THE LARGEST GROUPOID OF VARIETIES OF SEMIGROUPS WITH ZERO

T. A. Martynova

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It is well known [1] that with respect to the operation of multiplication, the set of all varieties of semigroups forms a partial groupoid. The structure of this is very complicated and still not known. In studying this groupoid the most natural problems are the following: 1) to describe all pairs of varieties whose product is a variety; 2) to find in this partial groupoid some important groupoids (in particular, semigroups) and to clarify their structure; 3) to distinguish maximal groupoids and semigroups. Some of these investigations have already been carried out. Thus, in [2] idempotents of this groupoid were described and a countable semigroup with nullary multiplication was distinguished, which consists of varieties of indempotent semigroups; in [3, 4] some conditions were given under which the product of two varieties of semigroups is a variety; in [5] a groupoid with the power of the continuum was mentioned, which consists of so-called 0-reduced varieties, that is, varieties of semigroups with zero 0, having an identity basis of the form w = 0, where w is a semigroup word; in [6] a number of important properties of this groupoid G were proved: it is cancellative and is the union of two nonintersecting subgroupoids H and L, where H is the largest subsemigroup, and L is an ideal of G. The structure of H has been completely clarified [7, 8]: it is the free product of a free commutative semigroup of countable rank and a free semigroup of the rank of the continuum with externally adjoined unity.

The main result of the paper is the following theorem.

THEOREM. The groupoid G is the maximal groupoid in the partial groupoid of all varieties of semigroups. In the partial groupoid of varieties of semigroups with signature zero, G is the largest groupoid.

We mention that this theorem gives a complete answer to the third question for the partial groupoid of semigroups with signature zero.

Before proceeding to the proof of the theorem, we give the necessary definitions and notation.

In this paper we use the generally accepted terminology (see [9, 10]). Our notation is also generally standard or follows the notation in [6].

Let X be a countable alphabet, and F a free semigroup over X. For a word a of F we denote by C(a) the set of all letters of X that occur in writing a, and by $|a|$ the length of the word a. A letter that occurs more than once in writing a will be called multiple, and the letters that appear in the first and last places in writing a will be called the beginning and end of a. For the graphic coincidences of the words a and b of F we shall write $a \equiv b$. The notation $b \leq a$ means that b is a subword of a; $b < a$ means that b is

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a proper subword of a. If $a \equiv bc$, then b will be called the initial and c the final segment of a. In particular, this terminology and notation will be used in those cases when b or c is an empty word. In the word $a \equiv u(b_1, b_2, \dots, b_n)$ the subwords b_i will be called b-blocks or simply blocks of a.

We denote the variety of all semigroups by $\mathfrak{M}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are subvarieties of $\mathfrak{M}$, we denote their $\mathfrak{M}$-product in the sense of Mal'tsev (see [1]) by $\mathfrak{A} \circ_{\mathfrak{M}} \mathfrak{B}$. The product $\mathfrak{A} \circ_{\mathfrak{M}} \mathfrak{A}$ will be denoted by $\mathfrak{A}^2$.

Let $\mathfrak{A}$ be a subvariety of $\mathfrak{M}$ and suppose that $S \in \mathfrak{M}$. A congruence ρ of S for which the factor semigroup S/ρ belongs to $\mathfrak{A}$ is called an $\mathfrak{A}$-congruence. The smallest $\mathfrak{A}$-congruence on S is called *verbal* and denoted by $\rho(\mathfrak{A}, S)$. If A is an identity basis of $\mathfrak{A}$, then as we know (see, for example, [11]), $\rho(\mathfrak{A}, S)$ is the congruence generated by the binary relation $\bar{\rho}$ consisting of just those pairs whose components are the values of some identity of A in S.

A variety $\mathfrak{A}$ of semigroups is called *periodic* if every semigroup of $\mathfrak{A}$ is periodic. If a monogenic semigroup that is free in $\mathfrak{A}$ has type (r, m), we say that $\mathfrak{A}$ has type (r, m).

Suppose that $A \subseteq F$. Then $\mathcal{J}(A)$ denotes an ideal generated by A, and $E(A)$ the completely characteristic ideal generated by A. The fact that $b \in E(A)$ will be denoted by $A \vdash b$ and we say that b follows from A. It is clear that if $b \in E(A)$, then $b \equiv fa\varphi g$, where $f, g \in F^1$, $a \in A$ and $a\varphi$ is the image of a under the endomorphism $\varphi: F \to F$. In particular, if $b \equiv a\varphi$, where $\varphi: F \to F$ is an automorphism, we shall say that a is equivalent to b, and write $a \sim b$. We shall denote the set $\{f \in F \mid f \sim a\}$ by $\bar{a}$. If ρ is a congruence of S and $a \in S$, then a^{ρ} is the congruence class of ρ that contains a.

Let T be a system of identities. Then the variety given by T will be denoted by [T]. We say that the identity $u_1 = v_1$ is equivalent to $u_2 = v_2$ with respect to T if $[T, u_1 = v_1] = [T, u_2 = v_2]$. This fact will be denoted thus: $u_1 = v_1 \underset{T}{\sim} u_2 = v_2$.

In this paper we also use the following notation:

$\mathfrak{A}$ is the variety of all commutative semigroups;

$\mathfrak{Z}$ is the variety of semigroups with nullary multiplication;

$\mathfrak{E}$ is the variety of one-element semigroups;

F^{01} is a free semigroup F with zero 0 and identity 1 adjoined;

$S/\mathcal{J}$ is the Riesz factor semigroup of S with respect to the ideal $\mathcal{J}$.

To prove the main theorem we need the following lemma.

LEMMA 1. If $\mathfrak{X}$ is a variety of null-semigroups that is not 0-reduced, then $\mathfrak{X}^2$ is not a variety.

Proof. Let T be the set of all nontrivial identities satisfied on each semigroup of $\mathfrak{X}$. Then $T = W \cup U$, where $W = \{w_i = 0 \mid i \in I\}$, $U = \{u_j = v_j \mid u_j \neq 0, v_j \neq 0, j \in J\}$. It is easy to see that $D = \{w_i \mid i \in I\}$ forms a completely characteristic ideal of F. It was proved in [6] that the system of identities of W has an irreducible basis of identities of W'. We denote the set $\{w_k \mid w_k = 0 \in W'\}$ by D' and put $m = \min\{|w_k| \mid w_k \in D'\}$. In view of the fact that $\mathfrak{X}$ is not 0-reduced, there is an identity $u_1 = v_1$ in U that is not equivalent to the system of identities $T_1 = T \setminus \{u_1 = v_1\}$ with respect to $\{u_1 = 0, v_1 = 0\}$. Clearly, in this case $v_1 \notin u_1, u_1 \notin v_1, C(u_1) = C(v_1)$ and $u_1 \not\equiv fw\varphi g$.

$v_1 \neq f w \varphi g$ for any $f, g \in F', w \in D'$ and endomorphism $\varphi : F \to F$. We denote the set $\{u_\ell = v_\ell \mid u_\ell = v_\ell \in U, u_\ell = v_\ell \neq \{u_\ell = 0, v_\ell = 0, \ell \in L\}$ by V and put $B = \{u_\ell, v_\ell \mid u_\ell = v_\ell \in V\}$, $n = \min\{|b_\ell| \mid b \in B\}$.

Let $A = \{w_k(d_{1k}, d_{2k}, \ldots, d_{nk}) = 0,\ u_\ell(d_{1\ell}, d_{2\ell}, \ldots, d_{p\ell}) = v_\ell(d_{1\ell}, d_{2\ell}, \ldots, d_{p\ell}) \mid w_k = 0 \in W', u_\ell = v_\ell \in V, d_{ij} \in D\}$, $\mathfrak{A} = [A]$ and let S be a semigroup of countable rank over X that is free in $\mathfrak{A}$. Clearly, $S = F^0/\sigma$, where σ is a verbal $\mathfrak{A}$-congruence. Since D is a completely characteristic ideal of F^0, any class of σ on F^0 that is not a one-element class is contained in D. Hence $\sigma \subset \rho(\mathfrak{X}, F^0)$ and $F^0/\sigma \notin \mathfrak{X}$.

Let $\rho = \rho(\mathfrak{X}, S)$. We show that $(0^\sigma)^\rho = D/\sigma$.

For since $\rho(\mathfrak{X}, F^0/\sigma) = \rho(\mathfrak{X}, F^0)/\sigma$ (see [11], for example), $S/\rho = F^0/\sigma/\rho(\mathfrak{X}, F^0/\sigma) = F^0/\rho(\mathfrak{X}, F^0)$. Hence $a^\sigma \rho b^\sigma \Leftrightarrow a \rho(\mathfrak{X}, F^0) b$, that is, $a^\sigma \rho 0^\sigma \Leftrightarrow a \rho(\mathfrak{X}, F^0) b \Leftrightarrow a \in D$.

