GROUPS WITH REGULAR ELEMENTARY 2-GROUPS OF AUTOMORPHISMS

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In the early 1960's there arose in the theory of finite groups the following

Conjecture. Suppose G is a finite solvable group, V is a subgroup of $Aut\,G$, $C_G(V)=1$, $(|V|,|G|)=1$, and $|V|$ is the product of n primes, not necessarily distinct. Then the nilpotent length of G is at most n.

It is well known that if the pair V, G satisfies the conditions of the conjecture and $|V|=2$, then G is Abelian. Shult [4] showed that the conjecture is true if V is an elementary 2-group. Bauman [2] proved that if V is a four-group, then the commutant of G is nilpotent. Many other cases of the problem have also been studied [3, 5, 6]. Almost all of the results depend on the theory of representations of finite groups. In the present paper we suggest another approach, which does not require that G be finite. Here consider the case where V is an elementary 2-group and G is periodic.

THEOREM 1. There exists a function $f(x,y)$ of two natural variables such that any k-step solvable, periodic group G admitting a regular elementary group of automorphisms of order 2^n has an invariant series

$$G=H_1\supseteq H_2\supseteq\ldots\supseteq H_{n+1}=1,$$

in which the factors are nilpotent and the nilpotent length of H_i/H_{i+1} is at most $f(i,k)$, $1\leq i\leq n$.

THEOREM 2. Suppose G is a periodic group admitting a regular elementary group of automorphisms of order 2^n. If some term of the derived series of G having a natural subscript is hypercentral, then G has an invariant series $G=H_1\supseteq H_2\supseteq\ldots\supseteq H_{n+1}=1$ in which all of the factors are hypercentral.

It is easy to see that these results are stronger than those of Shult and Bauman. Moreover, they show that in some cases the conjecture can be significantly strengthened.

In connection with Theorem 1 it is appropriate to mention that for any integers $n\geqslant 2$ and $k\geqslant 1$ there exists a k-step solvable, periodic group admitting a regular elementary group of automorphisms of order 2^n.

We also mention that the approach suggested in this paper enables us to obtain a generalization of the theorem of Kreknin and Kostrikin [1] which says that the nilpotent length of a k-step solvable Lie algebra admitting a regular automorphism of prime order p does not exceed some number $h(p,k)$ depending only on p and k. It can be shown that a k-step solvable Lie ring L admitting a regular elementary group of automorphisms of order 2^n has a system of ideals

$$L=L_1\supseteq L_2\supseteq\ldots\supseteq L_{n+1}=0,$$

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such that $\mathcal{L}_i^{f(i,\kappa)} \subseteq \mathcal{L}_{i+1}$ for $1 \le i \le n$.

This paper comprises three sections. In Sec. 1 we establish some general properties of an elementary 2-group of automorphisms, in Sec. 2 we prove the main results, and in Sec. 3 we make concrete the arguments of the preceding sections in the case of a four-groups of automorphisms.

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1. An Elementary 2-Group of Automorphisms

If x is an element of any group and $|x|$ is odd, then $x^{\frac{1}{2}}$ denotes an element y of $\langle x\rangle$ such that $y^2 = x$.

Proposition 1.1. Suppose V is a group of automorphisms of order 2^n of a periodic group G. Assume that each element of $C_G(V)$ has odd order. Then G has no involutions.

Proof. We proceed by induction on n. Suppose $n=1$ and v is an involution of V. Assume G contains an involution i. If $|i\cdot i^v|$ is even, then an involution of $\langle i\cdot i^v\rangle$ is contained in $C_G(V)$, which contradicts the hypothesis of the proposition. Suppose $|i\cdot i^v|$ is odd. Then $(i^v\cdot i)^{\frac{1}{2}}\cdot i$ is contained in $C_G(V)$ and is an involution.

Now suppose $n=\kappa\ge 2$ and the proposition has been proved for $n\le \kappa-1$. Suppose v is an involution of $Z(V)$. If G contains involutions, then, by what was proved above, some of them are contained in $C_G(v)$. Note that $H=C_G(v)$, is obviously a V-admissible subgroup of G, and V induces on it a group of automorphisms of order less than 2^n. Then, by the inductive assumption, $C_H(V)$ contains involutions, which proves the proposition.

Proposition 1.2. Suppose G is a periodic group without involutions and v is an automorphism of order 2. Let $J=\{x\in G;\ x^v = x^{-1}\}$. Then $G = C_G(v)\cdot J = J\cdot C_G(v)$.

Proof. Suppose x is any element of G. Put $g=(x^v x^{-1})^{-\frac{1}{2}}$, $g_1=(x^{-v}x)^{\frac{1}{2}}$, $h=(x^v x^{-1})^{\frac{1}{2}}x$. It is clear that $x=gh=hg_1$, and it is easy to see that $g, g_1\in J$, $h\in C_G(v)$.

Proposition 1.3. Suppose V is a regular elementary group of automorphisms of order 2^n of a periodic group G; W is a subgroup of index 2 in V, and $v\in V-W$. Then:

a) if $x\in C_G(W)$, then $x^v = x^{-1}$;

b) $C_G(W)$ is an Abelian subgroup of G;

c) each element x of $C_G(W)$ is weakly closed in $C_G(W)$, i.e., for any y in G, $x^y\in C_G(W)$ if and only if $x^y = x$.

Proof. Assertions "a" and "b" follow directly from the previous propositions. Let us prove "c."

Suppose $x, x^y\in C_G(W)$. If $n=1$, then G is Abelian by Proposition 1.3, "b," and $x^y = x$. Assume $n=\kappa\ge 2$ and the assertion is true when $n\le\kappa-1$. Suppose $w\in W^\#$, $H = C_G(w)$, and $\overline{V}$, $\overline{W}$ are the subgroups of $Aut\ H$ induced by the actions on H of the groups V, W, respectively. By Proposition 1.2, $y=hg$, where $h\in H$, $g^w = g^{-1}$. We have $x^y = x^{hg} = (x^{hy})^w = x^{hg^{-1}}$, hence $x^y = x^h$. If $\overline{V}=\overline{W}$, then $1 = C_H(\overline{V}) = C_H(\overline{W}) = C_G(W)$, hence $x=1$ and $x^y = x$. If $\overline{V}\ne\overline{W}$, then $[\overline{V}:\overline{W}]=2$. It is clear that $|\overline{V}|<|V|$, and therefore, by the inductive assumption, $x^h = x$ and $x^y = x$.

