ABELIAN GROUPS WITH ONE τ-ADIC RELATION

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In [1] one has introduced the algebra Q_τ of τ-adic numbers, depending on the type (of a torsion-free group of rank 1) τ and representing the generalization of the field of p-adic numbers and of Kulikov's ring of universal numbers [2]. Also there one has defined the τ-adic completion A_τ of an Abelian torsion-free group A, of finite rank, which is a Q_τ-module.

Since there exists a natural homomorphism $A \to A_\tau$, it follows that by fixing in A a maximal linearly independent system $x_1, \ldots, x_n$, in the Q_τ-module Q_τ^n one can define a submodule of τ-relations

$$\delta(A)=\{(\alpha_1,\ldots,\alpha_n)\in Q_\tau^n \mid \alpha_1 x_1+\ldots+\alpha_n x_n=0 \text{ in } A_\tau\}.$$

In the case when τ coincides with the type of the group of rational numbers $\delta(A)$ coincide with the Beaumont—Pierce invariants [3] of the quotient divisible groups A and with the Murley invariants [4] for a somewhat larger class of groups.

In [5] one has considered classes of groups which are determined by their modules of τ-relations to within a quasiisomorphism, and one has constructed a duality which generalizes the known dualities of Warfield [6] and Arnold [7].

In this paper we investigate the class $\mathcal{F}$ of torsion-free Abelian groups A of finite rank for which the module $\delta(A)$ of the τ-relations is cyclic, i.e., for the groups $A\in\mathcal{F}$ one has in A_τ an equality with τ-adic coefficients, $\alpha_1 x_1+\ldots+\alpha_n x_n=0$, from which all the remaining τ-relations are obtained by multiplication by τ-adic numbers; this entitles us to call the groups of class $\mathcal{F}$ groups with one τ-adic relation.

The class $\mathcal{F}$ contains as subclasses the classes of groups, investigated previously by the author, for which any subgroup of infinite index is free [8], and the groups for which any proper servant subgroup is free [1], and also the class of groups, pointed out by A. A. Kravchenko [9] at the solution of a problem due to J. Irwin. On the other hand, the class $\mathcal{F}$, dual in the sense of [10] to the class $\mathcal{E}$, is a natural generalization of Murley's class of groups [4]. The intersection $\mathcal{E}\cap\mathcal{F}$ coincides with the union of the class of torsion-free groups of rank 2, which benefits of a constant attention, and of the class of homogeneous, completely decomposable torsion-free groups of finite rank.

In Sec. 1 the investigated classes are defined with the aid of the Richman type [11] and, in addition, it is shown that each Abelian torsion-free group of finite rank is the sum of a finite number of groups of class $\mathcal{F}$, and also the intersection of a finite number of groups of class $\mathcal{E}$.

In Sec. 2 we enumerate the properties of the τ-adic numbers, used for the investigation of the groups of class $\mathcal{F}$.

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In Sec. 3 we give the description of groups of class $\mathcal{F}$, to within a quasiisomorphism (which, by virtue of duality [10], is also a description of the class $\mathcal{E}$), with the aid of τ-adic invariants, representing subspaces of the algebra Q_τ, finite-dimensional over the field of rational numbers.

In Secs. 4 and 5, with the aid of the constructed invariants, we describe the servant subgroups of class $\mathcal{F}$ and the quotient groups with respect to them, we compute the group $Q \otimes Hom(A, B)$ of quasihomormorphisms for arbitrary groups A and B of class $\mathcal{F}$, in particular, the algebra $Q \otimes End A$ of quasiendomorphisms, and we describe the lattice of completely characteristic servant subgroups of an arbitrary group of class $\mathcal{F}$.

We mention that the τ-adic invariants, introduced here, when applied to groups of rank 2, represent a new modification of the known Beaumont—Pierce invariants [12]. A vast literature has been devoted to torsion-free groups of rank 2, in particular to the problem of the description of the set of types of these groups [13]. Several questions on groups of rank 2 can be formulated in the language of τ-adic invariants and the results can be generalized to the classes $\mathcal{E}$ and $\mathcal{F}$. The results in Secs. 3 and 4, in connection with groups of rank 2, are partially known; however, the description of the group of quasihomomorphisms (Theorem 5.3) is new even for groups of rank 2.

In the paper we consider only commutative groups and we make use of the following notations: $\mathbb{Z}$ is the ring of integers, Q is the field of rational numbers, Z_p is the ring of integral p-adic numbers; if p is a prime number and m is a nonnegative integer or the symbol ∞, then $Z(p^m)$ is the cyclic group of the order p^m or the quasicyclic group; $\hat{A}_p$ is the p-adic completion of the group A; $\langle M \rangle_S$ is the S-submodule generated by the set M; gcd denotes the greatest common divisor and m.l.i.s. denotes maximal linearly independent system.

Everywhere in the paper, an expression depending on p and included between round parentheses will denote a sequence with respect to all prime numbers p. A sequence (k_p) of nonnegative integers and symbols ∞ is called a characteristic. Two characteristics are equivalent if they differ at most by a finite number of finite elements. An equivalence class of characteristics is called a type and it is denoted $\sigma = [(k_p)]$. If $\tau = [(m_p)]$ is another type, then $\sigma + \tau = [(k_p + m_p)]$, $\sigma \wedge \tau = [(min\{k_p, m_p\})]$, $\sigma \vee \tau = [(max\{k_p, m_p\})]$. If $\sigma \leq \tau$, i.e., $k_p \leq m_p$ for each prime number p, then $\tau - \sigma = [(m_p - k_p)]$, where one assumes that $\infty - \infty = 0$. The type $[(0, 0, \ldots)]$ (the type of the group Z) is called zero and it is denoted by the symbol 0. The types IT(A), OT(A) are the inner and the outer types of the group A (see [6]).

