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# STRUCTURE OF POWERS OF GENERALIZED INDEX SETS


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In this article we generalize the results of [1-3] in order to relate problems concerning index sets more closely with the theory of complete numerations [4] and treat ordinary multiple reducibility from a single viewpoint.

Let $A=(A,\alpha)$ be a numerated set and let the set $S$ be arbitrary. The set $Map(A,S)$ of all maps from $A$ into $S$ has two natural orderings, which we denote by $\leqslant_m$ and $\leqslant_M$ in order to emphasize their relationship with the corresponding concepts in [1]. Specifically, for $\varphi,\psi\in Map(A,S)$ we set $\varphi\leqslant_m\psi$ if $\varphi\circ\alpha\leqslant\psi\circ\alpha$ (here $\circ$ denote composition of maps and $\leqslant$ denotes reducibility of numerations), and $\varphi\leqslant_M\psi$ if $\varphi=\psi\circ\Phi$ for some morphism $\Phi$ from $A$ into $A$ $\varphi\leqslant_M\psi$ implies that $\varphi\leqslant_m\psi$. Among other things, we will study the preorders $\leqslant_m$ and $\leqslant_M$. We note that we recover the case of "ordinary" index sets by taking $S=\{0,1\}$, in which case we identify $Map(A,S)$ with the family of all subsets of $A$ and $\varphi\circ\alpha$ with the index set $\alpha^{-1}(\{a\in A\mid\varphi(a)=1\})$. We will use some of the terminology in [1].

## 1. AUXILIARY CONCEPTS

We introduce some concepts needed to study the preorders $\leqslant_m,\leqslant_M$. If $(P;\sqsubseteq)$ is a preordered set, then the closure of a set $X\subseteq P$ in $(P;\sqsubseteq)$ is the set $[X]\rightleftharpoons\{y\in P\mid\exists x\in X\,(x\sqsubseteq y\wedge y\sqsubseteq x)\}$. Let $\varphi,\psi$ be maps from $P$ into a preordered set $(P';\sqsubseteq')$; then $\varphi$ is equivalent to $\psi$ if $\forall x\in P\,(\varphi(x)\sqsubseteq'\psi(x)\wedge\psi(x)\sqsubseteq'\varphi(x))$. Two preordered sets $(P;\sqsubseteq)$ and $(P';\sqsubseteq')$ are equivalent if there exist monotone maps $\varphi:P\to P'$, $\varphi':P'\to P$ whose composite $\varphi'\circ\varphi$ is equivalent to the identity map of $P$, and $\varphi\circ\varphi'$ equivalent to the identity map of $P'$.

Let $I$ be a nonempty set. By a discrete generalized semilattice (more precisely, an $I$-discrete semilattice) we mean any algebraic system $(P;\sqsubseteq,\{P_i\}_{i\in I})$ satisfying the following conditions: 1) $\sqsubseteq$ is a preorder on $P$; 2) $\forall i\in I\,(P_i\subseteq P)$; 3) for all $i,i'\in I$, $i'\neq i$, the proposition $\forall x\forall y\exists z\forall t\,(z\in P_i\wedge x\sqsubseteq z\wedge y\sqsubseteq z\wedge(t\in P_i\wedge x\sqsubseteq t\wedge y\sqsubseteq t\to z\sqsubseteq t)\wedge(t\in P_{i'}\wedge t\sqsubseteq z\to t\sqsubseteq x\vee t\sqsubseteq y))$ is valid in $P$.

The element $z$, whose existence is asserted in 3), is defined uniquely up to equivalence in $(P;\sqsubseteq)$, so that we can define binary operations $u_i\,(i\in I)$ on $P$ $(u_i(x,y)\rightleftharpoons z)$ such that: $x,y\sqsubseteq u_i(x,y)$; if $x,y\sqsubseteq t$, $t\in[\operatorname{rng}u_i]$ then $u_i(x,y)\sqsubseteq t$; if $u_{i'}(x',y')\sqsubseteq u_i(x,y)$, $i'\neq i$, then $u_{i'}(x',y')$


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$\sqsubseteq x$ or $u_{i'}(x', y') \sqsubseteq y$ $(x, y, x', y' \in P,\ i, i' \in I)$. According to this definition $P_i = [rng\, u_i]$. Discrete generalized semilattices may thus be regarded as systems $(P; \sqsubseteq, \{u_i\}_{i \in I})$, where the $u_i$, which are called the generalized upper bound operations in $(P; \sqsubseteq)$, satisfy the above conditions. If such $u_i$ are given, it is easy to define generalized upper bound operations $u_i^{(n)}$ with any number $n \geq 2$ of arguments; for $n=4$, e.g., it suffices to set $u_i^{(4)}(x_1, x_2, x_3, x_4) \rightleftharpoons u_i(u_i(u_i(x_1, x_2), x_3), x_4)$. If $P = \bigcup_{i \in I} [rng\, u_i]$ and $x_1, \ldots, x_n \in P$, then the sets $\{x_1, \ldots, x_n\}$ and $\{u_i^{(n)}(x_1, \ldots, x_n) \mid i \in I\}$ form a discrete pair in the sense of [1, Sec. 1]. We also observe that if $(P; \sqsubseteq, \{u_i\})$ is a preorder with generalized upper bound operations, then $(P; \sqsubseteq, \{[rng\, u_i]\})$ is a discrete generalized semilattice.

We define yet another concept. A semilattice with discrete closures is a system $(Q; \sqsubseteq, \sqcup, \{\varphi_i\}_{i \in I})$ satisfying the conditions:

a) $\sqsubseteq$ is a preorder on the set $Q$;

b) $\forall x, y, z \in Q\ (x \sqsubseteq z \wedge y \sqsubseteq z \longleftrightarrow x \sqcup y \sqsubseteq z)$;

c) $\varphi_i\ (i \in I)$ is the closure operation in $(Q; \sqsubseteq)$, which by definition means that

$\forall x, y \in Q\ (x \sqsubseteq \varphi_i(x) \wedge (x \sqsubseteq y \longrightarrow \varphi_i(x) \sqsubseteq \varphi_i(y)) \wedge \varphi_i \varphi_i(x) \sqsubseteq \varphi_i(x))$;

d) $\forall i \in I\ \forall x, y, z \in Q\ (\varphi_i(x) \sqsubseteq y \sqcup z \longrightarrow \varphi_i(x) \sqsubseteq y \vee \varphi_i(x) \sqsubseteq z)$;

e) $\forall x, x' \in Q\ (\varphi_i(x) \sqsubseteq \varphi_{i'}(x') \longrightarrow \varphi_i(x) \sqsubseteq x')$ for all $i, i' \in I,\ i' \neq i$.

The above concepts are interrelated. Indeed, in any semilattice with discrete closures $(Q; \sqsubseteq, \sqcup, \{\varphi_i\})$ we can define the operations $u_i$: $u_i(x, y) \rightleftharpoons \varphi_i(x \sqcup y)$. If now $P$ is any subset of $Q$, closed with respect to all of the $u_i$, $i \in I$ (it suffices for this that $\bigcup_{i \in I} [rng\, u_i] \subseteq P$) then $(P; \sqsubseteq, \{u_i\})$ is a discrete generalized semiattice (the verification is rivial).

We will also need the following modifications of the above definitions. A system $(P; \sqsubseteq, \{P_{ij}\}_{i,j \in I})$ is called a 2-discrete generalized semilattice if it satisfies conditions 1)-3) (with i replaced by $ij$), and 3) is taken for all $i, j, i', j' \in I$ with $i' \neq i,\ j' \neq j$. A system $(Q; \sqsubseteq, \sqcup, \{\varphi_{ij}\}_{i,j \in I})$ is called a semilattice with 2-discrete closures if it satisfies conditions a)-e) (with i replaced by $ij$), where e) is taken over all $i, j, i', j' \in I,\ i' \neq i, j' \neq j$. As above, generalized upper bound operations $u_{ij}$ can be defined in any 2-discrete generalized semilattice, and any semilattice with 2-discrete closures will induce 2-discrete generalized semilattices.

## 2. THE OPERATIONS $\rho_S$

In this section we consider some questions involving complete numerations. Throughout this article, $S$ denotes an arbitrary set with at least two elements; $H(S)$ is the family of all maps from the natural numbers $N$ into $S$; $\leqslant$ is the reducibility relation in $H(S)$; $\oplus$ denotes direct sum of numerations on $H(S)$; $\mathbb{K} = (K, \varkappa)$ is a numerated Kleene set; $\Pi = (\Pi, \pi)$ is a numerated Post set; $\tilde{\varkappa}$ is a universal partial recursive function (p.r.f.), i.e., $\tilde{\varkappa}\langle x, y \rangle = \varkappa_x(y)$, where $\langle\,\rangle$ is a Cantor function used to encode pairs.