LEMMA 1.4. Suppose G is a periodic group without involutions, V is a four-subgroup of $Aut\, G$, and v_1, v_2, v_3 are involutions of V. Assume g is an element of G such that $g^{v_3} = g^{-1}$. Then there exist uniquely determined elements a, b of $C_G(v_1)$, $C_G(v_2)$, respectively, such that $a^{v_3} = a^{-1}$, $b^{v_3} = b^{-1}$, $g = bab$.

Proof. Put $G_i = C_G(v_i)$, $\mathcal{J}_i = \{x \in G \,;\, x^{v_i} = x^{-1}\}$, $1 \le i \le 3$. By Proposition 1.2, $g = g_2h_2$, where $g_2 \in \mathcal{J}_2$, $h_2 \in G_2$. Also, $h_2 = g_1h_1$, where $g_1 \in \mathcal{J}_1$, $h_1 \in G_1$, and in view of the V-admissibility of G_2 we may assume that $g_1 \in G_2$ and $h_1 \in G_2$. Since $g^{v_1} \cdot g^{v_2} = 1$, we have $(g_2g_1h_1)^{v_1} = (h_1^{-1} \cdot g_1^{-1}g_2^{-1})^{v_2}$, or

$$g_2^{v_1} g_1^{-1} h_1 = h_1^{-1} g_1^{-1} g_2. \tag{1}$$

Using $g_2^{v_1} \in \mathcal{J}_2$ and applying the automorphism v_2 to $h_1^{-1}g_1^{-1}g_2h_1^{-1}g_1$, we obtain the relation $g_1^{-1}h_1g_2^{-1}g_1h_1 = h_1^{-1}g_1^{-1}g_2^{-1}h_1^{-1}g_1$, or $g_2^{-1}g_1h_1g_1^{-1}h_1g_2 = h_1^{-1}g_1h_1^{-1}g_1^{-1}$. Since in a periodic group without involutions there is no nonidentity element conjugate to its inverse, it follows easily from the last equality that $h_1 = 1$. Then relation (1) assumes the form $g_2^{v_1}g_1^{-1} = g_1^{-1}g_2$, hence $g_2 \in C_G(g_1v_1)$. But $g_1v_1 = g_1^{\frac{1}{2}}v_1g_1^{-\frac{1}{2}}$, hence $g_2 \in g_1^{\frac{1}{2}}G_1g_1^{-\frac{1}{2}}$. Put $g_1^{\frac{1}{2}} = b$, $b^{-1}g_2b = a$. Then $g = bab$ and the pair a, b obviously satisfies the conclusion of the lemma.

Now assume the pair a_1, b_1 also satisfies the conclusion of the lemma. Then $b_1^{-1}babb_1^{-1} \in G_1$. Applying to $b_1^{-1}babb_1^{-1}$ the automorphism v_1, we obtain $b_1^{-1}babb_1^{-1} = b_1b^{-1}ab^{-1}b_1$. It follows easily that $bb_1^{-2}b$ is conjugate in G to its inverse, hence $bb_1^{-2}b = 1$. Then $b = b_1$ and and $a = a_1$. The lemma is proved.

Proposition 1.5. Suppose the conditions of Lemma 1.4 are satisfied and T is a V-admissible subgroup of G such that $g \in T$. Then a, $b \in T$.

Proof. This follows from the V-admissibility of T and the uniqueness of the pair a, b proved in Lemma 1.4.

LEMMA 1.6. Suppose G is a periodic group without involutions, V is an elementary subgroup of order 2^n of $Aut\, G$, and M is the set of all maximal subgroups of V. Then there exists an ordering $V_1, V_2, \ldots, V_{2^n-1}$ of the set M such that $G = C_G(V_1) \cdot C_G(V_2) \cdot \ldots \cdot C_G(V_{2^n-1})$.

Proof. When $n=1$ the lemma is obvious. Suppose $n=2$, $V^{\#} = \{v_1, v_2, v_3\}$; G_i and $\mathcal{J}_i$ are the same as in Lemma 1.4, $1 \le i \le 3$; and x is any element of G. By Proposition 1.2, for some g_3 in $\mathcal{J}_3$ we have $x \in g_3G_3$. Clearly $g_3^2 \in \mathcal{J}_3$, hence, by Lemma 1.4, $g_3^2 = ab^2a$, where a and b are elements of $G_1 \cap \mathcal{J}_3$ and $G_2 \cap \mathcal{J}_3$, respectively. Then $b^{-1}a^{-1}g_3 = bag_3^{-1}$. This shows that $b^{-1}a^{-1}g_3 \in G_3$, since $(b^{-1}a^{-1}g_3)^{v_3} = bag_3^{-1}$. Therefore $g_3 \in abG_3$ and $x \in G_1G_2G_3$, as required. Note that we have proved that when $n=2$ any ordering of the set M satisfies the conclusion of the lemma.

