in [2] for predicates, but such an enumeration cannot be achieved even for the type $(0|0)$ in certain admissible sets $\mathbb{A}$ (there where universal Σ-functions are not present).

<u>Remark 1.</u> The families of Σ_σ-predicates for predicates of type σ, constructed in [2], have a natural structure of complete f_A-spaces.

<u>Remark 2.</u> In the definitions of computability, instead of Σ-predicates over an admissible set one can use Σ-predicates over a Σ-admissible set [3].

Remark 3. The sets $\mathbb{A}_\tau$, $\tau \in PT$, constructed in [5], have a natural structure of complete f_A-spaces.

Remark 2 allows us to consider the complete f_0-spaces [1] as a special case of complete f_A-spaces for appropriate Σ-admissible sets $\mathbb{A}$.

Namely, let S be an arbitrary infinite set, let $\mathbb{A} = HF^{\Sigma}(S) \rightleftharpoons \langle HF(S); P(HF(S))\rangle$ be a Σ-admissible set (consisting of HF(S), i.e., all hereditarily-finite sets over S), in which the positive predicate variables run through all subsets of HF(S). Then any subset of HF(S) will be a Σ- (even Δ-) set.

We have the following obvious

Proposition 4. The category $\mathbb{F}^*_{\mathbb{A}}$ of complete f_A-spaces is equivalent to the category $F_0^{|S|}$ of complete f_0-spaces, having a basis of cardinality $\le |S|$.

Proposition 5. Let $\mathfrak{X}$ be a complete f_A-space and let $\mu: X \to X$ be a computable mapping. Then there exists a smallest fixed point $\xi_\mu \in X$ of this mapping. If $C(\mathfrak{X},\mathfrak{X})$ "exists," then the correspondence $\mu \mapsto \xi_\mu$ is a computable mapping from $C(\mathfrak{X},\mathfrak{X})$ into $\mathfrak{X}$.

We prove the first part of the proposition. By Gandy's theorem there exists a smallest set (which is a Σ-set) $P \subseteq B$ such that

1) $b_\perp \in P$ (here $b_\perp$ is such that $\nu b_\perp = \perp$);

2) $b_0 \in P$, $\langle b_0, b_1\rangle \in \angle_\mu \Rightarrow b_1 \in P$.

We verify that P satisfies properties 1 and 2 for B_ξ. Property 1 is obvious.

We verify property 2. We define a transfinite sequence of sets $P_\alpha \subseteq B$, α is an ordinal, such that:

$$P_0 \rightleftharpoons \{b \mid \langle b, b_\perp\rangle \in \angle_0\};$$

$$P_{\alpha+1} \rightleftharpoons \{b \mid \exists b_0 (b_0 \in P_\alpha \wedge \langle b, b_0\rangle \in \angle_\mu)\};$$

$$P_\beta \rightleftharpoons \bigcup_{\alpha<\beta} P_\alpha \quad \text{for a limiting } \beta .$$

We define a transfinite sequence of points ξ_α:

$$\xi_0 \rightleftharpoons \perp; \quad \xi_{\alpha+1} \rightleftharpoons \mu(\xi_\alpha), \quad \xi_\beta \rightleftharpoons \bigsqcup_{\alpha<\beta} \xi_\alpha$$

for a limiting β provided $\bigsqcup$ exists.

By induction with respect to ordinals one establishes that for $\alpha \in Ord(\mathbb{A})$ ξ_α exists and $P_\alpha = B_{\xi_\alpha}$.

Gandy's theorem asserts that

$$P = \bigcup_{\alpha \in Ord(\mathbb{A})} P_\alpha$$

and the set of $\mathcal{D} \rightleftharpoons \{\langle b,\alpha\rangle \mid \alpha \in Ord(\mathbb{A}),\ b \in P_\alpha\}$ is a Σ-set.

If $C \in P^{*}$, then

$$\forall b \in c\ \exists \alpha\ (\alpha \in Ord(\mathbb{A}) \wedge \langle b, \alpha \rangle \in \mathfrak{D});$$

by virtue of the Σ-sample there exists $d \subseteq Ord(\mathbb{A})$ such that

$$\forall b \in c\ \exists \alpha \in d\ (\langle b, \alpha \rangle \in \mathfrak{D}).$$

If $\alpha_0 \rightleftharpoons \bigcup d, \alpha_0 \in Ord(\mathbb{A})$ and $c \subseteq P_{\alpha_0} - B_{\xi_{\alpha_0}}$, then $c \in Con_{X,Y}$ and $b \in P_{\alpha_0} \subseteq P$ for $b \in B$ such that $\langle c, b \rangle \in \mathcal{L}$. Property 2 is verified and, consequently, P is B_{ξ} for some $\xi \in X$; it is easy to see that $\xi_{\mu} \rightleftharpoons \xi$ and it is the smallest fixed point for μ.

The computability of the correspondence $\mu \longmapsto \xi_{\mu}$, when there $C(\mathfrak{X}, \mathfrak{X}) \in \mathbb{F}^{*}_{\mathbb{A}}$ exists, is proved in a similar manner.

Remark. This rpoposition is valid also for Σ-admissible $\mathbb{A}$.

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LINEAR GROUPS OF SMALL DEGREES OVER THE FIELD OF ORDER 2

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UDC 512.542

In the "Kourovka Notebook" [4] V. D. Mazurov posed the following problem 8.39.a): to describe the irreducible subgroups of $GL_n(2)$ for $7 \leq n \leq 10$. The cases n = 7, 8, and 9 of this problem were considered in [2, 3] and in as yet unpublished work by the author. In the present paper the solution of the problem is completed modulo the classification of finite simple groups. We prove the following

THEOREM. Let V be a 10-dimensional vector space over $GF(2)$, $G = GL(V)$, H an irreducible subgroup of G all of whose composition factors are known simple groups. Then one of the following cases holds:

(1) $H \leq A \cong S_3 \wr S_5$, where A is the stabilizer in G of a decomposition of V into a direct sum of five two-dimensional subspaces. All subgroups isomorphic to A are conjugate in G and $A < Sp_{10}(2) < G$. If H is not solvable then $E_{3^4} \leftthreetimes A_5 \leq H$.

(2) $H \leq B \cong GL_5(2) \wr S_2$, where B is the stabilizer in G of a decomposition of V into a direct sum of two five-dimensional subspaces. All subgroups isomorphic to B are conjugate and maximal in G. If H is not solvable then either $H = B$, or $H \cong Aut(L_5(2))$.

Translated from Algebra i Logika, Vol. 25, No. 5, pp. 544-565, September-October, 1986. Original article submitted January 30, 1986.