To each element $s \in S$ we associate the unary operation $\rho_s$ on $H(S)$ by the following rule: if $\nu \in H(S)$, $x \in N$ then

$$(\rho_s(\nu))x \rightleftharpoons \begin{cases} s & \text{, if } x \notin dom\, \tilde{x}, \\ \nu \tilde{x}(x) & \text{otherwise.} \end{cases}$$

Clearly, $rng(\rho_s \nu) = rng\, \nu \cup \{s\}$ (for simplicity, we sometimes abbreviate $f(x)$ to $fx, f_x$). The operation $\rho_s$ may be regarded as a modification of the operation of taking the completion of a numerated set [4]. We state several properties of the $\rho_s$ (the obvious proofs are omitted).

The first property shows that the $\rho_s$ generalize the operation of $pm$-cylindrification. Numerations $\nu : N \to \{0,1\}$ are identified with the subsets $\{n \in N \mid \nu(n) = 1\}$.

1. If $S = \{0,1\}$, $\nu \in H(S)$, then $\rho_0(\nu) = pm(\nu), \rho_1(\nu) = \overline{pm(\bar{\nu})}$.

2. $\rho_s$ is the closure operation on $(H(S); \leq)$ for all $S$ and $s \in S$.

Let us verify, e.g., that $\nu, \nu' \in H(S), \nu \leq \nu' \to \rho_s(\nu) \leq \rho_s(\nu')$ (the assertions $\nu \leq \rho_s(\nu), \rho_s \rho_s(\nu) \leq \rho_s(\nu)$ are proved just as simply). Let $f$ be a generalized recursive function (g.r.f.) reducing $\nu$ to $\nu'$ and let $g$ be any g.r.f. satisfying

$$\tilde{x}_{g(n)} = \begin{cases} \varnothing & \text{, if } n \notin dom\, \tilde{x}, \\ \lambda y . f \tilde{x}(n) & \text{otherwise.} \end{cases}$$

Then the g.r.f. $\lambda n . \langle g(n), 0 \rangle$ reduces $\rho_s(\nu)$ to $\rho_s(\nu')$.

Remark. We note that the assertion $\forall \nu, \nu' (\nu \leq \rho_s(\nu) \wedge (\nu \leq \nu' \to \rho_s(\nu) \leq \rho_s(\nu')) \wedge \rho_s \rho_s(\nu) \leq \rho_s(\nu))$ holds effectively, i.e., if we are given, e.g., a g.r.f. reducing $\nu$ to $\nu'$, we can effectively find a g.r.f. reducing $\rho_s(\nu)$ to $\rho_s(\nu')$ for all $S$, $s \in S$, $\nu, \nu' \in H(S)$. Many of the other assertions in this paper are also effective in an analogous sense.

3. For any $s \in S$, $\nu \in H(S)$, $\rho_s(\nu)$ is the smallest numeration over $\nu$, which is complete with respect to the particular element $s$, i.e., $\rho_s(\nu)$ is complete relative to $s$, and if $\nu \leq \nu'$ and $\nu'$ is a complete numeration in $H(S)$ relative to $s$, then $\rho_s(\nu) \leq \nu'$.

4. The closure of the set $\{\rho_s(\nu) \mid s \in S, \nu \in H(S)\}$ in $(H(S); \leq)$ coincides with the set of all complete numerations in $H(S)$.

5. If $T$ is a set and $\varphi : S \to T$, $s \in S$, then $\varphi \circ \rho_s(\nu) = \rho_{\varphi(s)}(\varphi \circ \nu)$.

In order to formulate the next two results, we recall that numerations $\nu_1 \in H(S_1), \ldots, \nu_m \in H(S_m)$ can be put in correspondence with their product $\nu_1 \otimes \ldots \otimes \nu_m \in H(S_1 \times \ldots \times S_m)$ according to the rule: $(\nu_1 \otimes \ldots \otimes \nu_m) \langle x_1, \ldots, x_m \rangle \rightleftharpoons (\nu_1 x_1, \ldots, \nu_m x_m)$, where $\langle\ \rangle$ is the coding function for $m$-tuples. In addition, to each function $\varphi : S^m \to S$ we associate a function $\varphi^* : (HS)^m \to H(S)$ defined by $\varphi^*(\nu_1, \ldots, \nu_m) \rightleftharpoons \varphi \circ (\nu_1 \otimes \ldots \otimes \nu_m)$.

6. For arbitrary sets $S_1, \ldots, S_m$ and any $s_i \in S_i$, $\nu_i \in H(S_i)$ $(1 \leq i \leq m)$ we have: $\rho_{(s_1, \ldots, s_m)}(\rho_{s_1}(\nu_1) \otimes \ldots \otimes \rho_{s_m}(\nu_m)) \equiv \rho_{s_1}(\nu_1) \otimes \ldots \otimes \rho_{s_m}(\nu_m)$.

This follows from property 3 and the well-known fact that if $\nu_i' \in H(S_i)$ is complete with respect to $s_i \in S_i$ then $\nu_1' \otimes \ldots \otimes \nu_m'$ is complete with respect to $(s_1, \ldots, s_m)$.

7. The set of all complete numerations in $H(S)$ is closed under all the operations $\varphi^*$ $(\varphi : S^m \to S, m \geq 1)$. This follows from 3, 5, 6.

In order to formulate the next three results, we recall a few definitions. If $\nu \in H(S)$, $E \subseteq S$ then the set $E$ is said to be $\nu$-enumerable if $\nu^{-1}(E)$ is recursively enumerable. A numeration $\nu$ corresponds to a preorder $\leqslant_\nu$ on the set $S$ which is defined by $x \leqslant_\nu y$ if for every $\nu$-enumerable $E \subseteq S$, $x \in E$ implies that $y \in E$. It is clear that if $x \notin \mathrm{rng}\,\nu$ then the element $x \in S$ is not $\nu$-related in this way to the other elements in $S$. It follows from $\nu \leqslant \nu'$ that $\forall x \forall y\, (x \leqslant_\nu y \to x \leqslant_{\nu'} y)$. We now study the relationship between the preorders $\leqslant_\nu$ and $\leqslant_{\rho_s(\nu)}$ on $S$.

8. For any $E \subseteq S$, $s \in S$, $\nu \in H(S)$ we have

$$(\rho_s \nu)^{-1}(E) = \begin{cases} \mathrm{pm}(\nu^{-1}E), & \text{if } s \notin E, \\ \overline{\mathrm{pm}(\overline{\nu^{-1}E})}, & \text{if } s \in E. \end{cases}$$

This is a simple consequence of properties 1 and 5. Property 8 easily implies:

9. A set $E \subseteq S$ is $\rho_s(\nu)$-enumerable if and only if $\mathrm{rng}(\rho_s \nu) \subseteq E$ or $s \notin E$ and $E$ is $\nu$-enumerable.

10. For any $x, y, s \in S$, $\nu \in H(S)$ we have

$$x \leqslant_{\rho_s(\nu)} y \leftrightarrow x \leqslant_\nu y \vee (x \leqslant_\nu s \wedge y \in \mathrm{rng}\,(\rho_s \nu)),$$

i.e., $\leqslant_{\rho_s(\nu)}$ is the smallest preorder on $S$ which contains $\leqslant_\nu$, and is such that the element $s$ is less than or equal to all of the elements in $\mathrm{rng}\,\nu$.

If $x \leqslant_\nu y$, then $x \leqslant_{\rho_s(\nu)} y$ is implied by $\nu \leqslant \rho_s(\nu)$. Let $x \leqslant_\nu s$ and $y \in \mathrm{rng}(\rho_s \nu)$. Then $x \leqslant_{\rho_s(\nu)} s$ and property 9 implies that $s \leqslant_{\rho_s(\nu)} y$. It follows that $x \leqslant_{\rho_s(\nu)} y$. We now verify that $x \leqslant_{\rho_s(\nu)} y \to x \leqslant_\nu y \vee (x \leqslant_\nu s \wedge y \in \mathrm{rng}(\rho_s \nu))$. Let $x \leqslant_{\rho_s(\nu)} y$; if $x \leqslant_\nu s$, the assertion is obvious. Thus assume that $x \not\leqslant_\nu s$; we then have to verify that $x \leqslant_\nu y$. The assumption $x \not\leqslant_\nu s$ implies that there exists a $\nu$-enumerable subset $E \subseteq S$ such that $x \in E$, $s \notin E$. Let $D \subseteq S$ be $\nu$-enumerable and $x \in D$. We then have $s \notin E \cap D$ and $E \cap D$ is $\nu$-enumerable. By property 9, $E \cap D$ is $\rho_s(\nu)$-enumerable; but then $x \leqslant_{\rho_s(\nu)} y$ and $x \in E \cap D$ imply that $y \in E \cap D \subseteq D$. Thus, we derive that $y \in D$ from the assumption that $x \in D$, and $D$ is $\nu$-enumerable. Therefore, $x \leqslant_\nu y$.