It is now clear that $(0^\sigma)^\rho$ satisfies any identity of W' and V. Hence $(0^\sigma)^\rho \in \mathfrak{X}$, and $S \in \mathfrak{X}^2$.

We show that among the homomorphic images of S there is a $\bar{S}$ such that $\bar{S} \notin \mathfrak{X}^2$.

Everywhere below in the proof of Lemma 1 we shall denote by w a word of D' such that $|w| = m$, and by $u = v$ an identity of V such that $|u| = n$, and we shall assume that $w \equiv w(x_1, \ldots, x_k)$, $u \equiv u(x_1, \ldots, x_q)$, $\rho_1 = \rho(\mathfrak{X}, \bar{S})$ and $\tau : S \to \bar{S}$ is the natural homomorphism.

We now consider two cases.

1. Among the identities of V there is one $u_1 = v_1$ such that $v_1 \notin u_1^2$ for all

Since $\mathfrak{X}$ is the variety of null semigroups, in this case $v_1 \notin u_1^n$ for all $n \in N$, $n > 2$. Let $u_1 \equiv u_1(x_1, \ldots, x_p)$ and $w_i \equiv w(x_{p+(i-1)k+1}, \ldots, x_{p+ik})$ where $x_j \neq x_\ell$ if $j \neq \ell$.

We distinguish two subcases.

1.1. Suppose that $m < n$.

We show that in this case the semigroup $\bar{S} = S/\mathfrak{I}(v_1^\sigma)$ does not belong to $\mathfrak{X}^2$.

We observe that $w(w_1, w_2, \ldots, w_{k-1}, u_1) \neq f(h(d_1, d_2, \ldots, d_t)\varphi)g$ for any $f, g \in F'$, $h \in D' \cup B$, $d_j \in D$ and any endomorphism $\varphi : F \to F$.

For otherwise, since $u_1 \notin D$ and $w \in D'$, in each block of the word $w(w_1, w_2, \ldots, w_{k-1}, u_1)$ there is no more than one end of a d-block. Consequently, $|w| \geqslant |h|$. But if $|w| = |h|$, then from the equality $w(w_1, w_2, \ldots, w_{k-1}, u_1) = f(h(d_1, \ldots, d_t)\varphi)g$ it follows that some d-block either occurs in a u-block or it is a proper subword of w_i. The first contradicts the fact that $u_1 \notin D$, and the second contradicts the fact that $w \in D'$. Hence $|w| > |h|$, which is impossible, since $|w| = \min\{|h_i| \mid h_i \in D' \cup B\}$.

Clearly, $u_1^\sigma \tau \rho_1 0^\sigma \tau$, $w_i^\sigma \tau \rho_1 0^\sigma \tau$ and $w = 0 \in T$; hence, to prove that $\bar{S} \notin \mathfrak{X}^2$ it is sufficient to show that $w(w_1^\sigma \tau, \ldots, w_{k-1}^\sigma \tau, u_1^\sigma \tau) \neq 0^\sigma \tau$.

Clearly, $\mathfrak{I}(v_1^\sigma)$ is the complete inverse image of $0^\sigma \tau$ under the homomorphism $\tau : S \to \bar{S}$.

If $w(w_1^\sigma, \ldots, w_{k-1}^\sigma, u_1^\sigma) \in \mathfrak{I}(v_1^\sigma)$, then there is a finite sequence of words of F^0: $w(w_1, \ldots, w_{k-1}, u_1) \equiv a_1, a_2, \ldots, a_s \equiv f v_1 g$, where $f, g \in F^{01}$ in which either $a_1 \equiv f v_1 g$ or $s > 1$ and every identity $a_i = a_{i+1}$ is an immediate consequence of A (see [12]). The first contradicts the choice of $u_1 = v_1$ and w_i, the second contradicts the fact that $w(w_1, \ldots, w_{k-1}, u_1) \neq f(h(d_1, \ldots, d_t)\varphi)g$ for any $f, g \in F'$, $h \in D' \cup B$, $d_j \in D$ and endomorphism $\varphi : F \to F$.

1.2. Suppose that $m \geqslant n$.

Since $C(u) = C(v) = \{x_1, x_2, \ldots, x_q\}$ and $|u| = n$, there is a letter x_q in C(u) whose multiplicity in v is not less than that of the same letter in u. Consider the words $a^\sigma \equiv u(w_1^\sigma, \ldots,$

$w^{\sigma}_{q-1}, u^{\sigma}_1)$ and $b^{\sigma}=v(w^{\sigma}_1,\dots,w^{\sigma}_{q-1},u^{\sigma}_1)$, where x_i for $1\leqslant i<q$ takes the value w^{σ}_i, and x_q takes the value u^{σ}_1. We show that in this case the semigroup $\bar{S}=S/\mathfrak{I}(v^{\sigma}_1, b^{\sigma})$ does not belong to $\mathfrak{X}^2$.

Firstly, $u(w_1,\dots,w_{q-1},u_1)\neq f(h(d_1,\dots,d_r)\varphi)g$, since otherwise we would have $|u|>|h|$, since $u_1\notin D$, $w\in D'$. This contradicts the fact that $|u|=\min\{|h_i|\,|\,h_i\in D'\cup B\}$. Moreover, in view of the choice of $u_1=v_1$ and w_i we have $u(w_1,\dots,w_{q-1},u_1)\neq fv_1g$. Consequently, $a^{\sigma}\notin\mathfrak{I}(v^{\sigma}_1)$. If $a^{\sigma}\in\mathfrak{I}(b^{\sigma})$, then, since $u(w_1,\dots,w_{q-1},u_1)\neq f(h(d_1,\dots,d_r)\varphi)g$ we obtain $u(w_1,\dots,w_{q-1},u_1)=f'v(w_1,\dots,w_{q-1},u_1)g'$ for some $f',g'\in F^1$. Hence for $f'=g'=1$ we have $u(w_1,\dots,w_{q-1},u_1)=v(w_1,\dots,w_{q-1},u_1)$, which is impossible, since u = v is a nontrivial identity of V and $C(w_i)\cap C(u_1)=\emptyset$, $C(w_i)\cap C(w_j)=\emptyset$ for all $i\neq j$. If $f'\neq 1$ or $g'\neq 1$, then $|u(w_1,\dots,w_{q-1},u_1)|>|v(w_1,\dots,w_{q-1},u_1)|$. Taking account of this and the fact that $|u|\leqslant|v|$, and that the multiplicity of x_q in the word v is not less than that of x_q in u, it is easy to see that the multiplicity of x_q in v is greater than in u. This contradicts the fact that $u(w_1,\dots,w_{q-1},u_1)=f'v(w_1,\dots,w_{q-1},u_1)g'$, in view of the fact that $C(w_i)\cap C(u_1)=\emptyset$. Hence $a^{\sigma}\notin\mathfrak{I}(v^{\sigma}_1,b^{\sigma})$ and $u(w^{\sigma}_1\tau,\dots,w^{\sigma}_{q-1}\tau,u^{\sigma}_1\tau)\neq v(w^{\sigma}_1\tau,\dots,w^{\sigma}_{q-1}\tau,u^{\sigma}_1\tau)$. But since $u^{\sigma}_1\tau\rho_1 0^{\sigma}\tau$ and $w^{\sigma}_i\tau\rho_1 0^{\sigma}\tau$, we have $S/\mathfrak{I}(v^{\sigma}_1,b^{\sigma})\notin\mathfrak{X}^2$.

All the identities $u_\ell=v_\ell$ of V are such that $u_\ell\leqslant v^2_\ell$ and $v_\ell\leqslant u^2_\ell$. Suppose that $u_1=v_1$ is from V. Then in T there are identities $u_1y=v_1y$ and $yu_1=yv_1$, where $y\notin C(u_1)$. Assume that $u_1y=v_1y\in V$. Then $u_1y\leqslant(v_1y)^2$, and since $y\notin C(v_1)=C(u_1)$, we have $u_1y\leqslant v_1y$, that is, $v_1y\equiv fu_1y$ for $f\in F^1$. Hence $u_1y=fu_1y$. Hence, by virtue of the fact that $\mathfrak{X}$ is a null-variety, it follows that $u_1y=0$. This contradicts the definition of V. In exactly the same way we prove that $yu_1=yv_1\notin V$. Thus in this case all the identities $u_\ell=v_\ell$ of V are such that for any $y\in X$ we have $\{u_\ell y,\ yu_\ell,\ v_\ell y,\ yv_\ell\}\subseteq D$.