If A and B are torsion-free groups of finite rank, then QHom(A, B) and QEnd A denote the group of the quasihomomorphisms $Q \otimes Hom(A, B)$ and the algebra of quasiendomorphisms $Q \otimes End A$, respectively. The category, whose objects are torsion-free groups of finite rank while the morphisms are the quasihomomorphisms, is called the category of quasihomomorphisms of the torsion-free groups of finite rank. Direct-sum decompositions in this category are called quasidirect decompositions, while the indecomposable objects are called indecomposable groups.

Other definitions and notations are the conventional ones and agree with those of [14].

1. The Classes $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$

If F is a free subgroup of rank n of a torsion-free group A of rank n, then

$$A/F \cong \bigoplus_p \left(Z(p^{i_{1p}}) \oplus \ldots \oplus Z(p^{i_{np}})\right),$$

where $0 \le i_{1p} \le \ldots \le i_{np} \le \infty$ are nonnegative integers or the symbol ∞. The type $IT(A) = [(i_{1p})]$ is called the inner type of the group A and the type $OT(A) = [(i_{np})]$, the outer type [6]. The table $\| i_{kp} \|$ to within a finite number of finite elements does not depend on the choice of F and is called the Richman type [11] of the group A.

Making use of the Richman type, we select three classes of torsion-free groups, connected with the pair of types $\sigma = [(k_p)] \le \tau = [(m_p)]$.

1) $A \in \mathcal{D}_\sigma^\tau \iff$ for each prime number p we have $i_{1p}, \ldots, i_{np} \in \{k_p, m_p\}$.

Thus, for a group $A \in \mathcal{D}_\sigma^\tau$ and a suitable free subgroup F, the p-primary component of the quotient group A/F splits into the direct sum of s_p copies of the group $Z(p^{k_p})$ and $t_p = n - s_p$ copies of the group $Z(p^{m_p})$ for each prime number p.

2) $A \in \mathcal{E}_\sigma^\tau \iff A \in \mathcal{D}_\sigma^\tau$ and $s_p = 1$ for each prime number p;

3) $A \in \mathcal{F}_\sigma^\tau \iff A \in \mathcal{D}_\sigma^\tau$ and $t_p = 1$ for each prime number p.

According to definition, groups of rank 1 in the class $\mathcal{E}_\sigma^\tau$ have type σ and in the class $\mathcal{F}_\sigma^\tau$ type τ. If σ = τ, then $\mathcal{D}_\sigma^\tau = \mathcal{E}_\sigma^\tau = \mathcal{F}_\sigma^\tau$ is the class of completely decomposable, homogeneous, torsion-free groups of type τ of finite rank. If σ < τ, then $\mathcal{E}_\sigma^\tau \cap \mathcal{F}_\sigma^\tau$ is the class of torsion-free groups of rank 2 with inner type σ and outer type τ.

In the class $\mathcal{D}_\sigma^\tau$ one has a duality [10], similar to the Arnold duality [7] for the class of quotient divisible groups $\mathcal{D}_0^\infty$ (0 = type Z, ∞ = type Q). In the sense of this duality, the classes $\mathcal{E}_\sigma^\tau$ and $\mathcal{F}_\sigma^\tau$ are mutually dual; therefore, the investigation of the class $\mathcal{F} = \bigcup_{\sigma \le \tau} \mathcal{F}_\sigma^\tau$ yields information, in a dual manner, also on the class $\mathcal{E} = \bigcup_{\sigma \le \tau} \mathcal{E}_\sigma^\tau$.

The following proposition shows that from the groups of class $\mathcal{F}$ (class $\mathcal{E}$) one can form any torsion-free group of finite rank.

Proposition 1.1. Let A be an arbitrary torsion-free group of finite rank n and let F ⊂ A ⊂ D, where F and $\mathcal{D}$ are, respectively, a free and divisible group of rank n. Then there exist groups $F_1, \ldots, F_k \in \mathcal{F}$ and $E_1, \ldots, E_m \in \mathcal{E}$ such that

1) $F \subset F_i \subset A$, $i = 1, \ldots, k \le n$; $A \subset E_i \subset \mathcal{D}$, $i = 1, \ldots, m \le n$;

2) $A = \sum_{i=1}^{k} F_i$, $A = \bigcap_{i=1}^{m} E_i$;

3) $\left(\sum_{i \ne j} F_i\right) \cap F_j = F$, $\left(\bigcap_{i \ne j} E_i\right) + E_j = D$ for each j;

4) all the inner types of the groups $F_1, \ldots, F_k$ are zero and the outer types of $E_1, \ldots, E_m$ coincide with the type of the group Q.

Proof. Let

$$A/F = \bigoplus_p \left(Z(p^{i_{1p}}) \oplus \ldots \oplus Z(p^{i_{np}})\right).$$

Assume that F_j is the complete preimage of $\bigoplus_p Z(p^{i_p})$ under the canonical homomorphism $A \to A/F$. We represent $A/F = \bigoplus_p (Z(p^\infty)[p^{i_p}] \oplus \ldots \oplus Z(p^\infty)[p^{i_{np}}]) \subset \bigoplus_p (\bigoplus_n Z(p^\infty)) = D/F$, naturally, we do not add $[p^{i_{kp}}]$ if $i_{kp} = \infty$. We set $\bar{E}_j = \bigoplus_p (Z(p^\infty) \oplus \cdots \oplus Z(p^\infty)[p^{i_p}] \oplus \ldots \oplus Z(p^\infty))$. For E_j we take the complete preimage of $\bar{E}_j$ under the homomorphism $D \to D/F$.

The following lemma reduces the investigation, within the accuracy of a quasiisomorphism of the classes $\mathcal{D}_\sigma^\tau$, $\mathcal{E}_\sigma^\tau$, $\mathcal{F}_\sigma^\tau$ to the case when $\sigma = 0$.

LEMMA 1.2. The category of quasihomomorphisms $\mathcal{D}_\sigma^\tau$ is isomorphic to the category of quasihomomorphisms $\mathcal{D}_0^{\tau-\sigma}$.