11. We have $\rho_s(\nu) \leqslant \alpha \oplus \beta \to \rho_s(\nu) \leqslant \alpha \vee \rho_s(\nu) \leqslant \beta$ for all $s \in S$, $\nu, \alpha, \beta \in H(S)$.

This follows from 3 and [4, Proposition 10, p. 163].

12. If $s, s' \in S$, $\nu, \nu' \in H(S)$, $\rho_s(\nu) \leqslant \rho_{s'}(\nu')$ and $s' \neq s$, then $\rho_s(\nu) \leqslant \nu'$.

Let $f$ be a g.r.f. reducing $\rho_s(\nu)$ to $\rho_{s'}(\nu')$; let $g$ be any g.r.f. which satisfies

$$\mathscr{X}_{g(m,n,x)} = \begin{cases} \emptyset, & \text{if } f\langle x, n\rangle \notin \mathrm{dom}\,\tilde{\mathscr{X}}, \\ \mathscr{X}_m & \text{- otherwise} \end{cases}$$

for all $m, n, x \in N$. By the recursion theorem, there exists a g.r.f. $c(m,n)$ such that $\mathscr{X}_{c(m,n)} = \mathscr{X}_{g(m,n,c(m,n))}$. We claim that the function $h$, defined by $h\langle m,n\rangle \rightleftharpoons \tilde{\mathscr{X}} f\langle c(m,n), n\rangle$, is a g.r.f. reducing $\rho_s(\nu)$ to $\nu'$. Indeed, assume that $\langle m,n\rangle \notin \mathrm{dom}\,h$, i.e., $f\langle c(m,n), n\rangle \notin$

$dom\,\tilde{\mathscr{x}}$ . Then $(\rho_{s'}\nu')f\langle c(m,n),n\rangle = s'$. On the other hand, $\mathscr{x}_{c(m,n)}=\emptyset$ and therefore $\langle c(m,n),$ $n\rangle \notin dom\,\tilde{\mathscr{x}}$, $(\rho_s\nu)\langle c(m,n),n\rangle = s$. But we must have $(\rho_{s'}\nu')\, f\langle c(m,n),n\rangle = (\rho_s\nu)\langle c(m,n),n\rangle$ , which contradicts $s'\neq s$ . We have verified that $h$ is a g.r.f. It follows that $\mathscr{x}_{c(m,n)}=\mathscr{x}_m$ for all $m,n\in N$ and $(\rho_{s'}\nu')f\langle c(m,n),n\rangle = \nu'\tilde{\mathscr{x}}f\langle c(m,n),n\rangle = \nu' h\langle m,n\rangle$ . These equalities imply:

$$(\rho_s\nu)\langle m,n\rangle = (\rho_s\nu)\langle c(m,n),n\rangle = (\rho_{s'}\nu')f\langle c(m,n),n\rangle = \nu' h\langle m,n\rangle.$$

We now state a property that generalizes a property possessed by the $m$ -jump operation. First, some notation: If $\varphi: S\to S$, $\nu\in H(S)$ , then the symbols $\nu^{\varphi}$ denote the direct sum $\bigoplus_{k\geq 0}\nu_k$ of the following sequence of numerations: $\nu_0 \rightleftharpoons \nu$, $\nu_{k+1}\rightleftharpoons \varphi\circ\nu_k$ . The case of an $m$ -jump is recovered by specializing to $S=\{0,1\}$ , $\varphi(0)=1$, $\varphi(1)=0$.

13. If $\varphi: S\to S$ is a map without fixed points, two numerations $\rho_s(\nu^{\varphi}), \rho_{s'}(\nu^{\varphi})$ with $s'\neq s$ are not comparable, and therefore $\rho_s(\nu^{\varphi})\nleq \nu^{\varphi}$.

Assume that $\rho_s(\nu^{\varphi})\leq \rho_{s'}(\nu^{\varphi})$ . Then $\rho_s(\nu^{\varphi})\leq \nu^{\varphi}$ by property 12. Therefore (property 3), the numeration $\mu\rightleftharpoons\nu^{\varphi}$ is complete. In addition, we see easily that $\varphi\circ\mu\leq\mu$ . But it is easily seen that there exists no numeration $\mu$ with the property that $\varphi\circ\mu\leq\mu$ . Indeed, let the g.r.f. $f$ reduce $\varphi\circ\mu$ to $\mu: \varphi\circ\mu=\mu\circ f$ . By the recursion theorem for complete numerations [4, p. 161], $\mu f(c)=\mu c$ for some number $c$ . But then $\mu c=\mu f(c)=\varphi(\mu c)$ , i.e. $\mu c\in S$ is a fixed point of the map $\varphi$ , contrary to assumption.

The following important result follows from properties 2, 11, 12.

Proposition 1. The structure $(H(S);\leq,\oplus,\{\rho_s\}_{s\in S})$ is a semilattice with discrete closures.

COROLLARY. Let $H_s$ be the set of all complete (with respect to $s\in S$ ) numerations in $H(S)$; and let $H_1(S)\rightleftharpoons\bigcup_{s\in S}H_s$ be the set of all complete numerations in $H(S)$. Then $(H_1(S);\leq,\{H_s\}_{s\in S})$ is a discrete generalized semilattice.

We conclude this section by noting another application of the $\rho_s$ operations. That is, they can be used to find examples (more "explicit" than in [4, 5]) answering a question posed by A. I. Mal'tsev (it turns out that the first of these assertions was known to Yu. L. Ershov).

Proposition 2. 1) Let $\sigma$ be a $\Sigma_2^0$ -complete set regarded as a numeration $\sigma: N\to\{0,1\}$. Then the numeration $\sigma$ is complete relative to both 0 and 1. 2) The standard numeration of the class $\Sigma_2^0$ is complete relative to any subset of $\Sigma_2^0$.

The proof is a simple application of the Tarski–Kuratowski algorithm, together with property 3.

## 3. UNIVERSAL NUMERATED SETS

Consider a structure $(Map(A,S);\leq_m)$ for a class of numerated sets. The numerated set $\mathbb{A}=(A,\alpha)$ is said to be universal if the numeration $\alpha$ is complete and there exists an infinite computable sequence of nonempty pairwise disjoint $\alpha$ -enumerable subsets of $A$ . The map $\alpha: N\xrightarrow{\text{onto}}A$ induces an inclusion $\varphi\to\varphi\circ\alpha$ of $Map(A,S)$ onto $Map(N,S)=H(S)$ which we denote by $\mathscr{F}_{\alpha}$ .

THEOREM 1. The following conditions are equivalent for the numerated set $\mathbb{A}$:

1) $\mathbb{A}$ is a universal numerated set;

2) for every set $S$, the closure of the image of the set $Map(\mathbb{A}, S)$ in $(H(S);\leq)$ under the map $\mathcal{F}_\alpha$ coincides with the set of all complete numerations in $H(S)$.

It is easy to see that this is just a convenient reformulation for our purposes of the universality theorem in [4, p. 267].

We note also that the closure of the set $P_s \rightleftharpoons \{\varphi \in Map(\mathbb{A}, S) \mid \varphi(a)=s\}$ (where $a$ is a distinguished element in $\mathbb{A}$, $s \in S$) in $(H(S);\leq)$ coincides with the set of all numerations in $H(S)$ which are complete relative to the element $s \in S$. Together with Proposition 1 and its corollary, this gives:

COROLLARY. If $\mathbb{A}$ is universal then $(Map(\mathbb{A},S);\leq_m,\{P_s\}_{s\in S})$ is a discrete generalized semilattice which is equivalent to $(H_1(S);\leq,\{H_s\}_{s\in S})$.

The standard examples of universal numerated sets include $\mathbb{K}$ and the numerated set provided by the family $\{\varnothing,\{x\} \mid x\in N\}$ with a principal computable numeration. We note also that if $\mathbb{A}$ is a universal numerated set and $\mathbb{B}$ is a complete numerated set, then their product $\mathbb{A}\otimes\mathbb{B}$ is a universal numerated set.

## 4. THE OPERATIONS $q^t$

We now come to our main goal, which is to study $(Map(\mathbb{A},S);\leq_m)$ for another simple, natural class of numerated sets. However, this requires some preliminary work similar to that in Sec. 2.

Fix a creative set $\xi$. Then we can associate to each element $t\in S$ a unary operation $q^t$ on $H(S)$ defined by

$$(q^t\nu)\langle x,y\rangle \rightleftharpoons \begin{cases} \nu x, & \text{if } y\notin\xi, \\ t, & \text{if } y\in\xi. \end{cases}$$

Clearly, $rng(q^t\nu) = rng\,\nu \cup \{t\}$. We note some properties of these operations $q^t$, which will will be helpful in what follows. Most of them are dual (in an appropriate sense) to the properties of the operations $\rho_s$. In most cases, the proofs reduce to simple manipulations using the fact that $\xi$ is creative, and we therefore omit them.