Now suppose that $n \ge 3$ and the lemma has been proved for $|V| \le 2^{n-1}$. Let V_1 be any element of M. By the inductive assumption, there exists an ordering $W_1, W_2, \ldots, W_{2^{n-1}-1}$ of the set of maximal subgroups of V_1 satisfying the desired conditions. For each $i = 1, 2, \ldots, 2^{n-1}-1$ there exist two elements of V_1, different from M, containing W_i. We assign to one of them, arbitrarily, the number $2i$, and to the other the number $-2i+1$. It is easy to see that we obtain as a result an ordering of M. We will show it is the desired one. Put $G_i = C_G(V_i)$, $1 \le i \le 2^n - 1$. By the inductive assumption, $G = X_1 \cdot X_2 \cdot \ldots \cdot X_{2^{n-1}-1}$, where $X_i = C_G(W_i)$, $1 \le i \le 2^{n-1}-1$. Suppose U_i is a four-subgroup of V such that $W_i \cap U_i = 1$, $U_i^{\#} = \{t_i, u_i, v_i\}$, and $t_i \in V_1$, $u_i \in V_{2i}$, $v_i \in V_{2i+1}$, $1 \le i \le 2^{n-1} - 1$. By the assertion of the lemma for $n=2$ proved above we have $X_i = Y_i^1 Y_i^2 Y_i^3$, where

$$Y_i^1 = C_G(t_i) \cap X_i = C_G(V_i),$$
$$Y_i^2 = C_G(u_i) \cap X_i = C_G(V_{2i}),$$
$$Y_i^3 = C_G(v_i) \cap X_i = C_G(V_{2i+1}).$$

Thus we have shown that

$$G = G_1 \cdot G_2 \cdot G_3 \cdot G_1 \cdot G_4 \cdot G_5 \cdot \ldots \cdot G_1 \cdot G_{2^n-2} \cdot G_{2^n-1}.$$

Since when $n=2$ any ordering of M satisfies the conclusion of the lemma, we have $G_1 \cdot G_{2k} \cdot G_{2k+1} = G_{2k} \cdot G_{2k+1} \cdot G_1$ for any k in $\{1, 2, \ldots, 2^{n-1} - 1\}$. But then $G_1 G_{2k} G_{2k+1} G_1 = G_1 G_{2k} G_{2k+1}$, hence $G = G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot \ldots \cdot G_{2^n-1}$. The lemma is proved.

LEMMA 1.7. Suppose G is a periodic group without involutions and V is an elementary subgroup of order 2^n of $\mathrm{Aut}\, G$. If N is a normal V-admissible subgroup of G, then

$$C_{G/N}(V) = C_G(V) \cdot N/N.$$

Proof. We proceed by induction on n. Suppose $n=1$ and x is an element of G such that $xN \in C_{G/N}(V)$. If v is an involution of V, then $x^{-v}x \in N$. But $x = x(x^{-v}x)^{-\frac{1}{2}}(x^{-v}x)^{\frac{1}{2}}$, where $x(x^{-v}x)^{-\frac{1}{2}} \in C_G(v)$. Consequently, $xN \in C_G(V) \cdot N/N$. Now suppose $n = k \geq 2$ and the lemma is true for $n \le k-1$. Suppose v is an involution of V. Put $H = C_G(v)$ and let W be the subgroup of $\mathrm{Aut}\, H$ induced by V. Clearly $C_G(V) = C_H(W)$ and $|W| \le 2^{n-1}$. Assume x is an element of G such that $xN \in C_{G/N}(V)$. As above, in the equality $x = x(x^{-v}x)^{-\frac{1}{2}}(x^{-v}x)^{\frac{1}{2}}$ we have $(x^{-v}x)^{\frac{1}{2}} \in N$, $y = x(x^{-v}x)^{-\frac{1}{2}} \in H$ It is easy to see that $y \cdot (H \cap N) \in C_{H/H \cap N}(W)$, hence, by the inductive assumption, $y = ht$, where $h \in C_H(W)$, $t \in H \cap N$. But then $xN \in C_G(V) \cdot N/N$. Thus we have shown that $C_{G/N}(V) \subseteq C_G(V) \cdot N/N$. The reverse inclusion is obvious.

Recall that if a is an automorphism of an arbitrary group G, then $[a, G]$ denotes the subgroup of G generated by all elements of the form $x^{-a}x$, where $x \in G$. It is known that $[a, G]$ is always normal in G and can be defined as the smallest normal R-admissible subgroup a of G such that a induces the identity automorphism of the factor group G/R.

If A is a group of automorphisms of a group G, we put $G_A = \bigcap_{a \in A^{\#}} [a, G]$.

Proposition 1.8. Suppose A is a group of automorphisms of a group G and N is a normal A-admissible subgroup of G. Then $G_A \cdot N/N \subseteq (G/N)_A$.

LEMMA 1.9. Suppose G is a periodic group admitting a nontrivial regular elementary group of automorphisms of order 2^n. Then there exists a periodic group $\mathfrak{D}$ admitting a regular elementary 2-group of automorphisms of order at most 2^{n-1} having a subgroup isomorphic to the factor group G/G_V. If the K-th term of the derived series of G is trivial or hypercentral, then the K-th term of the derived series of $\mathfrak{D}$ is the same.

Proof. Suppose A is an abstract elementary group of order 2^{n-1}. For each v in $V^{\#}$ we define V_v to be the group of automorphisms of $G/[v,G]$ induced by the action of V on G. Clearly $|V_v| \leq 2^{n-1}$. We take as $\mathfrak{D}$ the direct product of the factor groups $G/[v,G]$, $v \in V^{\#}$. Suppose φ is a homomorphism from A into $Aut\,\mathfrak{D}$ such that the restriction of A^{φ} to each factor $G/[v,G]$ agrees with V_v. Using Lemma 1.7, it is easy to see that $C_{\mathfrak{D}}(A^{\varphi}) = 1$. Since by Remak's theorem G/G_V is embedded in $\mathfrak{D}$, the lemma is proved.

2. Main Results

In this section and the next, Lemma 1.7 will be used without explicit reference.

Suppose V is a regular elementary group of automorphisms of order 2^n, $n \geq 2$, of a periodic group G, and $V_1, V_2, \ldots, V_{2^n-1}$ is a fixed ordering of the set of maximal subgroups of V, and let $G_i = C_G(V_i)$, $0 \leq i \leq 2^n - 1$, where $V_0 = V$, $\Omega = \{0, 1, 2, \ldots, 2^n - 1\}$.