In the set B, among the words of minimal length we choose a word u such that if $b\vdash u$, where $b\in B$, then $b\sim u$. Since $uy\in D$, we have $uy\equiv fw_t\varphi g$ for some $f,g\in F^1$, $w_t\in D'$ and endomorphism $\varphi:F\to F$. Now in view of the fact that $u\notin D$ and $y\notin C(u)$ we have $g\equiv 1$ and $w_t\equiv w'_t y'$ where $y'\notin C(w'_t)$. Hence $w'_t\vdash u$.

In the set D' we choose a word w_1 that has smallest length among the words of D' that have no multiple letters. Clearly, $|w_1|\leqslant|w_t|\leqslant|uy|=n+1$.

To simplify the proof of the remaining subcases we first prove a number of auxiliary facts.

Suppose that $u\equiv u(x_1,x_2,\dots,x_n)$ is a word of F and $x^2_n\nleqslant u$, $v\notin D$, and that $w_1,\dots,w_{n-1}$ are such that $w_i\sim w'_i\in D'$, $C(v)\cap C(w_i)=\emptyset$, $C(w_i)\cap C(w_j)=\emptyset$ if $i\neq j$ and there is no multiple letter in w_i for $1\leqslant i<n$. Consider the word $u(w_1,\dots,w_{n-1},v)$, which is obtained from u by replacing x_i by w_i for $1\leqslant i<n$ and x_n by v, and the word $f(h(d_1,\dots,d_r)\varphi)g$, where $f,g\in F^1$, $h\in D'\cup B$, $d_j\in D$ and $\varphi:F\to F$ is an endomorphism.

<u>Fact 1.</u> If $u(w_1,\dots,w_{n-1},v)\equiv f(h(d_1,\dots,d_r)\varphi)g$, then $h\vdash u$ and $|h|<|u|$.

<u>Proof.</u> Since $d_j\varphi\nleqslant w_i$ and $d_j\varphi\nleqslant v$, each d-block of the word $f(h(d_1,\dots,d_r)\varphi)g$ contains the end of at least one a-block of the word $u(w_1,\dots,w_{n-1},v)$, where $a\in\{w_1,w_{n-1},v\}$. Suppose that the block $d_p\varphi$ contains the ends of the blocks $a_{1p},\dots,a_{\ell_p}$, and $d_q\varphi$ contains the ends of the blocks $a_{1q},\dots,a_{\ell_q}$ and that $d_p\varphi=d_q\varphi$. Then from the fact that $C(v)\cap C(w_i)=\emptyset$ and $C(w_i)\cap C(w_j)=\emptyset$ for $i\neq j$ it follows that $a_{1p}=a_{1q}$. If $a_{1p}\equiv w_i$, then in view of the fact that there is no multiple letter in w_i we see that the ends of the blocks a_{1p} and a_{1q} are

identically situated with respect to the beginnings of $d_p\varphi$ and $d_q\varphi$ respectively. (The proof of this fact is simple and is carried out in [6] in the proof of Lemma 4.) It follows that $l_p = l_q$ and $a_{1p} \equiv a_{1p}, \ldots, a_{l_p} \equiv a_{l_q}$. If $a_{1p} \equiv v$, then in view of the fact that $v^2 \nleq u(w_1, \ldots, w_{n-1}, v)$ and $C(v) \cap C(w_i) = \emptyset$ we again obtain $l_p = l_q$ and $a_{1p} \equiv a_{1q}, \ldots, a_{l_p} \equiv a_{l_q}$.

We now set up a correspondence between each block $d_j\varphi$ of the word $f(h(d_1, \ldots, d_r)\varphi)g$ and the letter $x_j \in X$, and between each letter x_j and the subword $y_{1j} \ldots y_{l_j}$ of u, if $d_j\varphi$ contains the ends of the blocks $a_{1j}, \ldots, a_{l_j}$. From the previous arguments it is clear that the subword $y_{11} \ldots y_{l_1} y_{12} \ldots y_{l_2} \ldots y_{l_m}$ of u, where $m = |h|$, follows from $x_1 x_2 \ldots x_m$. But the word $x_1 x_2 \ldots x_m$ is a consequence of h, obtained from it by possibly identifying letters. Hence $h \vdash u$. It follows that $|h| \leqslant |u|$. In the case $|h| = |u|$, from the equality $u(w_1, \ldots, w_{n-1}, v) \equiv f(h(d_1, \ldots, d_r)\varphi)g$ we see that either $d_j\varphi \leqslant v$ or $d_j\varphi < w_i$. This contradicts the hypothesis. Thus, $|h| < |u|$.

Fact 2. Suppose that $u \equiv u'x_n$ and $x_n \notin C(u')$. Then if $u(w_1, \ldots, w_{n-1}, v) \equiv f(h(d_1, \ldots, d_r)\varphi)g$, we have $h \vdash u'$.

Proof. Since $d_j\varphi \nleq v$ and $d_j\varphi \nleq w_i$, each d-block contains the end of at least one w-block. We set up a correspondence between each d-block and the system of w-blocks whose ends are in the given d-block. Then if $d_p\varphi \equiv d_q\varphi$, the systems of w-blocks corresponding to them are equal, since there is no multiple letter in w_i. It is now easy to see that some subword of u' is a consequence of h. Hence, $h \vdash u'$.

Fact 3. Suppose that $w_i \sim w \in D'$ and that there is no multiple letter in w. Then if $f_1 u(w_1, \ldots, w_n) \equiv f(h(d_1, \ldots, d_r)\varphi)g$, where $f_2 f_1 \sim w$ for $f_2 \not\equiv 1$, then $h \vdash u$. If $f \equiv 1$ and $g < w_i$ for $1 \leqslant i \leqslant n$ then $u \equiv h\psi$, where $\psi: F \to F$ is an endomorphism.

Proof. Since $d_j\varphi \nleq f_1$ and $d_j\varphi \nleq w_i$, each d-block contains the beginning of a least one w-block. We set up a correspondence between each d-block and the system of w-blocks whose beginnings are in the given d-block. Since there is no multiple letter in w, equal d-blocks correspond to equal systems of w-blocks. Hence, $h \vdash u$.

If $f \equiv 1$ and $g < w_i$, then the beginning of each w-block is in some d-block. It follows that $u \equiv h\psi$, where $\psi: F \to F$ is an endomorphism.

Fact 3'. Suppose that $w_i \sim w \in D'$ and that there is no multiple letter in w. Then if $u(w_1, \ldots, w_n)f_1 \equiv f(h(d_1, \ldots, d_r)\varphi)g$ where $f_1 f_2 \sim w$ and $f_2 \not\equiv 1$, then $h \vdash u$.

To prove this fact it is sufficient to repeat the proof of Fact 3, replacing "beginning" by "end."

Suppose that the w_i of F are such that $w_i \equiv w x_i$, where $x_i \notin C(w)$ and $x_i \neq x_j$ for $i \neq j$, and that the b of F is such that if $x \in C(b)$, than $x^2 \leqslant b$ and $h \equiv b\varphi$ for some endomorphism $\varphi: F \to F$. Consider the words $h(d_1, \ldots, d_r)$ and $a(w_1, \ldots, w_k)$, where $d_j \in D$ and $a \in F$. Clearly, a subword of $a(w_1, \ldots, w_k)$ has the form $fu(w_1, \ldots, w_n)g$, where f is a final segment of w_i and g is an initial segment of w_j.

Fact 4. If $fu(w_1, \ldots, w_n)g \equiv h(d_1, \ldots, d_r)$, then either $gf \equiv w_i$, or $gf \equiv 1$.

Proof. Since $h \equiv b\varphi \equiv b(x_1\varphi, \ldots, x_l\varphi)$, the word $h(d_1, \ldots, d_r) \equiv a_1 a_2 \ldots a_s$, where $a_i \equiv c_j\varphi(d_{1j}, \ldots, d_{pj})$. Clearly, $a_j^2 \leqslant fu(w_1, \ldots, w_n)g$ and $a_1 \equiv f w_{i_1} \ldots w_{i_l} g_1$, where g_1 is an initial segment of w_i. Moreover, if $a_1 \equiv a_p$, then since $w_i \equiv w x_i$, where $x_i \notin C(w)$, the subword a_p of $fu(w_1, \ldots, w_n)g$ can only be equal to $f w_{i_1} \ldots w_{i_l} g_1$. Now in view of the fact that $a_1^2 \leqslant fu(w_1, \ldots, w_n)g$, we have

$fw_1\ldots w_{i_1}g_1fw_1\ldots w_{i_1}g_1 \leq fu(w_1,\ldots,w_n)g$ If $f\not\equiv 1$ then $f\equiv f'x_i$ where $x_i\notin C(w')$. From this it is easy to see that $g_1f\equiv w_i$ for $1\leq i\leq n$. Hence, $a_2\equiv fw_{i_1+1}\ldots w_{i_2}g_2$. But since $a_2^2\leq fu(w_1,\ldots,w_n)g$, we again have $g_2f\equiv w_i$. Continuing similar arguments, we see that $a_s\equiv fw_{i_{k-1}+1}\ldots w_{i_k}g$ where $gf\equiv w_i$. If $f\equiv 1$, then $a_1\equiv w_1\ldots w_{i_1}g_1$, where $g_1<w_i$. Then, arguing similarly, we see that $g_1\equiv 1$, that is $a_1\equiv w_1\ldots w_{i_1}$, and furthermore $a_2\equiv w_{i_1+1}\ldots w_{i_2},\ldots,a_s\equiv w_{i_k+1}\ldots w_{i_k}$. It follows that $g\equiv 1$, that is, $gf\equiv 1$.