Proof. Let $A \in \mathcal{D}_\sigma^\tau$ and let $Z \subset R \subset Q$ be a group of rank 1 of type σ. The group Hom(R, A) is imbedded in A as the subgroup B of those elements whose characteristic in A is greater than or equal to the characteristic of 1 in R; moreover, the rank of B is equal to the rank of A since each element of the group A has type $\geq \sigma$ (see [10, Corollary 2.3]), $B \in \mathcal{D}_\varphi^\psi$, where φ is the idempotent type of the group $\operatorname{End} R$, $\psi = \tau - (\sigma - \varphi)$ (see [10, Lemma 2.2 and 13, Lemma 1.3]).

Further, let J be the set of prime numbers p for which $\varphi(p) = \infty$, let F be some free subgroup of the group B, whose rank coincides with the rank of the group A. We set $C = \bigcap_{p \in J} Q_p F \cap B$, where Q_pF is the subgroup of the divisible hull of F, consisting of elements of the form rf, r being rational numbers with denominator relatively prime with p, $f \in F$. The subgroup C is uniquely defined, to within quasiequalities, $C \in \mathcal{D}_0^{\tau-\sigma}$, $C \otimes R = A$.

Thus, the correspondence $A \mapsto C$ is a subfunctor of the identity functor, which performs the category isomorphism $\mathcal{D}_\sigma^\tau \to \mathcal{D}_0^{\tau-\sigma}$, and the inverse function is the tensor multiplication by a group of rank 1 and type σ.

We note that since the given functors preserve the upper and lower weights of the groups [10], it follows that to $\mathcal{E}$-groups there correspond $\mathcal{E}$-groups and to $\mathcal{F}$-groups there correspond $\mathcal{F}$-groups.

2. τ-adic Numbers

Assume that there is fixed a type $\tau = [(m_p)]$. We set $K_p = Z/p^{m_p}Z$ — the ring of the residues modulo p^{m_p}, if $m_p < \infty$, and $K_p = Z_p$ —the ring of the integral p-adic numbers, otherwise.

Definition 2.1 [1]. The algebra $Q_\tau = Q \otimes \prod_p K_p$ over the field of rational numbers is called the algebra of τ-adic numbers; its elements are called τ-adic numbers. The quotient ring of the ring $\prod_p K_p$ with respect to the ideal of periodic elements in the additive group is called the ring of τ-adic integers.

The ring of τ-adic integers is a subring of Q_τ under the natural imbedding $a \mapsto 1 \otimes a$ and coincides with it if the type τ does not contain symbols ∞. The definition of these rings does not depend on the selection of the characteristic (m_p) of type τ.

The set $\{p \mid m_p > 0\}$ of prime numbers will be called admissible [10] with respect to the given type $\tau = [(m_p)]$. Clearly, if we discard from an admissible set any finite set of prime numbers p, for which $m_p \neq \infty$, or if we adjoin an arbitrary finite set of prime numbers, then we obtain admissible sets.

Further, the τ-adic numbers can be represented in the form $\alpha = (r/s) \otimes (\alpha^{(p)})$ $s \neq 0$, $r \in Z$, $\alpha^{(p)} \in K_p$, where p runs through the admissible set of prime numbers. If $\beta = (r'/s') \otimes (\beta^{(p)}) \in Q_\tau$, then $\alpha = \beta \Leftrightarrow \{p / r's\beta^{(p)} = rs'\alpha^{(p)}\}$ is admissible.

For each prime number p, one defines in the ring K_p the p-height of an element $\alpha^{(p)} \in K_p$ as the largest exponent h for which p^h divides $\alpha(p)$ in K_p; we consider that the p-height of the zero element in K_p coincides with m_p. The collection of the p-heights of elements $\alpha^{(p)} \in K_p$ over all prime numbers p defines the type, which will be called the type of the τ-adic number $\alpha = (r/s) \otimes (\alpha^{(p)})$. The type of the τ-adic number α does not depend on the representation of the number α and it is less than or equal to τ. In the sequel we shall constantly make use of the following properties of τ-adic numbers (they follow from the definitions):

T1. If $\varphi \vee \sigma = \tau$ and $\varphi \wedge \sigma = 0$, then Q_τ splits into a direct sum of rings: $Q_\tau = Q_\varphi \oplus Q_\sigma$.

T2. For each τ-adic number α there exists a nonzero integer m such that $m\alpha$ is a τ-adic integer.

T3. For $\alpha \in Q_\tau$ type $\alpha \leq \tau$, type $\alpha = \tau \Leftrightarrow \alpha = 0$.

T4. Type $\alpha\beta$ = (type α + type $\beta) \wedge \tau$.

T5. $\alpha\beta = 0 \Longleftrightarrow$ type α + type $\beta \geq \tau$,

T6. α divides β, i.e., $\alpha\gamma = \beta$ for some $\gamma \in Q_\tau$ if and only if type $\alpha \leq$ type β.

T7. α is invertible if and only if it has type zero.

T8. For each finite collection of τ-adic numbers $\alpha_1, \dots, \alpha_n$ there exists their greatest common divisor and their least common multiple and, moreover, type gcd = type $\alpha_1 \wedge \dots \wedge$ type α_n, type lcm = type $\alpha_1 \vee \dots \vee$ type α_n.

T9. Any finitely generated ideal in Q_τ is principal.

T10. If $\alpha = \gcd(\alpha_1, \dots, \alpha_n)$, then there exist $\gamma_1, \dots, \gamma_n \in Q_\tau$ such that $\alpha = \alpha_1\gamma_1 + \dots + \alpha_n\gamma_n$.

T11. The type of the gcd $(\alpha_1, \dots, \alpha_n)$ will be called the type of the collection of τ-adic numbers $\alpha_1, \dots, \alpha_n$. If $M = \| r_{ij} \|$ is an $n \times n$ matrix with rational elements, then the collection of τ-adic numbers $(\alpha_1, \dots, \alpha_n)M = (r_{11}\alpha_1 + \dots + r_{n1}\alpha_n, \dots, r_{1n}\alpha_1 + \dots + r_{nn}\alpha_n)$ has type greater than or equal to the type of the collection $(\alpha_1, \dots, \alpha_n)$.