1. For the case when $S=\{0,1\}$ we have $q^0(\nu)=\nu\times\bar{\xi}$, $q^1(\nu)=\overline{\bar{\nu}\times\xi}$ (a numeration $\nu\in H(S)$ is identified with the corresponding subset of $N$, and for $\sigma,\tau\subseteq N$ we have $\sigma\times\tau \rightleftharpoons \{\langle x,y\rangle \mid x\in\sigma \wedge y\in\tau\}$).

2. $q^t$ is the closure operation on $(H(S);\leq)$ for all $S$ and $t\in S$.

In order to derive an analog of property 3 in Sec. 2, we introduce the following definition. A numeration $\nu\in H(S)$ is said to an element $t\in S$, if for arbitrary g.r.f. $f$ and recursively enumerable set (r.e.s.) $\sigma$ there exists a g.r.f. $g$ such that for all $x\in N$

$$\nu_{g(x)} = \begin{cases} \nu f(x), & \text{if } x\notin\sigma; \\ t, & \text{if } x\in\sigma. \end{cases}$$

A numeration is said to be cocomplete if it is cocomplete with respect to some $t\in S$.

3. For all $t\in S$, $\nu\in H(S)$ the numeration $q^t(\nu)$ is the smallest numeration over $\nu$ which is cocomplete relative to $t$.

4. If $T$ is a set and $\varphi: S \to T$, $t \in S$, then $\varphi \circ q^t(\nu) = q^{\varphi(t)}(\varphi \circ \nu)$.

5. For arbitrary sets $S_1, \ldots, S_m$ and arbitrary $t_i \in S_i$, $\nu_i \in H(S_i)$ $(1 \leq i \leq m)$ we have: $q^{(t_1,\ldots,t_m)}(q^{t_1}(\nu_1) \otimes \ldots \otimes q^{t_m}(\nu_m)) \equiv q^{t_1}(\nu_1) \otimes \ldots \otimes q^{t_m}(\nu_m)$.

It suffices to verify that $q^{(t_1,\ldots,t_m)}(q^{t_1}(\nu_1) \otimes \ldots \otimes q^{t_m}(\nu_m)) \leq q^{t_1}(\nu_1) \otimes \ldots \otimes q^{t_m}(\nu_m)$. Let $g$ be a g.r.f satisfying $u \in \xi \vee v \in \xi \to g(u,v) \in \xi$ for all $u, v \in N$. Then the g.r.f. taking the number $\langle\!\langle x_1, \ldots, x_m \rangle, y \rangle$ into $\langle\!\langle \ell x_1, g(\tau x_1, y) \rangle, \ldots, \langle \ell x_m, g(\tau x_m, y) \rangle\!\rangle$ (where $\ell$ and $\tau$ are g.r. functions inverse to the pair-coding function) gives the required reduction.

6. The set of all cocomplete numerations in $H(S)$ is closed under all the operations $\varphi^*$ $(\varphi: S^m \to S,\ m \geq 1)$.

The duality of the operations $\rho_S$ and $q^t$ can be seen in the following description of the preorder $\leq_{q^t(\nu)}$ on $S$.

7. For arbitrary $E \subseteq S$, $\nu \in H(S)$ we have

$$(q^t\nu)^{-1}(E) = \begin{cases} \nu^{-1}(E) \times \overline{\xi}, & \text{if } t \notin E, \\ \overline{\overline{\nu^{-1}(E)} \times \overline{\xi}}, & \text{if } t \in E. \end{cases}$$

8. The set $E \subseteq S$ is $q^t(\nu)$-enumerable if and only if $E \cap \operatorname{rng}(q^t\nu) = \emptyset$ or $t \in E$ and $E$ is $\nu$-enumerable.

9. For arbitrary $x, y, t \in S$, $\nu \in H(S)$ we have $x \leq_{q^t(\nu)} y \leftrightarrow x \leq_\nu y \vee (t \leq_\nu y \wedge x \in \operatorname{rng}(q^t\nu))$, i.e., $\leq_{q^t(\nu)}$ is the smallest preorder on $S$ that contains $\leq_\nu$ and is such that all elements in $\operatorname{rng}\nu$ are less than or equal to $t$.

If $x \leq_\nu y$ then $x \leq_{q^t(\nu)} y$ follows from $\nu \leq q^t(\nu)$. Let $t \leq_\nu y$, $x \in \operatorname{rng}(q^t\nu)$; then $t \leq_\nu y$ implies that $t \leq_{q^t(\nu)} y$ and $x \in \operatorname{rng}(q^t\nu)$ implies that $x \leq_{q^t(\nu)} t$, by property 8. Hence $x \leq_{q^t(\nu)} y$.

We now verify that $x \leq_{q^t(\nu)} y \to x \leq_\nu y \vee (t \leq_\nu y \wedge x \in \operatorname{rng}(q^t\nu))$. The case when $t \leq_\nu y$ is obvious. It therefore remains to prove that $x \leq_{q^t(\nu)} y$, $t \nleq_\nu y$ implies $x \leq_\nu y$. Since $t \nleq_\nu y$ here exists a $\nu$-enumerable set $E \subseteq S$ such that $t \in E$, $y \notin E$. Let $D \subseteq S$ be a $\nu$-enumerable set and let $x \in D$. Then $E \cup D$ is $\nu$-enumerable and $t \in E \cup D$. By property 8, the set $E \cup D$ is $q^t(\nu)$-enumerable. Since $x \leq_{q^t(\nu)} y$, $x \in E \cup D$, we have $y \in E \cup D$. But $y \notin E$, and therefore $y \in D$. Thus, the assumptions that $x \in D$, and $D$ is $\nu$-enumerable imply that $y \in D$.

## 5. THE OPERATIONS $\tau_S^t$

The compositions of the operations $\rho_S$ and $q^t$ $(S, t \in S)$, which we denote by $\tau_S^t$: $\tau_S^t \rightleftharpoons \rho_S \circ q^t$, will be important. We therefore note some properties of the operations $\tau_S^t$, all of which (except for one) follow easily from property 1 and the corresponding properties of the operations $\rho_S$, $q^t$.

1. For arbitrary $s, t \in S$, $\nu \in H(S)$ we have $\rho_S(q^t\nu) \equiv q^t(\rho_S\nu)$.

We indicate only the reducing functions, leaving the routine verification to the reader. We define the r.e.s. $\sigma$ by $\{x \in N \mid x \in dom\,\tilde{x} \wedge \tilde{x}(x) \in \xi\}$. Let $f$ be a g.r.f. which $m$-reduces $\sigma$ to $\xi$, and let $g$ be a g.r.f. satisfying

$$\varkappa_{g(x)} = \begin{cases} \emptyset, & \text{if } x \notin dom\,\tilde{\varkappa}, \\ \lambda z.\, \ell\tilde{\varkappa}(x) & \text{otherwise}. \end{cases}$$

Then the g.r.f. $\lambda x.\langle\langle g(x),0\rangle, f(x)\rangle$ reduces the numeration $\rho_s(q^t\nu)$ to the numeration $q^t(\rho_s\nu)$.

Let $h$ be any g.r.f. satisfying

$$\varkappa_{h(x)} = \begin{cases} \emptyset, & \text{if } r(x) \notin \xi \text{ and } \ell(x) \notin dom\,\tilde{\varkappa}; \\ \lambda z.\langle \tilde{\varkappa}\ell(x), r(x)\rangle, & \text{if } \ell(x) \in dom\,\tilde{\varkappa} \text{ and } \ell(x) \text{ is enumerated} \\ & \text{in } dom\,\tilde{\varkappa} \text{ before } r(x) \text{ is enumerated in } \xi; \\ \lambda z.\langle 0, r(x)\rangle & \text{– in all remaining cases.} \end{cases}$$

(we assume in such definitions that some method has been chosen to effectively enumerate the corresponding sets during the stepwise construction [in this case, $dom\,\tilde{\varkappa}$ and $\xi$ ]). Then the g.r.f. $\lambda x.\langle h(x), 0\rangle$ reduces $q^t(\rho_s\nu)$ to $\rho_s(q^t\nu)$.

Remark. It is easy to construct examples that show that the operations $\rho_s$ , $\rho_{s'}$ (and also $q^t, q^{t'}$ ) do not commute in general.

2. For $S=\{0,1\}$ we have: $q^0\rho_0(\nu)=\rho m(\nu)\times\bar{\xi}$, $q^0\rho_1(\nu)=\overline{\rho m(\bar{\nu})\times\bar{\xi}}$, $q^1\rho_0(\nu)=\overline{\overline{\rho m(\nu)}\times\bar{\xi}}$, $q^1\rho_1(\nu)=\overline{\rho m(\bar{\nu})\times\bar{\xi}}$ (cf. [1, Theorem 2]).