We now return immediately to the proof of the lemma. Thus, $w_1\in D'$ and it has smallest length among the words of D' in which there is no multiple letter. Then $|w_1|\leq n+1$.

Suppose that $u\equiv u(x_1,\ldots,x_q)$, $w_1\equiv w_1(x_1,\ldots,x_\ell)$, where x_ℓ is not multiple, and that $w_{1i}\equiv w_1(x_{q+(i-1)\ell+1},\ldots,x_{q+i\ell})$, where $x_i\not\equiv x_j$ for $i\neq j$. We consider four cases:

Case 1. In D' there is a word $w_2\equiv w_2(x_1,\ldots,x_p)$, such that $x_p^2\nleq w_2$ and $B\not\vdash w_2$.

We show that in this case the semigroup $\bar{S}=S/\mathcal{J}(v^\sigma)$ does not belong to $\mathfrak{X}^2$. Consider the word $w_2(w_{11}^\sigma\tau,\ldots,w_{1p-1}^\sigma\tau,u^\sigma\tau)$. Since $u^\sigma\tau\rho_1 0^\sigma\tau$ and $w_{1i}^\sigma\tau\rho_1 0^\sigma\tau$, to prove that $\bar{S}\in\mathfrak{X}^2$ it is sufficient to establish that $w_2(w_{11}^\sigma\tau,\ldots,w_{1p-1}^\sigma\tau,u^\sigma\tau)\neq 0^\sigma\tau$.

We first observe that for any $f,g\in F^1, h\in D'\cup B, d_i\in D$ and any endomorphism $\varphi: F\to F$ we have $w_2(w_{11},\ldots,w_{1p-1},u)\not\equiv f(h(d_1,\ldots,d_r)\varphi)g$.

For if $w_2(w_{11},\ldots,w_{1p-1},u)\equiv f(h(d_1,\ldots,d_r)\varphi)g$, then since $u\notin D, w_{1i}\sim w_1\in D'$, there is no multiple letter in w_1, and so by Fact 1 we have $h\vdash w_2$ and $|h|<|w_2|$. Hence, since D' is irreducible, we see that $h\in B$, that is, $B\vdash w_2$. This contradicts the hypothesis.

Moreover, since $C(w_1)\cap C(v)=\emptyset$ and $u^2\nleq w_2(w_{11},\ldots,w_{1p-1},u)$ we have $w_2(w_{11},\ldots,w_{1p-1},u)\not\equiv fvg$ for any $f,g\in F^1$. From this and the fact that $w_2(w_{11},\ldots,w_{1p-1},u)\not\equiv f(h(d_1,\ldots,d_r)\varphi)g$, it follows that $w_2(w_{11}^\sigma,\ldots,w_{1p-1}^\sigma,u^\sigma)\notin\mathcal{J}(v^\sigma)$, that is, $w_2(w_{11}^\sigma\tau,\ldots,w_{1p-1}^\sigma\tau,u^\sigma\tau)\neq 0^\sigma\tau$.

Case 2. In the set V there is a word in which there is no multiple letter.

Suppose that $u_1\equiv v_1\in V$ and that u_1 is a word of smallest length of B in which there is no multiple letter. Since $v_1\leq u_1^2$ and $C(v_1)=C(u_1)$, by the choice of u_1 we have $|v_1|\geq|u_1|$.

If $u_1\equiv u_1(x_1,\ldots,x_p)$, where x_p is not multiple, consider the words $a^\sigma\equiv u_1(w_{11}^\sigma,\ldots,w_{1p-1}^\sigma,u^\sigma)$ and $b^\sigma\equiv v_1(w_{11}^\sigma,\ldots,w_{1p-1}^\sigma,u^\sigma)$, where x_i for $1\leq i<p$ takes the value w_{1i}^σ, and x_p takes the value u^σ. We show that the semigroup $\bar{S}=S/\mathcal{J}(v_1^\sigma b^\sigma)$ does not belong to $\mathfrak{X}^2$.

Firstly, $u_1(w_{11},\ldots,w_{1p-1},u)\not\equiv f(h(d_1,\ldots,d_r)\varphi)g$, since otherwise by Fact 1 we should have $h\vdash u_1$ and $|h|<|u_1|$. Hence $h\in B$ and every letter is multiple in h. Then $u_1\equiv f_1h\varphi g_1$, where $f_1\not\equiv 1$ or $g_1\not\equiv 1$. Consequently, for $x\notin C(h)$ either $xh\vdash u_1$ or $hx\vdash u_1$. But $\{xh, hx\}\subseteq D$, and this contradicts the fact that $u_1\notin D$. Moreover, since $u^2\nleq u_1(w_{11},\ldots,w_{1p-1},u)$ and $C(w_{1i})\cap C(v)=\emptyset$, we have $u_1(w_{11},\ldots,w_{1p-1},u)\not\equiv fvg$. Finally, in view of the fact that $C(v_1)=C(u_1)$, $v_1\leq u_1^2$ and $|v_1|\geq|u_1|$, we have $|v_1(w_{11},\ldots,w_{1p-1},u)|\geq|u_1(w_{11},\ldots,w_{1p-1},u)|$. From this and the fact that $u_1=v_1$ is a nontrivial identity it follows that $u_1(w_{11},\ldots,w_{1p-1},u)\not\equiv fv_1(w_{11},\ldots,w_{1p-1},u)g$. Hence, $a^\sigma\notin\mathcal{J}(v_1^\sigma,b^\sigma)$, that is $u_1(w_{11}^\sigma\tau,\ldots,w_{1p-1}^\sigma\tau,u^\sigma\tau)\neq v_1(w_{11}^\sigma\tau,\ldots,w_{1p-1}^\sigma\tau,u^\sigma\tau)$. Thus, $\bar{S}\notin\mathfrak{X}^2$.

Case 3. In the set B there is a word $u_1\equiv u_1(x_1,x_2,\ldots,x_p)$ such that $x_p^2\nleq u_1$ and $B\setminus\bar{u}_1\not\vdash u_1$.

Suppose that $u_1=v_1\in V$. Consider the words $a^\sigma\equiv u_1(w_{11}^\sigma,\ldots,w_{1p-1}^\sigma,u^\sigma)$ and $b^\sigma\equiv v_1(w_{11}^\sigma,\ldots,w_{1p-1}^\sigma,u^\sigma)$. We observe that $u_1(w_{11},\ldots,w_{1p-1},u)\not\equiv f(h(d_1,\ldots,d_r)\varphi)g$, since otherwise in view of Fact 1 we should have $h\vdash u_1$ and $|h|<|u_1|$. Hence $h\in B$ and $B\setminus\bar{u}_1\vdash u_1$. This contradicts the choice of u_1. Moreover, $u_1(w_{11},\ldots,w_{1p-1},u)\not\equiv fvg$, since $u^2\leq u_1(w_{11},\ldots,w_{1p-1},u)$ and $C(w_{1i})\cap C(v)=\emptyset$. Finally,

$w_{1i+2},\ldots,w_{1p-1},u) \neq f v_1(w_{11},\ldots,w_{1p-1},u)g$, since otherwise from the fact that $C(w_{1i})\cap C(u)=\emptyset$ and $\cap C(w_{1j})=\emptyset$, it would follow that the u-blocks and w-blocks of the word $v_1(w_{11},\ldots,w_{1p-1},u)$ incide respectively with the u-blocks and w-blocks of the word $u_1(w_{11},\ldots,w_{1p-1},u)$. Hence v_1 is a subword of u_1 . This contradicts the fact that $u_1=v_1\in V$. Thus, $u_1(w_{11}^{\sigma}\tau,\ldots,w_{1p-1}^{\sigma}\tau, u^{\sigma}\tau)\neq v_1(w_{11}^{\sigma}\tau,\ldots,w_{1p-1}^{\sigma}\tau,u^{\sigma}\tau)$, and so in this case the semigroup $\bar{S}=S/\mathcal{I}(v^{\sigma},b^{\sigma})$ does not belong to $\mathfrak{X}^2$.