We introduce on Ω a binary operation $\circ$ as follows: If λ and μ are elements of Ω, then

$$\lambda \circ \mu = \mu \circ \lambda = \begin{cases} 0, & \text{if } \lambda = \mu; \\ \lambda, & \text{if } \mu = 0; \\ \nu, & \text{if } \lambda \neq \mu, \lambda \neq 0, \mu \neq 0; \end{cases}$$

where ν is defined by the conditions $(V_\lambda \cap V_\mu) \subset V_\nu$ and $\nu \notin \{\lambda, \mu\}$.

It can be verified directly that $(\Omega, \circ)$ is an elementary 2-group and $\Omega^{\#} = \{1, 2, \ldots, 2^n - 1\}$.

Suppose W is a subgroup of V. Put $\Omega(W) = \{\omega \in \Omega ; W \subseteq V_\omega\}$. It is easy to see that $\Omega(W)$ is a subgroup of Ω.

Proposition 2.1. Suppose Σ is any subgroup of Ω. If $W = \bigcap_{\sigma \in \Sigma} V_\sigma$, then $\Sigma = \Omega(W)$.

Proof. This follows from the definition.

Proposition 2.2. Suppose a and b are any elements of G_λ and G_μ, $\mu \neq 0$, respectively. Then $b^{-1}ab$ can be uniquely represented in the form $b^{-1}ab = ca_1c$, where $a_1 \in G_\lambda$, $c \in G_{\lambda \circ \mu}$.

Proof. It is clear that we need only consider the case $0 \notin \{\lambda, \lambda \circ \mu\}$. Suppose $W = V_\lambda \cap V_\mu$ and $v_1 \in V_{\lambda \circ \mu} - W$, $v_2 \in V_\mu - W$. Then $v_1 \cdot v_2 \in V_\lambda - W$ and $b^{-1}ab \in C_G(W)$. By Proposition 1.3, "a," $(b^{-1}ab)^{v_2} = b^{-1}a^{-1}b$, hence, by Lemma 1.4, there exist elements a_1 of $C_G(v_1)$ and c of $C_G(v_1 \cdot v_2)$ such that $b^{-1}ab = ca_1c$. By Proposition 1.5, a_1 and c are contained in $C_G(W)$ hence $a_1 \in C_G(V_\lambda), c \in C_G(V_{\lambda \circ \mu})$, as required.

The element c whose existence is asserted in Proposition 2.2 will be denoted by $a * b$. If A and B are nonempty subsets of G_λ and G_μ, $\mu \neq 0$, respectively, then we put $A * B = \{a * b; a \in A, b \in B\}$. Now suppose $A_1, A_2, \ldots, A_r$ are nonempty subsets of $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_r}$, respectively, where $\alpha_1 \in \Omega, \alpha_2, \ldots, \alpha_r \in \Omega^{\#}$. By induction we put $A_1 * A_2 * \ldots * A_r = (A_1 * A_2 * \ldots * A_{r-1}) * A_r$ for $r \geq 3$.

LEMMA 2.3. Suppose σ is an automorphism in $V^{\tau-1}$. Then

$$[\sigma, G] = \langle G_\alpha ; \alpha \in \Omega - \Omega(\langle\sigma\rangle)\rangle.$$

Proof. Suppose $\alpha \in \Omega - \Omega(\langle\sigma\rangle)$ and $\beta \in \Omega(\langle\sigma\rangle)$. Clearly $\alpha\circ\beta \in \Omega - \Omega(\langle\sigma\rangle)$. It follows easily from Proposition 2.2 that G_β normalizes the subgroup $\langle G_\alpha ; \alpha \in \Omega - \Omega(\langle\sigma\rangle)\rangle$. Since, by Lemma 1.6, G is generated by the subgroups of the form G_ω, $\omega \in \Omega$, we see that $\langle G_\alpha ; \alpha \in \Omega - \Omega(\langle\sigma\rangle)\rangle$ is normal in G. It is clear that σ induces the identity automorphism of the factor group $G/\langle G_\alpha ; \alpha \in \Omega - \Omega(\langle\sigma\rangle)\rangle$, hence $[\sigma, G] \subseteq \langle G_\alpha ; \alpha \in \Omega - \Omega(\langle\sigma\rangle)\rangle$. We will establish the reverse inclusion. Suppose α is an element of Ω such that $\sigma \notin V_\alpha$, and let a be any element of G_α. Then, by Proposition 1.3, "a," $a^\sigma = a^{-1}$. Since extraction of a square root in G is possible, we have $a = \sigma a^{-\frac{1}{2}} \sigma a^{\frac{1}{2}}$, hence $a \in [\sigma, G]$. The lemma is proved.

Proposition 2.4. Suppose a and b are elements of G_α and G_β, respectively, and let Q be the normal closure of $a * b$ in the group $\langle a, b\rangle$. Then Q is the commutant of $\langle a, b\rangle$.

Proof. Note first that, by Proposition 1.5, $a * b \in \langle a, b\rangle$, so that the proposition is properly formulated. By Proposition 1.3, "c," the elements a and b commute if and only if $a * b = 1$, which implies the desired result.

Proposition 2.5. Let $S = \langle G_\alpha * G_\beta ; \alpha, \beta \in \Omega^{\#}\rangle$. Then $\langle S^G\rangle$ is the commutant of G.

Proof. The inclusion $G' \subseteq \langle S^G\rangle$ is a direct consequence of the previous proposition and the commutativity of the subgroups G_α, $\alpha \in \Omega$. If the reverse inclusion is false, there exist elements a and b of G_α and G_β, respectively, such that $a * b \notin G'$, but this contradicts the previous proposition. The proposition is proved.

Suppose κ is a nonnegative integer, $G^{(\kappa)}$ is the κ-th term of the derived series of G, and ω is an element of Ω. Put $G_\omega^\kappa = G_\omega \cap G^{(\kappa)}$.