3. $\tau_s^t$ is the closure operation on $(H(S);\leq)$ for arbitrary $S$ and $s,t\in S$.

A numeration in $H(S)$ is said to be 2-complete relative to $s,t\in S$ if it is complete with respect to $s$ and cocomplete with respect to $t$ .

A numeration is said to be 2-complete if it is 2-complete with respect to some $s,t\in S$ .

4. $\tau_s^t(\nu)$ is the smallest numeration over $\nu$ which is 2-complete relative to $s,t$ .

5. If $T$ is a set and $\varphi: S\to T$, $s,t\in S$, then $\varphi\circ\tau_s^t(\nu)=\tau_{\varphi(s)}^{\varphi(t)}(\varphi\circ\nu)$.

6. For arbitrary sets $S_1,\dots,S_m$ and any $s_i,t_i\in S_i$, $\nu_i\in H(S_i)$ $(1\leq i\leq m)$ we have:

$$\tau_{(s_1,\dots,s_m)}^{(t_1,\dots,t_m)}(\tau_{s_1}^{t_1}(\nu_1)\otimes\dots\otimes\tau_{s_m}^{t_m}(\nu_m))\equiv\tau_{s_1}^{t_1}(\nu_1)\otimes\dots\otimes\tau_{s_m}^{t_m}(\nu_m).$$

7. The set of all 2-complete numerations in $H(S)$ is closed under all of the operations $\varphi^*$ $(\varphi: S^m\to S,\ m\geq 1)$.

8. $\leq_{\tau_s^t(\nu)}$ is the smallest preorder on $S$ which contains $\leq_\nu$ and is such that every element in $rng\,\nu$ is greater than or equal to $s$ and less than or equal to $t$ .

9. $\tau_s^t(\nu)\leq\alpha\oplus\beta\to\tau_s^t(\nu)\leq\alpha\vee\tau_s^t(\nu)\leq\beta$ for all $s,t\in S$, $\nu,\alpha,\beta\in H(S)$.

10. $\tau_s^t(\nu)\leq\tau_{s'}^{t'}(\nu')\to\tau_s^t(\nu)\leq\nu'$ for all $\nu,\nu'\in H(S), s,s',t,t'\in S$, $s'\neq s$, $t'\neq t$.

It follows from $\rho_s(q^t\nu)\leq\rho_{s'}(q^{t'}\nu')$ and $s'\neq s$ that $\rho_s(q^t\nu)\leq q^{t'}(\nu')$ (Sec. 2, property 12). Let the g.r.f. $f$ reduce $\rho_s(q^t\nu)$ to $q^{t'}(\nu')$, and let the recursively enumerable set $\sigma$ be defined by $\sigma\rightleftharpoons\{\langle x,y\rangle \mid (x\in dom\,\tilde{\varkappa}\wedge r\tilde{\varkappa}(x)\in\xi)\vee rf\langle y,r(x)\rangle\in\xi\}$.

Let $h$ be a g.r.f. $m$ -reducing $\sigma$ to $\xi$ and let $g$ be any g.r.f. satisfying

$$\varkappa_{g(x,y)} = \begin{cases} \lambda z.f\langle y, r(x)\rangle, & \text{if } rf\langle y,r(x)\rangle \text{ is enumerated in } \xi \\ & \text{before } x \text{ is enumerated in } dom\,\tilde{\varkappa}; \\ \lambda z.\langle \ell\tilde{\varkappa}(x), h\langle x,y\rangle\rangle & \text{otherwise} \end{cases}$$

(in particular, $\varkappa_{g(x,y)}=\emptyset$ if $x\notin dom\,\tilde{\varkappa}\wedge \tau f\langle y,\tau(x)\rangle\notin\xi$).

By the recursion theorem, there exists a g.r.f. $C$ such that $\varkappa_{c(x)}=\varkappa_{g(x,c(x))}$. We claim that the g.r.f. $\lambda x.\ell f\langle c(x),\tau(x)\rangle$ reduces $\rho_S(q^t\nu)$ to $\nu'$. In order to prove this, we first verify that

$$\forall x\,(\tau f\langle c(x),\tau(x)\rangle\notin\xi). \tag{1}$$

Proceeding by contradiction, assume that $\tau f\langle c(x),\tau(x)\rangle\in\xi$. If $\tau f\langle c(x),\tau(x)\rangle$ is computed in $\xi$ before $x$ is in $dom\,\tilde{\varkappa}$, then $\varkappa_{c(x)}(\tau x)=f\langle c(x),\tau(x)\rangle$ and therefore $t=(q^t\nu)f\langle c(x),\tau(x)\rangle=(q^t\nu)\varkappa_{c(x)}(\tau x)=(\rho_S q^t\nu)\langle c(x),\tau(x)\rangle=(q^{t'}\nu')f\langle c(x),\tau(x)\rangle=t'$, which contradicts the assumption $t'\neq t$. If $x$ is computed in $dom\,\tilde{\varkappa}$ before $\tau f\langle c(x),\tau(x)\rangle$ is computed in $\xi$, then $\varkappa_{c(x)}(\tau x)=\langle\ell\tilde{\varkappa}(x),h\langle x,c(x)\rangle\rangle$. By the definition of $\sigma,h$ we have $h\langle x,c(x)\rangle\in\xi$, and therefore $t=(q^t\nu)\langle\ell\tilde{\varkappa}(x),h\langle x,c(x)\rangle\rangle=(q^t\nu)\varkappa_{c(x)}(\tau x)=(\rho_S q^t\nu)\langle c(x),\tau(x)\rangle=(q^{t'}\nu')f\langle c(x),\tau(x)\rangle=t'$. We have thus verified (1).

It follows from (1) that $\forall x(\varkappa_{c(x)}(\tau x)=\langle\ell\tilde{\varkappa}(x),h\langle x,c(x)\rangle\rangle)$ and $\forall x(x\in dom\,\tilde{\varkappa}\wedge\tau\tilde{\varkappa}(x)\in\xi\leftrightarrow\langle x,c(x)\rangle\in\sigma\leftrightarrow h\langle x,c(x)\rangle\in\xi)$, whence

$$\forall x((\rho_S q^t\nu)x=(\rho_S q^t\nu)\langle c(x),\tau(x)\rangle). \tag{2}$$

Using (1) and (2), we find that $\forall x\,((\tau_S^t\nu)x=(\rho_S q^t\nu)\langle c(x),\tau(x)\rangle=(q^{t'}\nu')f\langle c(x),\tau(x)\rangle=\nu'\ell f\langle c(x),\tau(x)\rangle)$, as claimed.

The next proposition follows from Properties 3, 9, and 10.

Proposition 3. The structure $(H(S);\leqslant,\oplus,\{\tau_S^t\}_{s,t\in S})$ is a semilattice with 2-discrete closures.

COROLLARY. Let $H_S^t$ be the set of all 2-complete numerations in $H(S)$ with respect to $s,t\in S$; let $H_2(S)\rightleftharpoons\bigcup_{s,t\in S}H_S^t$ be the set of all 2-complete numerations in $H(S)$. Then $(H_2(S);\leqslant,\{H_S^t\}_{s,t\in S})$ is a 2-discrete generalized semilattice.

## 6. 2-UNIVERSAL NUMERATED SETS

We now consider $(Map(A,S);\leqslant_m)$ for another natural class of numerated sets. A numerated set $\mathbb{A}=(A,\alpha)$ is said to be 2-universal if the numeration $\alpha$ is 2-complete and there exist a g.r.f. $g$ and a computable sequence $\{E_\kappa\}$ of $\alpha$-enumerable subsets of $A$ such that $\alpha g(\kappa)\in E_\kappa\setminus\bigcup_{m\neq\kappa}E_m$ for every $\kappa\in N$.

Property 8 in Sec. 5 implies that the classes of universal and 2-universal numerated sets are disjoint. Examples of 2-universal numerated sets include $\Pi$ and the numerated set provided by the family $\{\emptyset,\{x\},N\mid x\in N\}$ with a principal computable numeration. If $\mathbb{A}$ and $\mathbb{B}$ are, respectively, a 2-universal and a 2-complete numerated set, then $\mathbb{A}\otimes\mathbb{B}$ is 2-universal.

THEOREM 2. The following conditions are equivalent for every numerated set $\mathbb{A}$:

1) $\mathbb{A}$ is a 2-universal numerated set;

2) for every set $S$, the closure of the image of the set $Map(A,S)$ in $(H(S);\leqslant)$ under the map $\mathcal{F}_\alpha$ (cf. Sec. 3) coincides with the set of all 2-complete numerations in $H(S)$.