Case 4. For w_j from D', either $x\in C(w_j)$ implies that $x^2\leqslant w_j$, or $B\vdash w_j$, and in any word u_i of B each letter is multiple and either $x\in C(u_i)$ implies that $x^2\leqslant u_i$, or $B\setminus\bar{u}_i\vdash u_i$.

We show that in this case any word $u_i\in B$ follows from some word $b_i\in B$, in which if $x\in C(b_i)$, then $x^2\leqslant b_i$.

For if there is a word u_i in B such that for some $x\in C(u_i)$ we have $x^2\nleqslant u_i$, then $B\setminus\bar{u}_i\vdash u_i$. Then $u_j\vdash u_i$, where $u_j\in B\setminus\bar{u}_i$. Suppose that $u_1\vdash u_i$ and $|u_1|=\min\{|u_j|\,|\,u_j\in B, u_j\vdash u_i\}$. We denote $|u_1|$ by p and put $B_p=\{u_j\,|\,u_j\in B, |u_j|=p\}$. If in u_1 we have $x^2\leqslant u_1$ for $x\in C(u_1)$, then $b_i\equiv u_1$. If this is not so, then $u_2\vdash u_1$, where $u_2\in B\setminus\bar{u}_1$. Clearly, $|u_2|\leqslant p$ and $u_2\vdash u_i$. Now if each letter x of $C(u_2)$ is such that $x^2\leqslant u_2$, we put $b_i=u_2$. If in $C(u_2)$ there is a letter x such that $x^2\nleqslant u_2$, then $|u_2|=p$ and in B there is a u_3 such that $u_3\vdash u_2$ and $u_3\in B\setminus\bar{u}_2\cup\bar{u}_1$. Since $\bar{u}_1\neq\bar{u}_2$, $\bar{u}_1,\bar{u}_2\subseteq B_p$ and the number of equivalence classes of B_p is finite, it is easy to see that in finitely many steps the process leads us to a word $u_k\in B_p$, where $u_k\vdash\ldots\vdash u_2\vdash u_1$, and either for any letter x of $C(u_k)$ we have $x^2\leqslant u_k$ or $u_{k+1}\vdash u_k$, where $|u_{k+1}|<p$. Then in the first case $b_i\equiv u_k$, and in the second case $b_i\equiv u_{k+1}$.

We recall that in the case under consideration $w_1\in D'$ and there is no multiple letter in w_1. Hence, $B\vdash w_1$, that is, $w_1\equiv fu_i\varphi g$ for $u_i\in B$. But since every letter is multiple in u_i, $w_1\equiv fu_i\varphi g$ implies that $f\neq 1$ or $g\neq 1$. Consequently, $|w_1|=n+1$. From this and the condition $|w_1|\leqslant n+1$ obtained earlier we have $|w_1|=n+1$. But since $|w_t|\geqslant n+1$ and the last letter is not multiple in w_t , we may suppose without loss of generality that $w_1\equiv w_t\equiv w_t' y$.

Suppose that $w_1\equiv w_1(x_1,\ldots,x_\ell)$ and $w_{1i}\equiv w_1(x_{(i-1)\ell+1},\ldots,x_{i\ell})$, where $x_i\neq x_j$ for $i\neq j$.

We now prove that the semigroup $\bar{S}=S/\mathcal{I}(v^{\sigma})$ does not belong to $\mathfrak{X}^2$. Consider the word $w_1(w_{11},\ldots,w_{1\ell-1},u)$, where the last (nonmultiple) letter of w_1 takes the value u. If $w_1(w_{11}^{\sigma}\tau,\ldots,w_{1\ell-1}^{\sigma}\tau,u^{\sigma}\tau)=0^{\sigma}\tau$, there is a finite sequence of words of F^0: $w_1(w_{11},\ldots,w_{1\ell-1},u)\equiv a_1, a_2,\ldots,a_s\equiv fvg$, where $f,g\in F^{01}$, in which either $a_1\equiv fvg$ or every identity $a_i=a_{i+1}$ is a direct consequence of A. Clearly, $a_1\not\equiv fvg$. Hence, $w_1(w_{11},\ldots,w_{1\ell-1},u)\equiv f(h(d_1,\ldots,d_r)\varphi)g$, where $f,g\in F'$, $h\in D'\cup B$, $d_j\in D$ and $\varphi: F\to F$ is an endomorphism. Hence, since $u\in D$, $w_{1i}\sim w_1\in D'$, and there is no multiple letter in w_1, by virtue of Fact 1 we have $h\vdash w_1$ and $|h|<|w_1|$. Consequently, $h\in B$ and $|h|=n$. Then the last letter in h is multiple. Hence, $f(h(d_1,\ldots,d_r)\varphi)\leqslant w_t'(w_{11},\ldots,w_{1\ell-1})$. But since $|w_t'|=n$ and $d_j\varphi\neq w_{1i}$, we have $f\equiv 1$ and $d_j\varphi=w_{1j}$. Hence $h_i\vdash w_t'$. But, as we showed earlier, $w_t'\vdash u$. Consequently, in view of the choice of u, we see that $u\sim h$ and $w_t'\sim u$. Hence $w_1\equiv\tilde{u}y'$, where $\tilde{u}\sim u$, and $y'\notin C(\tilde{u})$ and $w_1(w_{11},\ldots,w_{1q},u)\equiv\tilde{u}(w_{11},\ldots,w_{1q})u$. Thus, $a_1\equiv\tilde{u}(w_{11},\ldots,w_{1q})u$, where $C(w_{1i})\cap C(u)=\emptyset$ and $w_{1j}\sim\tilde{u}y'$.

It is now clear that $a_2\neq 0$ and $a_2\equiv\tilde{v}(w_{11},\ldots,w_{1q})u$, where $\tilde{u}=\tilde{v}\in V$. Hence $a_2\not\equiv fvg$.

If $a_2\equiv f(h_1(d_1,\ldots,d_r)\varphi)g$, where $h_1\in D'\cup B$, then $\tilde{v}y'(w_{11},\ldots,w_{1q},u)\equiv f(h_1(d_1,\ldots,d_r)\varphi)g$. Hence, by Fact 2 we obtain $h_1\vdash v$. Consequently, $h_1\in B$. Hence all the letters in h_1 are multiple.

But since $C(w_{1i}'')\cap C(u)=\varnothing$, we have $|g|\geq|u|$. If $|g|>|u|$, then since $xu\in D$, we obtain $g=d\in D$ and $f(h_1(d_1,\ldots,d_z)\varphi)g\equiv f(h_1 x(d_1,\ldots,d_z,d)\varphi_1)$. Now by Fact 2, from $\tilde{v}\varphi'(w_{11},\ldots,w_{1q},u)\equiv f(h_1 x(d_1,\ldots,d_z,d)\varphi_1)$ it follows that $h_1 x\vdash v$. But since $h_1 x\in D$, this contradicts the fact that $v\notin D$. Hence $|g|=|u|$, that is, $g\equiv u$, and $\tilde{v}(w_{11},\ldots,w_{1q})\equiv f(h_1(d_1,\ldots,d_z)\varphi)$. Now if $f\neq 1$, then by taking account of Fact 3 we obtain $xh_1\vdash v$. This again contradicts the fact that $v\notin D$. Hence, $\tilde{v}(w_{11},\ldots,w_{1q})\equiv h_1(d_1,\ldots,d_z)\varphi\equiv h_1\psi(d_1',\ldots,d_z')$, where ψ is an endomorphism of F such that $\psi\colon X\to X$. Then from Fact 3 it follows that $\tilde{v}\equiv h_1\psi\varphi'$. Consequently, $a_2\equiv h_1\psi(d_1',\ldots,d_z')u\equiv h_1\psi\varphi'(w_{11},\ldots,w_{1q})u$.

It is now clear that $a_3\neq 0$, $a_3\equiv b_1\psi\varphi'(w_{11},\ldots,w_{1q})u$, where $h_1=b_1\in V$. Hence, $a_3\neq fvg$.