LEMMA 2.6. Suppose Σ is a subgroup of Ω and Σ_1 is some coset of Σ. Suppose also that τ is a positive integer and $\Sigma_1 = A \cup B$ is a partition of Σ_1 into two disjoint subsets. If

$$L_\omega = \begin{cases} G_\omega^\tau & \text{if } \omega \in A; \\ G_\omega^{\tau-1} & \text{if } \omega \in B, \end{cases}$$

then $\langle G_\sigma^{\tau-1} ; \sigma \in \Sigma\rangle \subseteq N_G(\langle L_\omega ; \omega \in \Sigma_1\rangle)$.

Proof. Suppose $b \in G_\sigma$, where $\sigma \in \Sigma$, and $a \in L_\omega$, where $\omega \in \Sigma_1$. By Proposition 2.2, $b^{-1} a b = c a_1 c$, where $c \in G_{\sigma\circ\omega}, a_1 \in G_\omega$. On the other hand, $b^{-1} a b \in G^{(s)}$, where

$$s = \begin{cases} \tau - 1 & \text{if } \omega \in B; \\ \tau & \text{if } \omega \in A, \end{cases}$$

hence, by Proposition 1.5, $a_1 \in G_\omega^s$ and $c \in G_{\sigma\circ\omega}^s$. Since, by Proposition 2.5, $c \in (G^{(\tau-1)})^{(1)} = G^{(\tau)}$, we conclude that $a_1, c \in \langle L_\omega ; \omega \in \Sigma_1\rangle$ and $b^{-1} a b \in \langle L_\omega ; \omega \in \Sigma_1\rangle$, as required.

Suppose $Z = C_G(G^{(\kappa)})$ is the centralizer of the κ-th term of the derived series of G, and let $Z_\lambda = Z \cap G_\lambda$ $\lambda \in \Omega$. Assume that P is a nonempty subset of Z and α is an element of Ω. Then $\alpha[P]$ is defined to be the smallest integer m such that $P \subseteq C_G(G_\alpha^m)$.

If A is any subset of Ω, then by $A[P]$ we mean the following subset of A: $\{\alpha\in A; \alpha[P]\geq \beta[P]$ for each β in $A\}$.

If Σ is a nontrivial subgroup of Ω, then $\Sigma[P]$ is the largest number m such that Σ has a system of generators $\sigma_1,\sigma_2,\dots,\sigma_r$ for which $\sigma_j[P]\geq m$, $1\leq j\leq r$.

Suppose

$$\Omega=\Omega_1\supset\Omega_2\supset\dots\supset\Omega_n\supset\Omega_{n+1}=\{0\} \tag{2}$$

is any nest of distinct subgroups of Ω, denoted by $\mathfrak{Z}$. Then $(P,\mathfrak{Z})$ is the integral procession $(x_1,x_2,\dots,x_{2n-3})$ of dimension $2n-3$ defined as follows:

1) $x_1=\Omega[P]$;

2) $$x_{2m}=\begin{cases}0, & \text{if } x_{2m-1}=0,\\ |\overline{(\Omega_m-\Omega_{m+1})}[P]|, & \text{if } x_{2m-1}\neq 0,\end{cases}\quad 1\leq m\leq n-2;$$

3) $$x_{2m+1}=\begin{cases}0, & \text{if } x_{2m}=0;\\ \Omega_{m+1}[P], & \text{if } x_{2m}\neq 0,\end{cases}\quad 1\leq m\leq n-2.$$

If a nest $\mathfrak{Z}_1$ of the form (2) is such that for any other analogous nest $\mathfrak{Z}_2$ we have $(P,\mathfrak{Z}_1)\leq(P,\mathfrak{Z}_2)$, then instead of $(P,\mathfrak{Z}_1)$ we will write $\delta(P)$. Here the symbol $\leq$ is to be understood in the sense of the lexicographic order defined on the set of all integral processions of dimension $2n-3$.

Now suppose P is a nonempty subset of Z_λ for some λ in Ω and $\mathfrak{Z}$ is a nest of the form (2). By Proposition 1.5 and Proposition 2.2, for any μ in $\Omega^\#$ we have the inclusion $P*G_\mu\subseteq Z_{\lambda\circ\mu}$, hence the expression $(P*G_\mu,\mathfrak{Z})$ makes sense. We choose from each set $\overline{(\Omega_i-\Omega_{i+1})}[P]$ an element β_i, $1\leq i\leq n$. Viewing Ω as a vector space over the field of two elements, we note that $\beta_1,\beta_2,\dots,\beta_n$ is a basis of Ω. Let $\lambda=\beta_{i_1}\circ\beta_{i_2}\circ\dots\circ\beta_{i_s}$ be the representation of λ in this basis, $1\leq i_1<i_2<\dots<i_s\leq n$. Put $b_r=\beta_r[P]$, $1\leq r\leq n$; $a_q=b_{i_q}$, $\alpha_q=\beta_{i_q}$, $1\leq q\leq s$.

LEMMA 2.7. Suppose $0<b_1\leq b_2\leq\dots\leq b_{i_s}$. Then $(P,\mathfrak{Z})>(P*\Gamma_s*\dots*\Gamma_2,\mathfrak{Z})$ where $\Gamma_q=G_{\alpha_q}^{a_q-1}$, $2\leq q\leq s$.