We first prove that 1) $\to$ 2). Let the numeration $\alpha$ be 2-complete with respect to the elements $a, b \in A$; then $\tau_a^b(\alpha) \equiv \alpha$ by property 4, Sec. 5. If $\varphi$ is a map from $A$ into $S$, then $\varphi \circ \tau_a^b(\alpha) \equiv \varphi \circ \alpha$ and $\varphi \circ \tau_a^b(\alpha) = \tau_{\varphi(a)}^{\varphi(b)}(\varphi \circ \alpha)$ (property 5). The numeration $\varphi \circ \alpha \equiv \tau_{\varphi(a)}^{\varphi(b)}(\varphi \circ \alpha)$ is therefore 2-complete with respect to $\varphi(a), \varphi(b)$. It remains to show that if $\nu$ is a 2-complete numeration in $H(S)$ then $\varphi \circ \alpha \equiv \nu$ for a suitable $\varphi: A \to S$. Assume that $\nu$ is 2-complete with respect to $s, t \in S$; we then define the map $\varphi$ by

$$\varphi(x) \rightleftharpoons \begin{cases} s, & \text{if} \quad x \notin \bigcup_{k \geq 0} E_k; \\ \nu_k, & \text{if} \quad x \in E_k \setminus \bigcup_{m \neq k} E_m; \\ t, & \text{if} \quad \exists k, m\, (m \neq k \wedge x \in E_k \wedge x \in E_m). \end{cases}$$

It follows from the description of the preorder $\leqslant_{\tau_a^b(\nu)}$ (property 8, Sec. 5) that $\varphi(a) = s$, $\varphi(b) = t$. It remains to check that $\varphi \circ \alpha \equiv \nu$. We have $\forall k (\alpha g(k) \in E_k \setminus \bigcup_{m \neq k} E_m)$, and therefore $\forall k ((\varphi \circ \alpha) g(k) = \varphi(\alpha g(k)) = \nu_k)$, i.e., $g$ reduces $\nu$ to $\varphi \circ \alpha$. In order to prove that $\varphi \circ \alpha \leqslant \nu$ it suffices to verify that $\varphi \circ \alpha \leqslant \tau_s^t(\nu)$ (property 4, Sec. 5). Let the r.e.s. $\sigma$ be defined by $\sigma \rightleftharpoons \{y \in N \mid \exists k, m (m \neq k \wedge \alpha y \in E_k \cap E_m)\}$; let $h$ be a g.r.f. $m$-reducing $\sigma$ to $\xi$, and let $f$ be any g.r.f. satisfying

$$\varkappa_{f(y)} = \begin{cases} \emptyset, & \text{if} \quad \alpha y \notin \bigcup_{k \geq 0} E_k; \\ \lambda z.\langle k, h(y) \rangle - & \text{otherwise}, \end{cases}$$

where $k$ is the first number for which $y$ was enumerated in $\alpha^{-1}(E_k)$ in some simultaneous stepwise enumeration of the sequence $\{\alpha^{-1}(E_k)\}$. We verify without difficulty that the g.r.f. $\lambda y.\langle f(y), 0 \rangle$ reduces $\varphi \circ \alpha$ to $\tau_s^t(\nu)$.

We now prove that $S \rightleftharpoons A$ and let $\varphi$ be the identity map from $A$ into $S$. By condition 2, the numeration $\varphi \circ \alpha = \alpha$ is 2-complete. Now let $S = \{\emptyset, \{x\}, N \mid x \in N\}$, be a $\nu$-principal computable numeration of $S$. The numeration $\nu$ is 2-complete relative to $\emptyset, N$. By condition 2, there exists a map $\varphi: A \to S$ such that $\varphi \circ \alpha \equiv \nu$. Clearly, the sequence $\{D_k\}$, $D_k \rightleftharpoons \{\{k\}, N\}$ $(k \in N)$ and the g.r.f. $h$ satisfying the condition $\forall k\, (\nu h(k) = \{k\})$ demonstrate that the numerated set $(S; \nu)$ is 2-universal. Let the g.r.f. $f$ reduce $\nu$ to $\varphi \circ \alpha$. We then easily see from $\varphi \circ \alpha \equiv \nu$ that the existence of the sequence $\{E_k\}$, $E_k \rightleftharpoons \varphi^{-1}(D_k)$ and the g.r.f. $f \circ h$ prove that the numerated set $A$ is 2-universal.

The next result follows from the proof of Theorem 2 and the Corollary to Proposition 3.

COROLLARY. Let $A$ be 2-universal and $P_s^t \rightleftharpoons \{\varphi \in Map(A, S) \mid \varphi(a) = s, \varphi(b) = t\}$ (where $s, t \in S; a, b \in A$ are the elements with respect to which the numeration $\alpha$ is 2-complete). Then $(Map(A, S); \leqslant, \{P_s^t\}_{s,t \in S})$ is a 2-discrete generalized semilattice which is equivalent to $(H_2(S); \leqslant, \{H_s^t\}_{s,t \in S})$.

## 7. REFLECTIVE NUMERATED SETS

We will henceforth consider the structure $(Map(A, S); \leqslant_M)$ for two new natural classes of numerated sets. The numerated set $A = (A, \alpha)$ is said to be reflective if the numeration $\alpha$ is complete and there exist morphisms $\Phi_0, \Phi_0^*, \Phi_1, \Phi_1^*$ from $A$ into $A$ such that:

1) $\Phi_0^* \circ \Phi_0, \Phi_1^* \circ \Phi_1$ are the identity maps on $A$ ;

2) $rng\,\Phi_0$ , $rng\,\Phi_1$ are disjoint $\alpha$-enumerable sets.

Examples. $K$ is reflective. If $C$ is a finite family of finite subsets of $N$ such that $(C;\subseteq)$ has a smallest but not a largest element, then the numerated set $(A,\alpha)$ , formed by the family of all computable enumerations in $H(C)$ together with a principal computable numeration is reflective. If the numerated sets $A$ and $B$ are respectively reflective and complete, then $A\otimes B$ is reflective. The family $\{\emptyset,\{x\}\,|\,x\in N\}$ equipped with a principal computable numeration is not a reflective numerated set.

THEOREM 3. If $A$ is reflective and $P_s=\{\varphi \in Map(A,S)\,|\,\varphi(a)=s\}$ $a\in A$ (where $s\in S$, is the element with respect to which the numeration $\alpha$ is complete), then $(Map(A,S);\leqslant_M, \{P_s\}_{s\in S})$ is a discrete generalized semilattice.

We first define the binary operations $\theta_s(s\in S)$ on $Map(A,S)$. If $s\in S$, $\varphi_0,\varphi_1\in Map(A,S)$ then the element $\theta_s(\varphi_0,\varphi_1)\in Map(A,S)$ (more briefly, $\theta_s$ ) is defined for $(x\in A)$: by:

$$\theta_s(x) \rightleftharpoons \begin{cases} s\ , & \text{if} \quad x\notin rng\,\Phi_0 \cup rng\,\Phi_1\ , \\ \varphi_i\Phi_i^*(x)\ , & \text{if} \quad x\in rng\,\Phi_i\ ,\ i\in\{0,1\}\ . \end{cases}$$

We claim that $\theta_s(\varphi_0,\varphi_1)$ is the generalized upper bound of the elements $\varphi_0,\varphi_1$ in $(Map(A, S);\leqslant_M,\{P_s\})$ (cf. Sec. 1). Indeed, by property 10 in Sec. 2 and condition 2) in the above definition, we have $a\notin rng\,\Phi_0 \cup rng\,\Phi_1$ , and therefore $\theta_s\in P_s$ . We further have $\theta_s\Phi_i(x)= \varphi_i\Phi_i^*\Phi_i(x)=\varphi_i(x)$ i.e., the morphism $\Phi_i$ $M$-reduces $\varphi_i$ to $\theta_s$ $(\varphi_i=\theta_s\circ\Phi_i)$ . Let $\varphi_0,\varphi_1\leqslant_M\psi$, $\psi\in P_s$ and let $\Psi_i\,(i=0,1)$ be a morphism $M$ -reducing $\varphi_i$ to $\psi$ . Then the map $\Psi: A\longrightarrow A$ , defined by

$$\Psi(x) \rightleftharpoons \begin{cases} a\ , & \text{if} \quad x\notin rng\,\Phi_0 \cup rng\,\Phi_1\ ; \\ \Psi_i\,\Phi_i^*(x)\ , & \text{if} \quad x\in rng\,\Phi_i\ , \end{cases}$$

is a morphism from $A$ to $A$ which $M$ -reduces $\theta_s$ to $\psi$ (this follows easily from the reflectivity of $A$ ). Finally, let $\psi\in P_{s'}$, $s'\neq s$ and assume that the morphism $\Psi$ $M$-reduces $\psi$ to $\theta_s$ . It follows at once from $s'\neq s$ that $\Psi(a)\in rng\,\Phi_i$ for some $i\in\{0,1\}$ . Together with property 10, Sec. 2, and the fact that $\Psi$ is a monotone map from $(A;\leqslant_\alpha)$ into $(A;\leqslant_\alpha)$ [4, p. 111], this implies that $rng\,\Psi\subseteq rng\,\Phi_i$ , whence the morphism $\Phi_i^*\circ\Psi$ $M$ -reduces $\psi$ to $\varphi_i$ .