T12. If in the previous property the matrix M is invertible, then the types of the collections $(\alpha_1, \dots, \alpha_n)M$ and $(\alpha_1, \dots, \alpha_n)$ coincide.

T13. Let U be a finite-dimensional subspace of the algebra Q_τ over Q. The types of any two bases of U coincide. By type U we shall mean the type of any basis of U.

T14. If U is a subspace of the algebra Q_τ, finite-dimensional over Q, then the type of any system of generators of U is equal to the type of U; type $U=\tau \Leftrightarrow U=0$.

T15. Let U, V be finite-dimensional subsapces of the algebra Q_τ over Q. Then

1) type $(U+V)$ = type $U \wedge$ type V;

2) type $(U \cap V) \geq$ type $U \vee$ type V;

3) if $U \subset V$, then type $U \geq$ type V.

We introduce a few more definitions. Let A be a torsion-free group of finite rank and let $\tau=[(m_p)]$ be some type. We set $A(p)=A/p^{m_p}A$ if $m_p < \infty$ and $A(p)=\hat{A}_p$ the p-adic completion of the group A, otherwise. Then $A_\tau = Q \otimes \prod_p A(p)$ does not depend on the selection of the characteristic of type τ, it is called the τ-adic completion [1], and is a module over the ring Q_τ.

Let $a=(z/s)\otimes(a^{(p)}) \in A_\tau$. The collection of the p-heights of the elements $a(p)$ over all prime numbers p determines the type of the element a in A_τ; we consider that the p-height of the zero element of A(p) is equal to m_p. The type of an element $a \in A_\tau$ does not depend on the selection of the characteristic of type τ and on the representation of a. The canonical homomorphisms $\mu_p : A \to A(p)$ define the homomorphism $\mu : A \to A_\tau$ $(\mu(a)=1\otimes(\mu_p(a)))$, whose kernel is the set of elements having in A types greater than or equal to τ. If σ is the type of the element a in the group A, then in A_τ we have type $(\mu(a))=\sigma\wedge\tau$.

Let $\sigma=[(k_p)] \leq \tau=[(m_p)]$, i.e., $k_p \leq m_p$ for each prime number p and let A be a torsion-free group of finite rank or the ring Z. Then the natural group (ring) homomorphisms c_p: $A/p^{m_p}A \to A/p^{k_p}A$ for $m_p<\infty$, $c_p : \hat{A}_p \to \hat{A}_p/p^{k_p}\hat{A}_p = A/p^{k_p}A$ for $k_p < m_p = \infty$, and the identity homomorphisms for $k_p = m_p = \infty$ define a surjective group homomorphism $c_\sigma^\tau : A_\tau \to A_\sigma$ (ring homomorphism $c_\sigma^\tau : Q_\tau \to Q_\sigma$), $c_\sigma^\tau = id \otimes (c_p)$. The homomorphisms c_σ^τ will be called descent homomorphisms from type τ to type σ. For the sake of brevity we shall denote $a^c = c_\sigma^\tau(a)$ if the types τ and σ are fixed.

In the following properties we assume that A is a torsion-free group of finite rank:

T16. For $a \in A_\tau$ we have type $a \leq \tau$, type $a=\tau \Leftrightarrow a=0$.

T17. type αa = (type α + type a) $\wedge \tau$ for $\alpha \in Q_\tau$, $a \in A_\tau$.

T18. $\alpha a = 0 \Leftrightarrow$ type α + type $a \geq \tau$, $\alpha \in Q_\tau$, $a \in A_\tau$.

T19. If types are connected by the inequalities $\varphi \leq \sigma \leq \tau$ then $c_\varphi^\tau = c_\varphi^\sigma c_\sigma^\tau$.

T20. At the descent from type τ to type σ we have $(\alpha a)c = \alpha c a c$ in A_σ for $\alpha \in Q_\tau$, $a \in A_\tau$.

T21. For types $\sigma \leq \tau$ and a τ-adic number α of type σ, we denote $\sigma Q_\tau = \{\gamma \in Q_\tau \mid$ type $\gamma \geq \sigma\}$, $\alpha Q_\tau = \{\alpha\beta \mid \beta \in Q_\tau\}$ and, similarly, $\sigma A_\tau = \{a \in A_\tau \mid$ type $a \geq \sigma\}$, $\alpha A_\tau = \{\alpha a \mid a \in A_\tau\}$. Then $\sigma Q_\tau = \alpha Q_\tau$, $\sigma A_\tau = \alpha A_\tau$.

T22. The kernel of the descent homomorphism $Q_\tau \to Q_\sigma$ coincides with σQ_τ and the kernel of the descent homomorphism $A_\tau \to A_\sigma$ coincides with σA_τ.

T23. The module σQ_τ is a free cyclic module over $Q_{\tau-\sigma}$. If $\alpha \in \sigma Q_\tau$, $\gamma \in Q_{\tau-\sigma}$ then we set $\gamma\alpha = (C^{\tau}_{\tau-\sigma})^{-1}(\gamma)\cdot\alpha$ As a free generator of σQ_τ over $Q_{\tau-\sigma}$ one can take any τ-adic number of type σ.

T24. Let $\alpha = \gamma\beta$ in Q_τ, i.e., type $\alpha \geq$ type $\beta = \sigma$. In general, the quotient $\gamma \in Q_\tau$ is not defined in a unique manner. However, $\gamma^c \in Q_{\tau-\sigma}$ is uniquely defined by the τ-adic numbers α and β; therefore, we denote $\alpha/\beta = \gamma^c \in Q_{\tau-\sigma}$. Moreover, $\alpha = (\alpha/\beta)\cdot\beta$ in the sense of the previous property.