Proof. Note first that the statement of the lemma makes no sense if $s\leq 1$. We will show that under the conditions of the lemma $s\geq 2$. Suppose $s=0$. Then $\lambda=0$ and P consists of the identity element of G. But then $(P,\mathfrak{Z})$ is the zero procession, which contradicts the condition $0<b_1$. Suppose $s=1$. Then $\lambda=\beta_r$ for some r in $\{1,2,\dots,n\}$, whereas $\lambda[P]=0$ in view of the commutativity of G_λ, which again leads to a contradiction $0<b_1\leq 0$. Now suppose $s\geq 2$ and $(P,\mathfrak{Z})=(x_1,x_2,\dots,x_{2n-3})$, $(P*\Gamma_s*\dots*\Gamma_2,\mathfrak{Z})=(y_1,y_2,\dots,y_{2n-3})$. For each σ in $\Omega_r-\Omega_{r+1}$, where $1\leq r\leq i_s-1$, we put

$$\Delta_\sigma=\begin{cases}G_\sigma^{b_r}, & \text{if } \sigma\in\overline{(\Omega_r-\Omega_{r+1})}[P];\\ G_\sigma^{b_r-1}, & \text{if } \sigma\notin\overline{(\Omega_r-\Omega_{r+1})}[P].\end{cases}$$

Let $H_q = \langle L_G; G \in \Omega - \Omega_{i_q} \rangle, 2 \leq q \leq s$. It is not difficult to show that $P \subseteq C_G(H_q)$, $2 \leq q \leq s$. Since $b_1 \leq b_2 \leq \ldots \leq b_{i_2}$, we obtain from Lemma 2.6 the inclusion $\Gamma_q \subseteq N_G(H_q)$, $2 \leq q \leq s$. Consequently, $\langle P^{\Gamma_s} \rangle \subseteq C_G(H_s)$, hence, by Proposition 1.5, $P * \Gamma_s$ is contained in $C_G(H_s)$. By the same reasoning,

$$P * \Gamma_s * \Gamma_{s-1} \subseteq C_G(H_{s-1}), \ldots, P * \Gamma_s * \ldots * \Gamma_2 \subseteq C_G(H_2).$$

The last inclusion shows that for $r < i_2$ we have $\Omega_r[P] \geq \Omega_r[P * \Gamma_s * \ldots * \Gamma_2]$, and if $\Omega_r[P] = \Omega_r[P * \Gamma_s * \ldots * \Gamma_2]$, then $(\overline{\Omega_r - \Omega_{r+1}})[P] \supseteq (\overline{\Omega_r - \Omega_{r+1}})[P * \Gamma_s * \ldots * \Gamma_2]$. Thus for $1 \leq r \leq i_2$ we have $x_{2r-1} \geq y_{2r-1}$, and in the case $x_{2r-1} = y_{2r-1}$ we also have $x_{2r} \geq y_{2r}$.

Let us now assume the lemma is false and $(P, \mathfrak{Z}) \leq (P * \Gamma_s * \ldots \Gamma_2, \mathfrak{Z})$. Then, by what was proved above, $x_j = y_j$, if $1 \leq j \leq 2(i_2 - 1)$. Let m be the largest number such that $\lambda \in \Omega_m$. It is easy to see that it is also the largest such that $\alpha_1 \in \Omega_m$. Suppose first that $m < n-1$. The equality $x_{2m} = y_{2m}$ means that $(\overline{\Omega_m - \Omega_{m+1}})[P] = (\overline{\Omega_m - \Omega_{m+1}})[P * \Gamma_s * \ldots * \Gamma_2]$, hence $\alpha_1 \in (\overline{\Omega_m - \Omega_{m+1}})[P * \Gamma_s * \ldots * \Gamma_2]$. But, by Proposition 2.2, the set $P * \Gamma_s * \ldots * \Gamma_2$ is contained in $G_{\lambda \cdot \lambda_s \cdot \ldots \cdot \lambda_2} = G_{\alpha_1}$, hence $\alpha_1[P * \Gamma_s * \ldots * \Gamma_2] = 0$. Then $\alpha_1[P] = 0$, which contradicts $0 < b_1 \leq b_2$. The proof for $m = n-1$ differs from the above only in that we cannot use the equality $x_{2m} = y_{2m}$, since it is absent. This is inessential, since when $|\Omega_m| = 4$ the same argument yields $s = 2$ and $\Omega_m[P] > \Omega_m \times [P * \Gamma_2]$. The lemma is proved.

Suppose n and k are nonnegative integers, $n \geq 2$. We denote by $\mathfrak{M}_{n,k}$ the set of all integral processions $X = (x_1, x_2, \ldots, x_{2n-3})$ of dimension $2n-3$ whose coordinates satisfy these conditions:

1) if $x_1 = 0$, then $x_2 = x_3 = \ldots = x_{2n-3} = 0$;
2) $0 \leq x_1 \leq k$;
3) if $x_1 \neq 0$, then $1 \leq x_{2m-1} \leq x_{2m+1} \leq k$; $1 \leq m \leq n-2$;
4) if $x_{2m-1} \neq 0$, then $1 \leq x_{2m} \leq 2^{n-m}$; $1 \leq m \leq n-2$.

It can be shown that $|\mathfrak{M}_{n,k}| = 1 + \Delta(n,k)$, where

$$\Delta(n,k) = 2^{\frac{(n+1)(n-2)}{2}} \cdot \binom{k+n-2}{n-1}.$$

LEMMA 2.8. If P is a subset of $C_G(G^{(k)})$, then

$$\delta(P) \in \mathfrak{M}_{n,k}.$$

Proof. Suppose $\delta(P) = (x_1, x_2, \ldots, x_{2n-3})$. It is clear that to prove the lemma it suffices to establish the inequalities $x_{2m-1} \leq x_{2m+1}$, $1 \leq m \leq n-2$. If $x_1 = 0$, then $\delta(P)$ is the zero procession and the lemma is true. Suppose $x_1 > 0$ and m is a number for which $x_{2m-1} > x_{2m+1}$. Let $\mathfrak{Z}$ be a nest of the form (2) such that $(P, \mathfrak{Z}) = \delta(P)$. Our aim is to show that under these assumptions there exists a nest $\mathfrak{Z}_1$ of the form (2) such that $(P, \mathfrak{Z}) > (P, \mathfrak{Z}_1)$. We would then, of course, have a contradiction to the definition of $\delta(P)$. Suppose $\alpha \in (\overline{\Omega_m - \Omega_{m+1}})[P]$. Then for any β in $\Omega_{m+1} - \Omega_{m+2}$ we have $\beta[P] < \alpha[P]$. Put $\Sigma = \langle \Omega_{m+2}, \alpha \rangle$, and as the desired nest $\mathcal{B}_1$ take the sequence $\Omega_1 \supset \Omega_2 \supset \ldots \supset \Omega_m \supset \Sigma \supset \Omega_{m+2} \supset \ldots \supset \Omega_{n+1}$. Suppose $(P, \mathcal{B}_1) = (y_1, y_2, \ldots, y_{2n-3})$. It is easy to see that $y_1 = x_1$, $y_2 = x_2, \ldots, y_{2m-1} = x_{2m-1}$, $y_{2m} < x_{2m}$. The lemma is proved.