We note some additional properties of reflective numerated sets $A$.

1. If $X\subseteq A$ is an $\alpha$-enumerable set, then its image $\Phi_i(X)$ under the map $\Phi_i$ $(i=0,1)$ is also $\alpha$-enumerable, and a $\pi$-index for the set $\alpha^{-1}(\Phi_i(X))$ can be found effectively in terms of a $\pi$-index of the set $\alpha^{-1}(X)$.

This follows from the readily verified assertion

$$\forall x\in A\ (x\in\Phi_i(X) \longleftrightarrow x\in rng\,\Phi_i \wedge \Phi_i^*(x)\in X).$$

We define a sequence $\{\Psi_\kappa\},\{\Psi^*_\kappa\}$ of morphisms from $\mathbb{A}$ into $\mathbb{A}$ by

$$\Psi_0 \rightleftharpoons \Phi_0,\ \Psi_{\kappa+1} \rightleftharpoons \Phi_1 \circ \Psi_\kappa;\ \Psi^*_0 \rightleftharpoons \Phi^*_0,\ \Psi^*_{\kappa+1} \rightleftharpoons \Psi^*_\kappa \circ \Phi^*_1.$$

2. The sequences of morphisms $\{\Psi_\kappa\}, \{\Psi^*_\kappa\}$ are computable; $\Psi^*_\kappa \circ \Psi_\kappa$ is the identity on $A$ for every $\kappa \in N$.

3. The sets $D_\kappa, D_\kappa \rightleftharpoons \operatorname{rng} \Psi_\kappa$ $(\kappa \in N)$ are pairwise disjoint.

It suffices to prove that $\forall \kappa\, \forall m\ (\kappa < m \to D_\kappa \cap D_m = \emptyset)$. This can be done by a simple induction on $\kappa$.

4. Every reflective numerated set is universal.

It follows from 1-3 that $\{D_\kappa\}$ is a computable sequence of nonempty disjoint $\alpha$-enumerable subsets of $A$.

5. If $T$ is a set and $\psi: S \to T$, $s \in S$, $\varphi_0, \varphi_1 \in Map(A,S)$, $\quad \psi \circ \theta_s(\varphi_0,\varphi_1) = \theta_{\psi(s)}(\psi \circ \varphi_0, \psi \circ \varphi_1)$.

Property 4 and the Corollary to Theorem 1 imply that generalized upper bound operations can be defined, in addition to the operations $\theta_s$ on $(Map(A,S); \leqslant_m, \{P_s\}_{s \in S})$, for a reflective numerated set $\mathbb{A}$. These operations are closely related.

6. For any $s \in S$, $\varphi_0, \varphi_1 \in Map(A,S)$ the enumerations $\theta_s(\varphi_0,\varphi_1)\circ\alpha$ and $\rho_s((\varphi_0\circ\alpha)\oplus(\varphi_1\circ\alpha))$ are equivalent.

We also note that property 2 can be used to define the operations $\theta_s$ even for infinite sequences of elements in $Map(A,S)$, which is useful in some problems. Indeed, if $\varphi_\kappa \in Map(A,S), \kappa \in N$, then $\theta_s = \theta_s(\varphi_0, \varphi_1, \ldots)$ is defined by

$$\theta_s(x) \rightleftharpoons \begin{cases} s & \text{, if } \quad x \notin \bigcup_{\kappa \geqslant 0} D_\kappa, \\ \varphi_\kappa \Psi^*_\kappa(x) & \text{, if } \quad x \in D_\kappa. \end{cases}$$

In this case we also have $\theta_s(\varphi_0,\varphi_1,\ldots)\circ\alpha \equiv \rho_s(\bigoplus_{\kappa \geqslant 0}(\varphi_\kappa \circ \alpha))$.

## 8. 2-REFLECTIVE NUMERATED SETS

We now consider another class of numerated sets. A numerated set $\mathbb{A} = (A,\alpha)$ is said to be 2-reflective if the numeration $\alpha$ is 2-complete and there exist morphisms $\Phi_0, \Phi^*_0, \Phi_1, \Phi^*_1$ from $\mathbb{A}$ into $\mathbb{A}$ and $\alpha$-enumerable subsets $B_0, C_0, B_1, C_1 \subseteq A$ such that:

1) the maps $\Phi^*_0 \circ \Phi_0, \Phi^*_1 \circ \Phi_1$ are the identity on $A$;

2) $B_i \supseteq C_i$, $\operatorname{rng} \Phi_i = B_i \setminus C_i$ $(i=0,1)$, $B_0 \cap B_1 = C_0 \cap C_1$.

Condition 1) implies that $\Phi_0, \Phi_1$ and injective, and 2) says that $\operatorname{rng} \Phi_0 \cap \operatorname{rng} \Phi_1 = \emptyset$.

<u>Examples.</u> The set $\Pi$ is 2-reflective. If $C$ is a finite family of finite subsets of $N$ that contains at least two elements and is such that $(C; \subseteq)$ has a minimal element and a maximal element, then the numerated set defined by $C$ as in the corresponding example in Sec. 7 is 2-reflective. If $\mathbb{A}$ is 2-reflective and $\mathbb{B}$ is 2-complete, the $\mathbb{A} \otimes \mathbb{B}$ is 2-reflective. The numerated set $\{\emptyset, \{x\}, N \mid x \in N\}$ with a principal computable numeration is not 2-reflective.

<u>THEOREM 4.</u> If $\mathbb{A}$ is 2-reflective and $P^t_s = \{\varphi \in Map(A,S) \mid \varphi(a)=s, \varphi(b)=t\}$ (where $s,t \in S; a$, and $b \in A$ are the elements with respect to which the numeration $\alpha$ is 2-complete), then

$(Map(A,S); \leqslant_M, \{P_s^t\}_{s,t\in S})$ is a 2-discrete generalized semilattice.

We first define binary operations $\theta_s^t$ on $Map(A,S)$. If $s,t \in S, \varphi_0, \varphi_1 \in Map(A,S)$ we define the map $\theta_s^t = \theta_s^t(\varphi_0, \varphi_1)$ by:

$$\theta_s^t(x) \rightleftharpoons \begin{cases} s, & \text{if } x \notin B_0 \cup B_1, \\ \varphi_i \Phi_i^*(x), & \text{if } x \in B_i \setminus C_i, \\ t, & \text{if } x \in C_0 \cup C_1. \end{cases}$$

The map $\theta_s^t(\varphi_0,\varphi_1) \in Map(A,S)$ is well defined, since the 2-reflectivity of $A$ implies that $(\overline{B_0 \cup B_1}, B_0 \setminus C_0, B_1 \setminus C_1, C_0 \cup C_1)$ is a nontrivial decomposition of the set $A$. The map $\theta_s^t(\varphi_0,\varphi_1) \in P_s^t$, since $a \in \overline{B_0 \cup B_1}, b \in C_0 \cap C_1$. Following the proof of Theorem 3 and using the appropriate properties in Sec. 5, we verify without difficulty that the $\theta_s^t$ are generalized upper bound operations in $(Map(A,S); \leqslant_M, \{P_s^t\})$.

We note some additional properties of 2-reflective numerated sets $\mathbb{A}$.

1. If $X \subseteq A$ is $\alpha$-enumerable then the same is true of $\Phi_i(X) \cup C_i$ $(i=0,1)$, and a $\pi$-index for the set $\alpha^{-1}(\Phi_i(X) \cup C_i)$ is given effectively in terms of a $\pi$-index for the set $\alpha^{-1}(X)$.

This follows from the easily verified assertion

$$\forall x \in A\ (x \in \Phi_i(X) \cup C_i \leftrightarrow x \in B_i \wedge (\Phi_i^*(x) \in X \vee x \in C_i)).$$

The sequences $\{\Psi_k\}, \{\Psi_k^*\}, \{D_k\}$ are defined as in Sec. 7 and also possess properties 2, 3 in Sec. 7. We also define the sequences $\{E_k\}, \{F_k\}$ of subsets of $A$ by $E_0 \rightleftharpoons B_0$, $E_{k+1} \rightleftharpoons \Phi_1(E_k) \cup C_1$; $F_0 \rightleftharpoons C_0$, $F_{k+1} \rightleftharpoons \Phi_1(F_k) \cup C_1$. It follows from 1 above that

2. The sequences $\{E_k\}, \{F_k\}$ of $\alpha$-enumerable subsets of $A$ are computable.

3. $E_k \supseteq F_k$, $D_k = E_k \setminus F_k$ for every $k \in N$.