Since $b_1<h_1^2$, that is, $b_1\equiv h_{12}h_{11}x_1$, where h_{12} is the final and $h_{11}x_1$ the initial segment of h_1, we have $b_1\psi(d_1',\ldots,d_z')<[h_1\psi(d_1',\ldots,d_z')]^2$ and $b_1\psi(d_1',\ldots,d_z')\equiv(h_{12}h_{11}x_1)\psi(d_1',\ldots,d_z')$. From what was said above, $h_1\psi(d_1',\ldots,d_z')\equiv h_1\psi\varphi'(w_{11},\ldots,w_{1q})$, where $h_1\psi\varphi'\equiv\tilde{v}$ by Fact 3. But in the proof of this, each d-block corresponds to a system of w-blocks whose beginnings are in the given d-block. Hence, either $b_1\psi(d_1',\ldots,d_z')\equiv f_1[(h_{12}h_{11})\psi\varphi'(w_{11},\ldots,w_{1q})]g_1$ where $f_1\neq 1$ or $g_1\neq 1$ and f_1 is the final and g_1 the initial segment of the word w_{1i} for $1\leq i\leq q$, or $b_1\psi(d_1',\ldots,d_z')\equiv(h_{12}h_{11}x_1)\psi\varphi'(w_{11},\ldots,w_{1q})$. We denote $(h_{12}h_{11}x_1)\psi\varphi'$ by cx, and $(h_{12}h_{11})\psi\varphi'$ by c. We observe that $cx\notin D$, since otherwise $cx\equiv(h_{12}h_{11}x_1)\psi\varphi'\equiv b_1\psi\varphi'\in D$. Hence, $h_1\psi\varphi'\equiv\tilde{v}\in D$. But this contradicts the fact that $u=v\in V$.

Suppose that $b_1\psi(d_1',\ldots,d_z')\equiv f_1c(w_{11},\ldots,w_{1q})g_1$. Then since $h_1<b_1^2$, we have $h_1\psi(d_1',\ldots,d_z')<[b_1\psi(d_1',\ldots,d_z')]^2$, that is $h_1\psi\varphi'(w_{11},\ldots,w_{1q})<f_1c(w_{11},\ldots,w_{1q})g_1f_1c(w_{11},\ldots,w_{1q})g_1$. Now in view of the fact that $w_{1i}\sim\tilde{u}y'$, where $y\notin C(\tilde{u})$, and $C(w_{1i})\cap C(w_{1j})=\varnothing$ for $i\neq j$, the beginning of the w-block of the word $h_1\psi\varphi'(w_{11},\ldots,w_{1q})$ coincides with either the beginning of the w-blocks of the word $c(w_{11},\ldots,w_{1q})$ or with the beginning of g_1. But since $h_1\psi\varphi'(w_{11},\ldots,w_{1q})\leq c(w_{11},\ldots,w_{1q})$, we have $g_1f_1\equiv w_{1i}$ for $1\leq i\leq q$. Thus, $a_3\equiv f_1c(w_{11},\ldots,w_{1q})g_1u$, where $g_1f_1\equiv w_{1i}$, $g_1\neq 1$, $f_1\neq 1$ and $C(w_{1i}'')\cap C(u)=\varnothing$, or $a_3\equiv cx(w_{11},\ldots,w_{1q})u$.

Suppose that $f_1c(w_{11},\ldots,w_{1q})g_1u\equiv f(h_2(d_1,\ldots,d_t)\varphi_1)g$, $C\equiv C'C''$, $C''\neq 1$, $h_2\equiv h_2'x_t$. Then for $|f|>|f_1|$ and $|g|\geq|u|$ and in view of Fact 3' we see that $h_2\vdash c$, that is, $h_2x\vdash cx$, where $x\notin C(h_2)$. For $|f|>|f_1|$, $|g|<|u|$ and $|(d_t\varphi_1)g|\geq|g_1u|$ we see that x_t is not a multiple letter of h_2 and in view of Fact 3, $h_2'\vdash c'$, that is, $h_2\vdash c$ and $h_2x\vdash cx$. This contradicts the fact that $cx\notin D$. If $|(d_t\varphi_1)g|<|g_1u|$ for $|f|>|f_1|$, then x_t is not multiple in h and by Fact 3' we obtain $h_2'\vdash c$, that is, $h_2\vdash cx$. Consequently, $h_2\in B$. This contradicts the fact that in any word of B every letter is multiple. Hence, $|f|\leq|f_1|$. Now for $|g|<|u|$ and $|(d_t\varphi_1)g|\geq|g_1u|$ we find that x_t is not a multiple letter in h_2, and in view of Fact 3, $h_2'\vdash c$, that is, $h_2\vdash cx$. Hence, $h_2\in B$. This contradicts the fact that in any word of B every letter is multiple. If $|(d_t\varphi_1)g|<|g_1u|$ the letter x_t is again not multiple in h_2. Hence, $h_2\in D'$ and $f_1'c(w_{11},\ldots,w_{1q})g_1'\equiv h_2'(d_1,\ldots,d_{t-1})\varphi_1\equiv h_2'\psi_1(d_1',\ldots,d_{t-1}')$, where $f_1\equiv ff_1'$, $g_1\equiv g_1'g_1''$, $g_1'\neq 1$, $g_1''\neq 1$, since $g_1''u\equiv(d_t\varphi_1)g$ and $u\notin D$. Now since $h_2\in D'$ and there is no multiple letter in h_2, we have $B\vdash h_2$, that is, $h_2\equiv fb\varphi g$, where $b\in B$. But in any word of B every letter is multiple, and in h_2 the last letter is not multiple. Hence, $g\neq 1$ and $b\vdash b_2'$, and by hypothesis $b_1\vdash b$, where $x\in C(b_1)$ for $x^2\leq b_1$. Now in view of Fact 4, $f_1'c(w_{11},\ldots,w_{1q})g_1'\equiv h_2'(d_1,\ldots,d_{t-1})\varphi_1$ implies that $g_1'f_1'\equiv w_{1i}$, where $f_1'\leq f_1$ and $g_1'<g_1$. This contradicts the fact that $g_1f_1\equiv w_{1i}$. Moreover, for $|g|>|u|$ and $|g|\geq|g_1u|$, in view of Fact 3 we have $h_2\vdash c$, that is, $h_2x\vdash cx$. This contradicts the fact that $cx\notin D$. Finally, if $|u|\leq|g|<|g_1u|$, then $f_1c(w_{11},\ldots,w_{1q})g_1u\equiv f(h_2(d_1,\ldots,$

$d_t)\varphi_1)g$ implies that $f_1'c(w_{11},\dots,w_{1q})g_1' \equiv h_2(d_1,\dots,d_t)\varphi_1 \equiv h_2\psi(d_1',\dots,d_t')$ for $f_1 \equiv ff_1'$ and $g_1 \equiv g_1'g_1''$, where $g_1''u \equiv g$. Hence, in view of Fact 3, $cx \equiv h_2\psi_1\varphi_1'$ and, in view of Fact 4, $g_1'f_1' \equiv w_{1i}$, where $f_1' \leq f_1$ and $g_1' \leq g_1$. But since $g_1f_1 \equiv w_{1i}$, we have $g_1 \equiv g_1'$ and $f_1 \equiv f_1'$. Consequently $g \equiv u$ and $f \equiv 1$. Hence in this case if $a_3 \equiv f(h_2(d_1,\dots,d_t)\varphi_1)g$, then $a_3 \equiv h_2(d_1,\dots,d_t)\varphi_1 u$, where $h_2(d_1,\dots,d_t)\varphi_1 \equiv h_2\psi_1(d_1',\dots,d_t') \equiv f_1c(w_{11},\dots,w_{1q})g_1$ and $cx \equiv h_2\psi_1\varphi_1'$.

Suppose that $cx(w_{11},\dots,w_{1q})u \equiv f(h_2(d_1,\dots,d_t)\varphi_1)g$ and $c \equiv x_1c'$. Then for $|g| > |u|$ from Fact 3' we obtain $h_2 \vdash c$, that is, $h_2 x \vdash cx$. This contradicts the fact that $cx \notin D$. For $|g| < |u|$ we find that the block $d_t\varphi_1$ is not multiple. Consequently, $h_2 \equiv h_2'x_t$ where $x_t \notin C(h_2')$. Now since $|(d_t\varphi)g| > |u|$, in view of Fact 3' we have $h_2' \vdash c$, that is, $h_2 \vdash cx$. Hence, $h_2 \in B$. This contradicts the fact that in any word of B every letter is multiple. Consequently, $g \equiv u$. Now for $f \not\equiv 1$ we have $h_2 \vdash c'x$, that is, $xh_2 \vdash cx$, which is impossible. Hence, $f \equiv 1$, and in this case, if $a_3 \equiv f(h_2(d_1,\dots,d_t)\varphi_1)g$, then $a_3 \equiv h_2(d_1,\dots,d_t)\varphi_1 u$, where $h_2(d_1,\dots,d_t)\varphi_1 \equiv h_2\psi_1(d_1',\dots,d_t') \equiv cx(w_{11},\dots,w_{1q})$ and $cx \equiv h_2\psi_1\varphi_1'$.