Let

$$C_1(G) = C_G(G_V),$$

$$C_{i+1}(G)/C_i(G) = C_{G/C_i(G)}((G/C_i(G))_V),\ i = 1, 2, \ldots .$$

<u>Proposition 2.9.</u> Suppose m is a natural number. Then $C_m(G) \cap G_V$ is contained in the m-th hypercenter of G_V.

<u>Proof.</u> This follows from Proposition 1.8.

Suppose $X_0 < X_1 < \ldots < X_{\Delta(n,k)}$ is the lexicographic ordering of the set $\mathfrak{M}_{n,k}$.

<u>LEMMA 2.10.</u> If P is a nonempty subset of Z_λ, where $\lambda \in \Omega$, and $\delta(P) = X_t$, where $0 \le t \le \Delta(n,k)$, then $P \subseteq C_{n^t}(G)$.

<u>Proof.</u> We proceed by induction on t. If $t = 0$, then $\delta(P)$ is the zero procession and $\Omega[P] = 0$. Therefore, Ω contains a proper subgroup Σ such that any element μ of Ω such that $\mu[P] > 0$ is contained in Σ. By Proposition 2.1 and Lemma 2.3, there exists an automorphism v in $V^\#$ such that $[v, G] \subseteq \langle G_\omega, \omega \in \Omega - \Sigma \rangle$ hence $P \subseteq C_G([v, G]) \subseteq C_1(G)$. Suppose $t > 0$ and assume that if Q is a subset of Z_γ, where $\gamma \in \Omega$, L is any normal V-admissible subgroup of G, $\bar{G} = G/L$, and $\bar{Q}$ is the image of Q in $\bar{G}$ with $\delta(\bar{Q}) < \delta(P)$, then $\bar{Q} \subseteq C_{n^{t-1}}(\bar{G})$. Suppose $\mathcal{B}$ is a nest of the form (2) such that $(P, \mathcal{B}) = \delta(P)$, and let $\beta_1, \beta_2, \ldots, \beta_n$ be a basis of Ω such that $\beta_i \in (\overline{\Omega_i - \Omega_{i+1}})[P]$, $1 \le i \le n$. Then, by Lemma 2.8, $\beta_1[P] \le \beta_2[P] \le \ldots \le \beta_n[P]$, and $0 < \beta_1[P]$, since $t \ne 0$. Let $\lambda = \beta_{i_1} \circ \beta_{i_2} \circ \ldots \circ \beta_{i_s}$ be the representation of λ in this basis, $1 \le i_1 < i_2 < \ldots < i_s \le n$. Put $\alpha_q = \beta_{i_q}$, $a_q = \alpha_q[P]$, $\Gamma_q = G_{\alpha_q}^{a_q - 1}$ $1 \le q \le s$, $\mathcal{D}_0 = P$, $\mathcal{D}_m = \mathcal{D}_{m-1} * \Gamma_{s-m+1}$, where $1 \le m \le s$. Then, by Lemma 2.7, $(\mathcal{D}_{s-1}, \mathcal{B}) \le \delta(P)$, hence, by the inductive assumption, $\mathcal{D}_{s-1} \subseteq C_{n^{t-1}}(G)$. By Proposition 2.4, the commutant $[\Gamma_2, \mathcal{D}_{s-2}]$ is contained in $C_{n^{t-1}}(G)$. Using this fact and arguing as in the proof of Lemma 2.8, we obtain $\delta(\bar{\mathcal{D}}_{s-2}) < \delta(P)$, where $\bar{\mathcal{D}}_{s-2}$ is the image of $\mathcal{D}_{s-2}$ in $\bar{G} = G/C_{n^{t-1}}(G)$. Again by the inductive assumption, $\bar{\mathcal{D}}_{s-2} \subseteq C_{n^{t-1}}(\bar{G})$, or $\mathcal{D}_{s-2} \subseteq C_{2 \cdot n^{t-1}}(G)$. Repeating this argument $s-2$ times, we obtain $P \subseteq C_\tau(G)$, where $\tau = s \cdot n^{t-1}$. Since $s \le n$, it follows that $P \subseteq C_{n^t}(G)$. The lemma is proved.

<u>COROLLARY 2.11.</u> $C_G(G^{(k)}) \subseteq C_\tau(G)$, where $\tau = n^{\Delta(n,k)}$.

<u>COROLLARY 2.12.</u> If $G^{(k)} = 1$, then G_V is nilpotent with nilpotent length at most $\sum_{s=1}^{k} n^{\Delta(n,k-s)}$.

Theorem 1 and Theorem 2 follows from Corollaries 2.11, 2.12, Proposition 2.9, and Lemma 1.9.

<u>Remark.</u> It is easy to see that, in fact, we have also proved

<u>Proposition 2.13.</u> Suppose V is a regular elementary 2-group of automorphisms of a periodic group G. Assume there is a natural number k such that $C_G(G^{(k)}) \ne 1$. Then there exists an automorphism v in $V^\#$ such that $C_G([v, G]) \ne 1$.

3. A Four-Group of Automorphisms

There is no doubt that the estimate for $f(n,k)$ indicated in Corollary 2.12 is not the best possible. In the present section, without setting for ourselves the goal of obtaining an unimprovable estimate, we will show that

$$f(2,k)\leq 2^{k}-k-1.$$

In this section, V is a regular four-group of automorphisms of a periodic group G, $V^{\#}=\{v_1,v_2,v_3\}$, $G_i=C_G(v_i)$, $1\leq i\leq 3$, and $Z_n(G^{(m)})$ is the n-th hypercenter of the m-th term of the derived series of G.