We give the proof by induction on $k$. The assertion is obvious for k = 1. Let $E_k \supseteq F_k$, $D_k = E_k \setminus F_k$; then $\Phi_1(E_k) \supseteq \Phi_1(F_k)$, whence $E_{k+1} = \Phi_1(E_k) \cup C_1 \supseteq \Phi_1(F_k) \cup C_1 = F_{k+1}$. We also have $E_{k+1} \setminus F_{k+1} = (\Phi_1(E_k) \cup C_1) \setminus (\Phi_1(F_k) \cup C_1) = \Phi_1(E_k) \setminus \Phi_1(F_k)$ since $\operatorname{rng} \Phi_1 \cap C_1 = \emptyset$. Further, $\Phi_1(E_k) \setminus \Phi_1(F_k) = \Phi_1(E_k \setminus F_k)$, since $\Phi_1$ is injective. But $E_k \setminus F_k = D_k$, and therefore $E_{k+1} \setminus F_{k+1} = \Phi_1(D_k) = D_{k+1}$

4. For any $k, m \in N$ with $k \neq m$, we have $E_k \cap E_m = F_k \cap F_m$.

By 3, if suffices to verify that $\forall k \forall m\ (k < m \to E_k \cap E_m \subseteq F_k \cap F_m)$. This is also proved by a simple induction on $k$.

5. Every 2-reflective numerated set is 2-universal.

We define the sequence $\{d_k\}$ of elements of $A$ by $d_k \rightleftharpoons \Psi_k(a)$ $(k \in N)$. The computabilit of the sequence of morphisms $\{\Psi_k\}$ implies the existence of a g.r.f. $g$ such that $\forall k\ (d_k = \alpha g(k))$. Properties 3 and 4 then easily imply that $\forall k\ (d_k \in E_k \setminus \bigcup_{m \neq k} E_m)$, which together with property 2 gives the required result.

The analogs of the other remarks made at the end of Sec. 7 are also valid.

## 9. MULTIPLE REDUCIBILITY

The above results also make it easy to study multiple ($m$- and $M$-) reducibility. Let $\mathbb{A}=(A,\alpha)$ -be a numerated set, $\Lambda$ a nonempty set, and let $F,G\in Map(\Lambda, Map(A,S))$. We say that $F$ is multiply $m$-reducible to $G$ ($F\leq^*_m G$) if there exists a g.r.f. $f$ such that $f$ $m$-reduces $F(\lambda)$ to $G(\lambda)$ for every $\lambda\in\Lambda$. The relation $\leq^*_M$ of multiple $M$-reducibility is defined analogously (with the g.r.f. $f$ replaced by a morphism $\phi:\mathbb{A}\to\mathbb{A}$).

We first analyze the special case when $\mathbb{A}=N$ and $\alpha$ is the identity numeration of $N$ in which case $\leq^*_m$ and $\leq^*_M$ coincide. The "usual" multiple $m$-reducibility [4] is recovered by specializing to $S=\{0,1\}$. We define the binary operation $\oplus$ on $Map(\Lambda, Map(N,S))$ and the unary operations $\overset{*}{\rho}_\varphi$, $\overset{*}{\tau}{}^\psi_\varphi$ for $\varphi,\psi\in Map(\Lambda,S)$ as follows: if $F,G\in Map(\Lambda, Map(N,S))$, $\lambda\in\Lambda$, then $(F\oplus G)\lambda\rightleftharpoons F(\lambda)\oplus G(\lambda)$, $(\overset{*}{\rho}_\varphi F)\lambda\rightleftharpoons\rho_{\varphi(\lambda)}(F\lambda)$, $(\overset{*}{\tau}{}^\psi_\varphi F)\lambda\rightleftharpoons\tau^{\psi(\lambda)}_{\varphi(\lambda)}(F\lambda)$.

Proposition 4. The algebraic systems $(Map(\Lambda, Map(N,S)); \leq^*_m, \oplus, \{\overset{*}{\rho}_\varphi\}, \{\overset{*}{\tau}{}^\psi_\varphi\})$ and $(Map(N, Map(\Lambda,S)); \leq_m, \oplus, \{\rho_\varphi\}, \{\tau^\psi_\varphi\})$ are naturally isomorphis.

Here the word "natural" means that the isomorphism is given by mutually inverse maps $Map(\Lambda, Map(N,S))\rightleftarrows Map(N, Map(\Lambda,S))$, whose composition "interchanges the arguments." The verification is trivial.

COROLLARY. The structure $(Map(\Lambda, Map(N,S)); \leq^*_m, \oplus, \{\overset{*}{\rho}_\varphi\})$ is a semilattice with discrete closures; $(Map(\Lambda, Map(N,S)); \leq^*_m, \oplus, \{\overset{*}{\tau}{}^\psi_\varphi\})$ is a semilattice with 2-discrete closures (cf. secs. 1,2,5).

If we are given a numerated set $\mathbb{A}=(A,\alpha)$ then the numeration $\alpha: N\longrightarrow A$ induces an inclusion $\mathcal{F}_\alpha: Map(A,S)\to Map(N,S)$ for every set $S$; the imbedding $\mathcal{F}_\alpha$ in turn induces an inclusion $\mathcal{G}_\alpha: Map(\Lambda, Map(A,S))\longrightarrow Map(\Lambda, Map(N,S))$ for every $\Lambda$.

THEOREM 5. If $\mathbb{A}$ is universal, then the closure of the image of the set $Map(\Lambda, Map(A,S))$ in $(Map(\Lambda, Map(N,S)); \leq^*_m)$ under the map $\mathcal{G}_\alpha$ coincides with the closure of the set $\{\overset{*}{\rho}_\varphi(F)\mid\varphi\in Map(\Lambda,S), F\in Map(\Lambda, Map(N,S))\}$. The same result is valid when "universal" is replaced by "2-universal" and $\overset{*}{\rho}_\varphi$ is replaced by $\overset{*}{\tau}{}^\psi_\varphi$.

Consider the diagram

$$\begin{tikzcd}
Map(\Lambda, Map(A,S)) \arrow[r, shift left] \arrow[d, "\mathcal{G}_\alpha"'] & Map(A, Map(\Lambda,S)) \arrow[l, shift left] \arrow[d, "\mathcal{F}_\alpha"] \\
Map(\Lambda, Map(N,S)) \arrow[r, shift left] & Map(N, Map(\Lambda,S)), \arrow[l, shift left]
\end{tikzcd}$$

where the horizontal maps are natural equivalences. It is easily verified that the diagram commutes and that $F\leq^*_m G\leftrightarrow\mathcal{G}_\alpha(F)\leq^*_m\mathcal{G}_\alpha(G)$ for every $F,G\in Map(\Lambda, Map(A,S))$. The required result follows from this, Theorems 1, 2, and Proposition 4.

Remark. The following generalization can easily be proved by using properties 5 in Secs. 2, 5, Let $T\subseteq Map(\Lambda,S)$, $V\rightleftharpoons\{F\in Map(\Lambda, Map(A,S))\mid\forall a\in A$ (the function $\lambda\mapsto F(\lambda)(a)$ is contained in $T$)$\}$. Then the closure of the image of the set $V$ in $(Map(\Lambda, Map(N,S)); \leq^*_m)$ coincides with the closure of the set $\{\overset{*}{\rho}_\varphi(F)\mid\varphi\in T, F\in Map(\Lambda, Map(N,S))\}$ if $\mathbb{A}$ is universal.

In particular $(V; \leqslant_m^*)$ has a natural discrete generalized semilattice structure. Similar results hold for 2-universal sets $\mathcal{A}$ and for the operations $\overset{*}{\iota}{}_\varphi^\psi$.

The relation $\leqslant_M^*$ can be analyzed in the same way. We define the binary operations $\overset{*}{\theta}_\varphi$ on $Map(\Lambda, Map(A,S))$ for the case when $A$ is reflective [and the binary operations $\overset{*}{\theta}{}_\varphi^\psi$ if $A$ is 2-reflective] as follows:

$$(\overset{*}{\theta}_\varphi(F,G))\lambda \rightleftharpoons \theta_{\varphi(\lambda)}(F\lambda, G\lambda), \quad (\overset{*}{\theta}{}_\varphi^\psi(F,G))\lambda \rightleftharpoons \theta_{\varphi(\lambda)}^{\psi(\lambda)}(F\lambda, G\lambda)$$

for arbitrary $\varphi, \psi \in Map(\Lambda, S)$, $F, G \in Map(\Lambda, Map(A,S))$, $\lambda \in \Lambda$.

THEOREM 6. If $\mathcal{A}$ is reflective then $(Map(\Lambda, Map(A,S)); \leqslant_M^*, \{\overset{*}{\theta}_\varphi\})$ is a discrete generalized semilattice. If $\mathcal{A}$ is 2-reflective then $(Map(\Lambda, Map(A,S)); \leqslant_M^*, \{\overset{*}{\theta}{}_\varphi^\psi\})$ is a 2-discrete generalized semilattice.

The analog of the remark to Theorem 5 is also valid.

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