Thus, $a_3 \equiv f(h_2(d_1,\dots,d_t)\varphi_1)g \equiv h_2\psi_1(d_1',\dots,d_t')u \in \{cx(w_{11},\dots,w_{1q})u,\ f_1c(w_{11},\dots,w_{1q})g_1u \mid g_1f_1 \equiv w_{1i},\ cx \equiv h_2\psi_1\varphi_1' \in B\}$.

It is easy to see that in both cases $a_4 \not\equiv 0$ and $a_4 \equiv b_2\psi_1(d_1',\dots,d_t')u$, where $h_2 = b_2 \in V$ and $C(d_i') \cap C(u) = \emptyset$. Since $b_2 < h_2^2$, that is, $b_2 \equiv h_{22}h_{21}x_2$, where h_{22} is the final and $h_{21}x_2$ the initial segment of the word h_2, we have $(h_{22}h_{21}x_2)\psi_1(d_1',\dots,d_t') < [h_2\psi_1(d_1',\dots,d_t')]^2$. Taking account of the fact that $h_2\psi_1(d_1',\dots,d_t') \in \{cx(w_{11},\dots,w_{1q});\ f_1c(w_{11},\dots,w_{1q})g_1 \mid g_1f_1 \equiv w_{1i},\ 1 \leq i \leq q\}$, we obtain either $b_2\psi_1(d_1',\dots,d_t') \equiv f_2[(h_{22}h_{21})\psi_1\varphi_1'(w_{11},\dots,w_{1q})]g_2$, where $f_2 \not\equiv 1$ or $g_2 \not\equiv 1$ and f_2 is the final and g_2 the initial segment of w_{1i}, or $b_2\psi_1(d_1',\dots,d_t') \equiv (h_{22}h_{21}x_2)\psi_1\varphi_1'(w_{11},\dots,w_{1q})$. Suppose that $(h_{22}h_{21}x_2)\psi_1\varphi_1' \equiv c_1x$, and $(h_{22}h_{21})\psi_1\varphi_1' \equiv c_1$. We observe that $c_1x \notin D$, since otherwise $b_2\psi_1\varphi_1' \in D$, consequently $h_2\psi_1\varphi_1' \in D$, which implies that $v \in D$. But this contradicts the fact that $u = v \in V$. Moreover, if $b_2\psi_1(d_1',\dots,d_t') \equiv f_2c_1(w_{11},\dots,w_{1q})g_2$, then, arguing as in the similar case with a_3, we obtain $g_2f_2 \equiv w_{1i}$. Thus, $a_4 \equiv f_2c_1(w_{11},\dots,w_{1q})g_2u$, where $g_2 \not\equiv 1$, $f \not\equiv 1$, $g_2f_2 \equiv w_{1i}$ or $a_4 \equiv c_1x(w_{11},\dots,w_{1q})u$.

Suppose that $a_4 \equiv f(h_3(d_1,\dots,d_p)\varphi_2)g$. Then if $a_4 \equiv f_2c_1(w_{11},\dots,w_{1q})g_2u$, repeating the arguments given for the similar case with a_3, we see that $g \equiv u$, $f \equiv 1$ and $h_3(d_1,\dots,d_p)\varphi_2 \equiv h_3\psi_2(d_1',\dots,d_p') \equiv f_2c_1(w_{11},\dots,w_{1q})g_2$, where $c_1x \equiv h_3\psi_2\varphi_2'$. If $a_4 \equiv c_1x(w_{11},\dots,w_{1q})u$, then $g \equiv u$, $f \equiv 1$, and $h_3(d_1,\dots,d_p)\varphi_2 \equiv h_3\psi_2(d_1',\dots,d_p') \equiv c_1x(w_{11},\dots,w_{1q})$, where $cx \equiv h_3\psi_2\varphi_2'$.

Thus, $a_4 \equiv f(h_3(d_1,\dots,d_p)\varphi_2)g \equiv h_3\psi_2(d_1',\dots,d_p')u \in \{c_1x(w_{11},\dots,w_{1q})u,\ f_2c_1(w_{11},\dots,w_{1q})g_2u \mid g_2f_2 \equiv w_{1i},\ c_1x \equiv h_3\psi_2\varphi_2' \in B\}$.

It is now clear that for any $f, g \in F^{01}$ we have $a_4 \not\equiv f0g$, $a_5 \not\equiv f0g$ and so on. Consequently, in any finite sequence of words of F^0: $w_1(w_{11},\dots,w_{1\ell-1},u) \equiv a_1, a_2,\dots,a_{s-1},a_s$, where each identity $a_i = a_{i+1}$ is an immediate consequence of A, we have $a_s \not\equiv f0g$. Hence, $w_1(w_{11}^{\sigma}\tau,\dots,w_{1\ell-1}^{\sigma}\tau, u^{\sigma}\tau) \neq 0^{\sigma}\tau$, and consequently $S/\mathfrak{I}(v^{\sigma}) \notin \mathfrak{X}^2$.

Lemma 1 is proved.

The proof of the next lemma is actually contained in [11]. We give it here for the reader's convenience.

<u>LEMMA 2.</u> If $\mathfrak{X}$ is a proper nonperiodic variety, then $\mathfrak{X}^2$ is not a variety.

<u>Proof.</u> Since $\mathfrak{X}$ is a nonperiodic variety, a free monogenic subgroup of $\mathfrak{X}$ is infinite cyclic. Now from the fact that a free commutative semigroup is embedded in some power of an

infinite cyclic semigroup, it follows that $\mathfrak{A} \subseteq \mathfrak{X}$. For an absolutely free semigroup F of any cardinality there is a congruence ρ such that F/ρ is a free commutative semigroup. But since there are no idempotents in F/ρ, $F \in \mathfrak{X}^2$. On the other hand, among the homomorphic images of free semigroups there are all the simple semigroups, and any semigroup is enclosed in some simple semigroup. Consequently, if $\mathfrak{X}^2$ is a variety, then any simple semigroup P belongs to $\mathfrak{X}^2$. Then $P \in \mathfrak{X}$, since any congruence ρ on P is either universal or the equality relation. Hence, $F \in \mathfrak{X}$, which contradicts the hypothesis.

LEMMA 3. If $\mathfrak{X}$ is a periodic variety of semigroups that is not a null-variety, then $\mathfrak{Z} \circ \mathfrak{X}$ is not a variety.

Proof. Suppose that $\mathfrak{X}$ has type (r,m) and that T is the set of all nontrivial identities that are satisfied on every semigroup of $\mathfrak{X}$.

If ρ is a verbal $\mathfrak{X}$-congruence on F and x and y are distinct elements of X, then for x^r and y^r we can present the following two cases: $x^r \rho y^r$ and $x^r \bar{\rho} y^r$. Let us consider them.

1. Suppose that $x^r \rho y^r$. Then in T there is an identity $x^r = y^r$. If $r=1$, then $\mathfrak{X} = \mathfrak{Z}$, contrary to hypothesis. If $r>1$, we take a free semigroup F_2 of rank 2 generated by elements a and b. Let $\rho = \rho(\mathfrak{X}, F_2)$. Since there is an identity $x^r = y^r$ in T, from which it follows that $x^r y = y x^r$ and $x^r = x^{2r}$, we have $a^r b a^r \rho b a^r$ and $a^r b^r \rho a^r$. Moreover, we have $a^r b \bar{\rho} a^r$ and $b a^r \bar{\rho} a^r$.

For if $a^r b \rho a^r$, then in T there is an identity $x^r y = x^r$, and so $y^{r+1} = y^r$. From this and the fact that $x^r = y^r$, we have $yx^r = y^{r+1}$, $yx^r = y^r$, $yx^r = x^r$. Thus, $yx^r = x^r$ and $x^r y = x^r$ are identities of T. Hence, $\mathfrak{X}$ is a null-variety. This contradicts the hypothesis.

The assumption $b a^r \rho a^r$ again leads to the same contradiction.

Finally we show that $b a^{r-1} \bar{\rho} a^r$ and $a^{r-1} b \bar{\rho} a^r$.

For if $b a^{r-1} \rho a^r$, then in T there are identities $xy^{r-1} = y^r$ and $xy^r = y^{r+1}$. But since there is an identity $x^r = y^r$ in T, we have $xy^r = x^{r+1}$. Hence, $x^{r+1} = y^{r+1}$ is an identity of T. From this and the fact that $yx^r = y^{r+1}$, we have $y^{r+1} x = y^{r+2}$. Consequently, $x^{r+2} = y^{r+2}$. Now $yx^{r+1} = y^{r+2}$, $y^{r+2} x = y^{r+3}$, $x^{r+3} = y^{r+3}$, $\ldots$, $x^{r+m-1} = y^{r+m-1}$.