T25. Assume that $x \in A_\tau$ has type σ in A_τ. Then the Q_τ-module homomorphism $Q_\tau \to A_\tau$, under which $\alpha \mapsto \alpha x$, passes through the descent homomorphism $Q_\tau \to Q_{\tau-\sigma}$ and, therefore, $\alpha x = \alpha c x$, where the descent is carried out to the type $\tau - \sigma$.

T26. If U is a finite-dimensional subsapce over Q of the algebra Q_τ, then under a descent to type σ we have type (U^c) = type $(U)\wedge\sigma$, where $U^c = \{\alpha^c \mid \alpha \in U\}$.

T27. Let $Z \subset R \subset Q$ be a subgroup of Q of type τ. Then $Q_\tau = Q \otimes \mathrm{End}(R/Z)$, $A_\tau = Q \otimes \mathrm{Ext}(R/Z, A)$.

3. Description of Groups of Class $\mathcal{F}$ to within a Quasiisomorphism

The class of servant free groups [1] is a subclass of the class $\mathcal{F}$. In this section, the description of the servant free groups, obtained in [1], is generalized to the class $\mathcal{F}$.

Let A be an arbitrary torsion-free group of finite rank with a m.l.i.s. $x_1, \ldots, x_n$, and let τ be a fixed type. The images of the elements of the group A under the homomorphism $\mu\colon A \to A_\tau$, defined in Sec. 2, will be denoted in the sequel as the elements themselves. With a m.l.i.s. of the group A one associates the Q_τ-module

$$\delta = \{(\alpha_1, \ldots, \alpha_n) \in Q^n_\tau \mid \alpha_1 x_1 + \ldots + \alpha_n x_n = 0 \text{ in } A_\tau\} \subset Q^n_\tau,$$

which will be called the module of the τ-relations. The invariants, based on the modules of τ-relations, have been considered in [5].

From the results of [10, Corollary 2.3, Lemma 2.4] there follows directly

LEMMA 3.1. Let $A \in \mathcal{F}^{\tau}_{\sigma}$ and let $x_1, \ldots, x_n$ be some m.l.i.s. of the group A. Then

1) the type of any element of the group A is greater than or equal to σ and, therefore, by virtue of T25, $\langle A\rangle_{Q_\tau}$ is a $Q_{\tau-\sigma}$-module;

2) the module of $(\tau-\sigma)$-relations

$$\delta = \{(\alpha_1, \ldots, \alpha_n) \in Q^n_{\tau-\sigma} \mid \alpha_1 x_1 + \ldots + \alpha_n x_n = 0 \text{ in } A_\tau\}$$

is a cyclic free $Q_{\tau-\sigma}$-module;

3) for any generating element $(\alpha_1, \ldots, \alpha_n)$ of the module δ, the collection of $(\tau-\sigma)$-adic numbers $\alpha_1, \ldots, \alpha_n$ has type zero.

Definition 3.2. An equality in A_τ $\alpha_1 x_1+\ldots+\alpha_n x_n=0$ with $(\tau-\sigma)$-adic coefficients will be called a generating relation if $(\alpha_1, \ldots, \alpha_n)$ is a generating element of the module δ of $(\tau-\sigma)$-relations of the group $A\in\mathcal{F}_\sigma^\tau$ with respect to the m.l.i.s. $x_1, \ldots, x_n$.

Any two generating relations of the group $A\in\mathcal{F}_\sigma^\tau$ with respect to the same m.l.i.s. differ only by an invertible $(\tau-\sigma)$-adic factor. If in A_τ we have the equality $\beta_1 x_1+\ldots+\beta_n x_n=0$ with $(\tau-\sigma)$-adic coefficients and the collection of coefficients has type zero, then the given relation is generating.

In Sec. 5 we need the following

LEMMA 3.3. Let $A\in\mathcal{F}_\sigma^\tau$; let $x_1, \ldots, x_n$ be a m.l.i.s. of A, and let $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ be a generating relation. If $\psi\le\tau$ is an arbitrary type and $\beta_1, \ldots, \beta_n$ are ψ-adic numbers for which $\beta_1 x_1+\ldots+\beta_n x_n=0$ in A_ψ, then there exists a unique $(\psi-\psi\wedge\sigma)$-adic number α such that $(\beta_1^c,\ldots,\beta_n^c)=\alpha(\alpha_1^c,\ldots,\alpha_n^c)$, where the descent of the number is carried out, respectively, from the types ψ and $\tau-\sigma$ to the type $\psi-\psi\wedge\sigma=\psi\vee\sigma-\sigma$.

Proof. Since the types of the elements $x_1, \ldots, x_n$ in A is greater than or equal to σ, it follows that their types in A_ψ are greater than or equal to $\sigma\wedge\psi$. Therefore, by virtue of T25, we have $\beta_1 x_1+\ldots+\beta_n x_n=\beta_1^c x_1+\ldots+\beta_n^c x_n$ in A_ψ, where the descent is carried out to the type $\psi-\psi\wedge\sigma$.

Let $\gamma_1, \ldots, \gamma_n$ be $(\tau-\sigma)$-adic numbers, which under a descent to the type $\psi\vee\sigma-\sigma=\psi-\psi\wedge\sigma$ go into $\beta_1^c,\ldots,\beta_n^c\in Q_{\psi-\psi\wedge\sigma}$, respectively. Since $\beta_1^c x_1+\ldots+\beta_n^c x_n=0$ in A_ψ and type $(x_1), \ldots,$ type$(x_n)\ge\sigma$ in A, from [10, Lemma 2.4] it follows that there exist $(\tau-\sigma)$-adic numbers $\gamma, \xi_1, \ldots, \xi_n$ such that type $(\gamma)\ge\psi\vee\sigma-\sigma$ and

$$\gamma_1 x_1+\ldots+\gamma_n x_n=\gamma(\xi_1 x_1+\ldots+\xi_n x_n) \text{ in } A_\tau.$$