LEMMA 3.1. $G^{(1)}=G_v$.

Proof. By Lemma 1.9, the factor group G/G_v has a regular automorphism of order 2 and therefore, by Proposition 1.3, "b," is Abelian, hence $G^{(1)}\subseteq G_v$. We will prove the reverse inclusion. By Lemma 1.6, $G_V=N_1N_2N_3$, where $N_i=G_i\cap G_V$, $1\leq i\leq 3$. By Lemma 2.3, $[v_1,G]=\langle G_2,G_3\rangle$. Now suppose $a\in N_1$. Since $a\in[v_1,G]$, there exist elements $b_1,b_2,\dots,b_s$ and $c_1,c_2,\dots,c_s$ of G_2 and G_3, respectively, such that $a=b_1c_1b_2c_2\dots b_sc_s$. Using Proposition 1.3, "a," and applying to a the automorphism v_1, we obtain $a=b_1^{-1}c_1^{-1}b_2^{-1}c_2^{-1}\dots b_s^{-1}c_s^{-1}$ or $a^2=b_1c_1b_2\times c_2\dots b_sc_sb_1^{-1}c_1^{-1}b_2^{-1}c_2^{-1}\dots b_s^{-1}c_s^{-1}$, hence $a^2\in G^{(1)}$. But, by Proposition 1.1, $|a|$ is odd, hence $a\in G^{(1)}$. Since a was chosen from N_1 arbitrarily, we conclude that $N_1\subseteq G^{(1)}$. Analogously, $N_2,N_3\subseteq G^{(1)}$, and the lemma is proved.

LEMMA 3.2. Suppose k is a nonnegative integer. Then:

a) $C_G([v_i,G^{(k)}])\subseteq C_{2^k}(G)$; $1\leq i\leq 3$;

b) if $k\geq 1$, then $C_G(G^{(k)})\subseteq C_t(G)$, where $t=2^k-1$.

Proof. We will first prove "a." Since when $k=0$ it is obvious, we may assume $k=s\geq 1$ and assertion "a" holds for $k\leq s-1$. Suppose, for definiteness, that $i=1$. Put $C_G([v_1,G^{(s)}])\cap C_m=Q_m$; $G^{(s)}\cap G_m=S_m$; $G^{(s-1)}\cap G_m=R_m$; $1\leq m\leq 3$. By Lemma 2.3, $[v_1,G^{(s)}]=\langle S_2,S_3\rangle$. Clearly $Q_1\subseteq C_C(\langle R_1,S_2\rangle)$. By Lemma 2.6, $R_3\subseteq N_G(\langle R_1,S_2\rangle)$, hence $\langle Q_1^{R_3}\rangle\subseteq C_G(\langle R_1,S_2\rangle)$, and therefore, by Proposition 2.2 and Proposition 1.5, $Q_1*R_3\subseteq C_G(\langle R_1,S_2\rangle)\cap G_2$. Since, by Proposition 1.3, "b," G_2 is Abelian, we have $Q_1*R_3\subseteq C_G(\langle R_1,R_3\rangle)$. But, by Lemma 2.3, $\langle R_1,R_2\rangle$ is precisely $[v_3,G^{(s-1)}]$. Therefore, by the inductive assumption, $Q_1*R_3\subseteq C_{2^{s-1}}(G)$. Then, by Proposition 2.4, the commutant $[Q_1,R_3]$ is contained in $C_{2^{s-1}}(G)$. Using the commutativity of G_1, it is easy to see that $C_{2^{s-1}}(G)$ also contains the commutant $[Q_1,\langle R_1,R_3\rangle]$. Since, by Lemma 2.3, $\langle R_1,R_3\rangle=[v_2,G^{(s-1)}]$, on applying the inductive assumption to the factor group $\overline{G}=G/C_{2^{s-1}}(G)$ we obtain $\overline{Q}_1\subseteq C_{2^{s-1}}(\overline{G})$, where $\overline{Q}_1$ is the image of Q_1 in $\overline{G}$. But then $Q_1\subseteq C_{2^s}(G)$. By the same reasoning, Q_2 and Q_3 are contained in $C_{2^s}(G)$, which, in view of Lemma 1.6, proves "a."

Let us turn to the proof of "b." When $k=1$ it follows immediately from Lemma 3.1. Suppose $k=s\geq 2$ and assume that "b" holds if $k\leq s-1$. Put $P_m=C_G(G^s)\cap G_m$; $1\leq m\leq 3$. It is clear that $P_m\subseteq Q_m$, hence $[P_1,P_3]\subseteq C_{2^{s-1}}(G)$. Analogously, $[P_1,P_2]\subseteq C_{2^{s-1}}(G)$, hence, in view of the commutativity of G_1 and Lemma 1.6, $[P_1, G^{(s-1)}] \subseteq C_{2^{s-1}}(G)$. By the inductive assumption, applied to $\overline{G}$, we have $\overline{P}_1 \subseteq C_t(\overline{G})$, where $t = 2^{s-1}$ and $\overline{P}_1$ is the image of P in $\overline{G}$. Consequently, $P_1 \subseteq C_{2^{s-1}}(G)$. Analogously, $P_2, P_3 \subseteq C_{2^{s-1}}(G)$, hence, by Lemma 1.6, $C_G(G^{(s)}) \subseteq C_{2^{s-1}}(G)$.

COROLLARY 3.3. If G is a periodic group admitting a regular four-group of automorphisms, then $Z_1(G^{(k)}) \subseteq Z_{2^k-1}(G^{(1)})$ for $k = 1, 2, \ldots$.

COROLLARY 3.4. The commutant of a k-step solvable, periodic group admitting a regular four-group of automorphisms is nilpotent of length at most $2^k - k - 1$.

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