In this case, from $y_1 x^r = y_1^{r+1}$ it follows that $y_1 x^r = y^{r+1}$, $y_2 y_1 x^r = y_2^{r+2}$, $y_2 y_1 x^2 = y_3^{r+2}$, $\ldots$, $y_m \cdots y_2 y_1 x^r = y_m^{r+m}$ and $y_m \cdots y_2 y_1 x^r = x^r$. Similarly from $x^r y_m = y_m^{r+1}$ it follows that $x^r y_m = y_{m-1}^{r+1}$, $\ldots$, $x^r y_m \cdots y_2 y_1 = x^r$.

Thus in T there are identities $x^r y_m \cdots y_2 y_1 = x^r$, $y_m \cdots y_2 y_1 x^r = x^r$ and $x^{r+m} = x^r$. Hence, $x^r y = x^{r+m} y = x^r x \ldots xy = x^r$ and $yx^r = yx^{r+m} = yx x \ldots x x^r = x^r$. It follows that $\mathfrak{X}$ is a null-variety, contrary to hypothesis.

If we assume that $a^{r-1} b \rho a^r$, then similar arguments again lead to the same contradiction.

Moreover, in each class C_i^{ρ} that is a subsemigroup and contains an element u for which $|u| \geq 2r$ we fix an element e_i such that $e_i \notin \{a^r b a^r, a^{r-1} b a^r, a^r b a^{r-1}\} = B$ and $|e_i| = \min\{|u| \mid u \in C_i^{\rho}, |u| \geq 2r\}$. We denote the set of elements u of F_2 such that $|u| \geq 2r$, $u \notin B$ and $u \rho e_i$ by $\bar{e}_i$. Clearly, if $\bar{e}_i \neq \emptyset$, then $\bar{e}_i$ is an ideal of C_i^{ρ}. Let σ be a relation on F_2 defined as follows: if $u, v \in \bigcup_{i \in I} \bar{e}_i$, then $u \sigma v \iff u, v \in \bar{e}_i$, and if $u \notin \bigcup_{i \in I} \bar{e}_i$ or $v \notin \bigcup_{i \in I} \bar{e}_i$, then $u \sigma v \iff u = v$. We show that σ is a congruence on F_2.

Obviously, σ is an equivalence relation. It is stable, since if $u\sigma v$, then $u\rho v$. Hence, for any c of F_2 we have $uc\rho vc$. In the case $u \neq v$ we have $|u| \geq 2r$, $|v| \geq 2r$ and $u, v \notin B$. Consequently, $|uc| > 2r$, $|vc| > 2r$ and $uc, vc \notin B$. Hence, $uc\sigma vc$.

Let $S = F_2/\sigma$. Then $S = \bigcup_{i \in I} \overline{c}_i$, where

$$\overline{c}_i = \begin{cases} c_i^{\rho}/\overline{e}_i, & \text{if } \overline{e}_i \neq \emptyset, \\ c_i^{\rho}, & \text{if } \overline{e}_i = \emptyset. \end{cases}$$

It is easy to see that this partition is an $\mathfrak{X}$-congruence on S. We denote it by $\sim$ and show that if $\overline{c}_i$ is a subsemigroup of S then $\overline{c}_i$ is a semigroup with nullary multiplication.

For if $\overline{c}_i$ is a subsemigroup of S and $|\overline{c}_i| = 1$, then the proof is obvious. If $|\overline{c}_i| > 1$, then $|c_i^{\rho}| > 1$. Moreover, since $\mathfrak{X}$ is a periodic variety of type (r, m), $u_1, u_2 \in \overline{c}_i$ implies that $|u_1| \geq r$, $|u_2| \geq r$. Consequently, $|u_1 u_2| \geq 2r$. But in view of the fact that $a^r \not\rho\, a^r b$, $a^r \not\rho\, b a^r$ and $a^r \not\rho\, b a^{r-1}$, $a^r \not\rho\, a^{r-1} b$, we have $u_1, u_2 \notin B$. Hence, $u_1 u_2 = \overline{e}_i$. It is now clear that $S \in \mathfrak{Z} \circ \mathfrak{X}$.

Furthermore, suppose that $e_1 \sim a^r$, $e_2 \sim a^r b$ and that A is the ideal of S generated by e_1 and e_2. Clearly, $a^r, a^r b, a^r b a^r \notin A$ and in $\overline{S} = S/A$ the class $0^{\rho(\mathfrak{X},\overline{S})}$ contains a^r, $a^r b$ and $a^r b a^r$. Consequently, $0^{\rho(\mathfrak{X},\overline{S})} \notin \mathfrak{Z}$ and $S \notin \mathfrak{Z} \circ \mathfrak{X}$.

2. Suppose that $x^r \overline{\rho}\, y^r$. For $a^r b^r$ of F_2 we can present the following subcases: $a^r b^r \notin (a^r)^{\rho} \cup (b^r)^{\rho}$, $a^r b^r \in (a^r)^{\rho} \cup (b^r)^{\rho}$.

2.1. $a^r b^r \notin (a^r)^{\rho} \cup (b^r)^{\rho}$.

In each class c_i^{ρ} of the congruence ρ on F_2 that is a subsemigroup and contains an element u for which $|u| \geq 2r$, we fix an element e_i such that $e_i \notin \{a^r b^r, b^r a^r\}$. We denote the set of elements u of F_2 such that $|u| \geq 2r$, $u \notin \{a^r b^r, b^r a^r\}$ and $u \rho e_i$ by $\overline{e}_i$. Let σ be the relation on F_2 defined as follows: if $u, v \in \bigcup_{i \in I} \overline{e}_i$, then $u\sigma v \Leftrightarrow uv \in \overline{e}_i$, and if $u \notin \bigcup_{i \in I} \overline{e}_i$ or $v \notin \bigcup_{i \in I} \overline{e}_i$, then $u\sigma v \Leftrightarrow u = v$. Then σ is a congruence on F_2.

Let $S = F_2/\sigma$. Then $S = \bigcup_{i \in I} \overline{c}_i$, where

$$\overline{c}_i = \begin{cases} c_i^{\rho}/\overline{e}_i, & \text{if } e_i \neq \emptyset, \\ c_i^{\rho}, & \text{if } e_i = \emptyset. \end{cases}$$

It is easy to see that this partition is an $\mathfrak{X}$-congruence on S. We denote it by $\sim$. Moreover, repeating the arguments of part 1, we can show that if $\overline{c}_i$ is a subsemigroup of S, then $\overline{c}_i$ is a semigroup with nullary multiplication. Hence, $S \in \mathfrak{Z} \circ \mathfrak{X}$.

Suppose that $e_1 \sim a^r$, $e_2 \sim b^r$, $e_3 \sim a^r b^r$ and that A is the ideal of S generated by the elements e_1, e_2, e_3. Then a^r, b^r and $a^r b^r \notin A$ and in $\overline{S} = S/A$ the class $0^{\rho(\mathfrak{X},\overline{S})}$ contains a^r, b^r and $a^r b^r$. Consequently, $\overline{S} \notin \mathfrak{Z} \circ \mathfrak{X}$.

2.2. $a^r b^r \rho\, a^r$ or $a^r b^r \rho\, b^r$.

For definiteness we shall take the first and observe that if $a^r b^r \rho\, a^r$, then $b^r a^r \rho\, b^r$.

On F_2 we define a congruence σ in the same way as in case 2.1, and consider the semigroup $S = F_2/\sigma$. The partition $S = \bigcup_{i \in I} \overline{c}_i$ defines an $\mathfrak{X}$-congruence $\sim$ in which a class that is a subsemigroup of S is a semigroup with nullary multiplication. Hence, $S \in \mathfrak{Z} \circ \mathfrak{X}$.

Now if $e_1 \sim a^2, e_2 \sim b^2$ and A is the ideal of S generated by the elements e_1 and e_2, then a^2, b^2, a^2b^2 and $b^2a^2 \notin A$. But in the semigroup $\overline{S} = S/A$ the class $0^{\rho(\mathfrak{X},S)}$ contains a^2, b^2 and a^2b^2. Consequently, $\overline{S} \notin \mathfrak{Z} \circ \mathfrak{X}$.

Thus, for any periodic variety of semigroup $\mathfrak{X}$ that is not a null-variety, in the class $\mathfrak{Z} \circ \mathfrak{X}$ there is a semigroup S of which there is a homomorphic image $\overline{S}$ that does not belong to $\mathfrak{Z} \circ \mathfrak{X}$. Consequently, $\mathfrak{Z} \circ \mathfrak{X}$ is not a variety.

Lemmas 1-3 and the fact that the subgroupoids of a partial groupoid of varieties of semigroups with signature zero consist of varieties of null-semigroups [13] prove the theorem.

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