By Lemma 3.1 for some $\beta\in Q_{\tau-\sigma}$ we obtain

$$(\gamma_1-\gamma\xi_1,\ldots,\gamma_n-\gamma\xi_n)=\beta(\alpha_1,\ldots,\alpha_n).$$

By a descent to type $\psi\vee\sigma-\sigma$ since $\gamma^c = 0$, we have $(\beta_1^c,\ldots,\beta_n^c)=\beta^c(\alpha_1^c,\ldots,\alpha_n^c)$. The desired $\alpha = \beta^c$ is uniquely determined since $\alpha_1^c, \ldots, \alpha_n^c$ is a collection of zero type and from the equality $\alpha(\alpha_1^c,\ldots,\alpha_n^c)=\alpha'(\alpha_1^c,\ldots,\alpha_n^c)$ there follows that $(\alpha-\alpha')(\varepsilon_1\alpha_1^c+\ldots+\varepsilon_n\alpha_n^c)=\alpha-\alpha'=0$ for appropriate $\varepsilon_1,\ldots,\varepsilon_n\in Q_{\psi\vee\sigma-\sigma}$ having the property $\varepsilon_1\alpha_1^c+\ldots+\varepsilon_n\alpha_n^c=1$.

The following lemmas can be obtained directly from the corresponding statements of [10, Sec. 2] or, with the use of Lemma 1.2 in a more general situation they have been proved in [5].

LEMMA 3.4. Let $\tau\ge\sigma$ be arbitrary types and let $\alpha_1, \ldots, \alpha_n$ be an arbitrary collection of $(\tau-\sigma)$-adic numbers of zero type, let $V=Qx_1\oplus\ldots\oplus Qx_n$ be a vector space over Q with basis $x_1,\ldots,x_n$, $n\ge 1$. Then in the additive group V there exists a subgroup $A\in\mathcal{F}_\sigma^\tau$ unique up to quasiequalities, with generating relation $\alpha_1 x_1+\ldots+\alpha_n x_n=0$.

LEMMA 3.5 (quasihomomorphism criterion). Assume that $A\in\mathcal{F}_\sigma^\tau$ is defined by a generating relation $\alpha_1 x_1+\ldots+\alpha_n x_n=0$ and let $y_1, \ldots, y_n$ be elements of an arbitrary torsion-free group of rank B, $n\ge 1$. The mapping $x_1\mapsto y_1,\ldots,x_n\mapsto y_n$ can be extended to a group quasihomomorphism $A \to B$ if and only if the types of the elements $y_1, \ldots, y_n$ in B are greater than or equal to σ and in B_τ we have the equality $\alpha_1 y_1 + \ldots + \alpha_n y_n = 0$.

Definition 3.6. Let U and V be subsapces of zero type of the algebra Q_τ, finite-dimensional over Q. We consider that U is equivalent to V if there exists an invertible τ-adic number α such that $U = \alpha V$. The equivalence class $[U]$ of the zero type space $U \subset Q_\tau$, of dimension n over Q, will be called the τ-adic invariant of dimension n.

THEOREM 3.7. The group rank $n > 1$, the pair of types $\sigma \leq \tau$, the dimension $0 < k \leq n$ and the $(\tau-\sigma)$-adic invariant of dimension k form a full and independent system of invariants of the groups of class $\mathcal{F}$ of rank greater than 1 to within a quasiisomorphism.

Proof. In terms of a group A of class $\mathcal{F}$ of rank n there are uniquely defined the types $\sigma \leq \tau$ so that $A \in \mathcal{F}_\sigma^\tau$ with the exception of the case $n = 1$ in which the type σ is not defined. As the $(\tau-\sigma)$-adic invariant we take $[U]$, where $U \subset Q_{\tau-\sigma}$ is generated over Q by the coefficients of any generating relation $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$, $U = \langle \alpha_1, \ldots \alpha_n \rangle_Q \subset Q_{\tau-\sigma}$ and, clearly, $k = \dim U \leq n$.

Let $f: A \to B$ be a quasiisomorphism and let $\beta_1 y_1 + \ldots + \beta_n y_n = 0$ be a generating relation in the group B. Then there exists an invertible $n \times n$ matrix M with rational elements, for which $(fx_1, \ldots, fx_n) = (y_1, \ldots, y_n) M$. By the quasihomomorphism criterion,

$$(fx_1, \ldots, fx_n) \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_n \end{pmatrix} = 0.$$

From here

$$(y_1, \ldots, y_n) M \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_n \end{pmatrix} = 0$$

and, by Lemma 3.1,

$$M \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_n \end{pmatrix} = \alpha \begin{pmatrix} \beta_1 \\ \cdots \\ \beta_n \end{pmatrix},$$

i.e., $\alpha \langle \beta_1, \ldots, \beta_n \rangle = U$ for an invertible $\alpha \in Q_{\tau-\sigma}$ since by T12 the collection of numbers $M \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_n \end{pmatrix}$ has type zero and generates U. Thus, the $(\tau-\sigma)$-adic invariants of A and B coincide and do not depend on the selection of the m.l.i.s.

Assume that there are given in an arbitrary manner the types $\sigma \leq \tau$, the positive numbers $k \leq n$, and a $(\tau-\sigma)$-adic invariant $[U]$ of dimension k. We select an arbitrary system of generators $\alpha_1, \ldots, \alpha_n$ of the space U. It has zero type since U is a space of zero type (T14). By Lemma 3.4 there exists a group $A \in \mathcal{F}_\sigma^\tau$ of rank n with generating relation $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ and the $(\tau - \sigma)$-adic invariant of the group A coincides with $[U]$. If $\beta_1, \ldots, \beta_n$ is a system of generators of αU for some invertible $\alpha \in Q_{\tau-\sigma}$ then we can select an invertible $n \times n$ matrix M, with rational elements, such that

$$M \begin{pmatrix} \beta_1 \\ \cdots \\ \beta_n \end{pmatrix} = \alpha \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_n \end{pmatrix}.$$

Then by the quasihomomorphism criterion, the mapping f, defined by the equality $(fy_1,\dots,fy_n)=(x_1,\dots,x_n)M$, is a group quasiisomorphism, defined by the generating relation $\beta_1 y_1+\dots+\beta_n y_n=0$ on the group A. Thus, to each system of invariants there corresponds a group of class $\mathcal{F}$, unique to within a quasiisomorphism.

Remark 3.8. A group of rank 1, belonging to the class $\mathcal{F}_\sigma^\tau$, has, according to definition, type τ and its $(\tau-\sigma)$-adic invariant coincides with [Q], if $\sigma<\tau$.

4. Servant Subgroups of Groups of Class $\mathcal{F}$ and Their Quotient-Groups

In this section we investigate the servant subgroups of groups of class $\mathcal{F}_0^\tau$ and the quotient groups relative to them; by virtue of Lemma 1.2, the results can be carried over to arbitrary classes $\mathcal{F}_\sigma^{\sigma+\tau}$.

THEOREM 4.1. Assume that $A\in\mathcal{F}_0^\tau$ is defined by the generating relation $\alpha_1 x_1+\dots+\alpha_n x_n=0$; let B be the servant hull in A of the elements $x_1, \dots, x_k$, $0<k<n$; let σ be the type of the collection of τ-adic numbers $\alpha_{k+1},\dots,\alpha_n$; α is an arbitrary τ-adic number of type σ.

Then 1) $B\in\mathcal{F}_0^\sigma$ and it is defined by the generating relation $\alpha_1^c x_1+\dots+\alpha_k^c x_k=0$, where the descent of the coefficients is carried out from the type τ to the type σ;

2) $A/B\in\mathcal{F}_0^{\tau-\sigma}$ and it is defined by the generating relation $(\alpha_{k+1}/\alpha)\bar{x}_{k+1}+\dots+(\alpha_n/\alpha)\bar{x}_n=0$, where α_i/α are the $(\tau-\sigma)$-adic numbers, uniquely defined in T24, $\bar{x}_i=x_i+B$, $i=k+1,\dots,n$.

Proof. By virtue of T2, the coefficients $\alpha_1, \dots, \alpha_n$ can be considered to be integral τ-adic numbers and, since $A\in\mathcal{F}_0^\tau$ is defined to within quasiequalities, we can assume that

$$A/\langle x_1,\dots,x_n\rangle \cong \oplus_p Z(p^{m_p}) \text{ and } (\alpha_1^{(p)})x_1+\dots+(\alpha_n^{(p)})x_n=0$$

in $\prod_p A(p)$, where $\alpha_1=(\alpha_1^{(p)}),\dots,\alpha_n=(\alpha_n^{(p)})$ is the representation of the τ-adic coefficients in $\prod_p K_p$ corresponding to the characteristic (m_p) of type τ.

We denote C = A/B; $\oplus_p \bar{B}_p$ and $\oplus_p \bar{C}_p$ are the decompositions into direct sums of p-primary components of the groups $B/\langle x_1,\dots,x_k\rangle$ and $C/\langle \bar{x}_{k+1},\dots,\bar{x}_n\rangle$ and we consider the commutative 3 × 3 diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \langle x_1,\dots,x_k\rangle \arrow[r] \arrow[d] & \langle x_1,\dots,x_n\rangle \arrow[r] \arrow[d] & \langle \bar{x}_{k+1},\dots,\bar{x}_n\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_p \bar{B}_p \arrow[r] \arrow[d] & \oplus_p Z(p^{m_p}) \arrow[r] \arrow[d] & \oplus_p \bar{C}_p \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The lower row induces for each prime number p an exact sequence of p-primary groups:

$$0\to \bar{B}_p \to Z(p^{m_p}) \to \bar{C}_p \to 0 .$$

From here we conclude at once that $\bar{B}_p$ and $\bar{C}_p$ are cyclic and if $\bar{B}_p\cong Z(p^{k_p})$ then $\bar{C}_p\cong Z(p^{m_p-k_p})$, $0\le k_p\le m_p$, i.e., $B\in\mathcal{F}_0^\sigma$ and $C\in\mathcal{F}_0^{\tau-\sigma}$, provided $\sigma=[(k_p)]$.

For the proof of $\sigma=[(k_p)]$ we consider two cases: 1) $k_p = m_p$; 2) $k_p < m_p$. In the first case we select integers $a_1 \equiv \alpha_1^{(p)} \pmod{p^r}, \ldots, a_n \equiv \alpha_n^{(p)} \pmod{p^r}$ for $0 \le r \le m_p$. Then $a_1x_1+\ldots+a_nx_n$ is divisible by p^r in A and, consequently, $a_{k+1}\bar{x}_{k+1}+\ldots+a_n\bar{x}_n$ is divisible by p^r in C; but since $\bar{C}_p=0$ we have $p^{-r}(a_{k+1}\bar{x}_{k+1}+\ldots+a_nx_n)\in\langle\bar{x}_{k+1},\ldots,\bar{x}_n\rangle$ and, therefore, $a_{k+1}\equiv\ldots\equiv a_n\equiv 0 \pmod{p^r}$, i.e., $\alpha_{k+1}^{(p)}=\ldots=\alpha_n^{(p)}=0$, and the p-height of the elements $\alpha_{k+1}^{(p)},\ldots,\alpha_n^{(p)}\in K_p$ coincides with $k_p = m_p$.

In the second case we select the integers $a_1\equiv\alpha_1^{(p)} \pmod{p^{k_p+1}}, \ldots, a_n\equiv\alpha_n^{(p)} \pmod{p^{k_p+1}}$. Not all these numbers are divisible by p since the collection $\alpha_1, \ldots, \alpha_n$ has zero type. The element $a_1x_1+\ldots+a_nx_n$ is divisible in A by p^{k_p+1} and $y=p^{-k_p-1}(a_1x_1+\ldots+a_nx_n)$ goes according to the diagram in $Z(p^{m_p})$ into an element of order p^{k_p+1}. Then y goes into a nonzero element from $\bar{C}_p$, while py goes into 0. This means that $p^{-k_p}(a_{k+1}\bar{x}_{k+1}+\ldots+a_n\bar{x}_n)\in\langle\bar{x}_{k+1},\ldots,\bar{x}_n\rangle$ while $p^{-k_p-1}(a_{k+1}\bar{x}_{k+1}+\ldots+a_n\bar{x}_n)\notin\langle\bar{x}_{k+1},\ldots,\bar{x}_n\rangle$, i.e., the numbers $a_{k+1},\ldots,a_n$ are divisible by p^{k_p} in Z and not divisible in their totality by p^{k_p+1}.

Thus, for every prime number p, k_p coincides with the minimum of the p-heights of the elements $\alpha_{k+1}^{(p)},\ldots,\alpha_n^{(p)}$, i.e., $[(k_p)]$ is the type of the greatest common divisor of the τ-adic numbers $\alpha_{k+1},\ldots,\alpha_n$.

Now, descending the relation $\alpha_1x_1+\ldots+\alpha_nx_n=0$ from A_τ into A_σ, we obtain $\alpha_1^cx_1+\ldots+\alpha_k^cx_k=0$, since $\alpha_{k+1}^c=\ldots=\alpha_n^c=0$ in Q_σ. By virtue of the fact that B is servant, the obtained relation is satisfied also in B_σ, the collection of the coefficients of this relation has zero type and, therefore, it is generating for the group B.

In C_τ we have the equality $\alpha_{k+1}\bar{x}_{k+1}+\ldots+\alpha_n\bar{x}_n=0$ by virtue of T6, $\alpha_{k+1}=\alpha\beta_{k+1},\ldots,\alpha_n=\alpha\beta_n$ for τ-adic numbers $\beta_{k+1},\ldots,\beta_n$, in general, not uniquely defined. By T18, $\beta_{k+1}\bar{x}_{k+1}+\ldots+\beta_n\bar{x}_n\in(\tau-\sigma)C_\tau$ and by a descent from type τ to type $\tau-\sigma$ we obtain $\beta_{k+1}^c\bar{x}_{k+1}+\ldots+\beta_n^c\bar{x}_n=0$ in $C_{\tau-\sigma}$, where, by T24, the coefficients $\beta_{k+1}^c=\alpha_{k+1}/\alpha,\ldots,\beta_n^c=\alpha_n/\alpha\in Q_{\tau-\sigma}$ are uniquely defined from $\alpha,\alpha_{k+1},\ldots,\alpha_n$. Since, by assumption, type $(\gcd(\alpha_{k+1}, \ldots, \alpha_n))$ = type $\alpha=\sigma$, we have $\alpha = \gcd(\alpha_{k+1}, \ldots, \alpha_n)$ and the collection of the coefficients of the relation $(\alpha_{k+1}/\alpha)\bar{x}_{k+1}+\ldots+(\alpha_n/\alpha)\bar{x}_n=0$ in $C_{\tau-\sigma}$ has zero type; therefore, it is a generating relation for the group $C = A/B$.

<u>COROLLARY 4.2.</u> The dimension of the τ-adic invariant of a group A of class $\mathcal{F}_0^\tau$ is strictly smaller than the rank of the group A if and only if A has nonzero free direct summands.

<u>Proof.</u> If the dimension k of the τ-adic invariant of the group A is smaller than the rank n of the group A, then for some m.l.i.s. $x_1, \ldots, x_n$ the τ-adic relation of the group A has the form

$$\alpha_1x_1+\ldots+\alpha_kx_k+0x_{k+1}+\ldots+0x_n=0.$$

Let B be the servant hull of the elements $x_1, \ldots, x_k$. By Theorem 4.1, $A/B\in\mathcal{F}_0^0$ is a free group. Consequently, $A\cong B\oplus F$, where F is a free group of rank $n-k$.

Conversely, if A has a nonzero free direct summand, then there exists a nonzero homomorphism $f: A\to Z$. By the homomorphism criterion, if $\alpha_1x_1+\ldots+\alpha_nx_n=0$ is a τ-adic relation in A, then $\alpha_1 f(x_1)+\ldots+\alpha_n f(x_n)=0$ in Z_τ or $z_1\alpha_1+\ldots+z_n\alpha_n=0$ in Q_τ, where not all integers $z_1=f(x_1),\ldots,z_n=f(x_n)$ are equal to zero, i.e., the rank of the collection of τ-adic numbers $\alpha_1, \ldots, \alpha_n$, equal to the dimension of the τ-adic invariant, is less than the rank n of the group A.

COROLLARY 4.3. The group class $\mathcal{F}$ is closed with respect to taking servant subgroups and torsion-free quotient groups. The class $\mathcal{F}_0=\bigcup_\tau \mathcal{F}_0^\tau$ is closed with respect to taking any subgroups and torsion-free quotient groups.

Proof. We note that if $A\in\mathcal{F}_0^\tau$ and B is a subgroup of A of the same rank, then $(B+F)/F\subset A/F=\bigoplus_p Z(p^{m_p})$ for some free subgroup F of the same rank and, therefore, $(B+F)/F\cong\bigoplus_p Z(p^{k_p})$, i.e., B + F, quasiequal to B, belongs to the class $\mathcal{F}_0^\sigma$, where $\sigma=[(k_p)]\leq\tau$. From this remark, Theorem 4.1, and Lemma 1.2 there follows the given assertion. The corollary is proved.

It is easy to see that a homogeneous, completely decomposable group of type σ is isolated in a group of class $\mathcal{F}_\sigma^\tau$ by a quasidirect summand if and only if it is isolated by a direct summand.

Definition 4.5. A group $A\in\mathcal{F}_\sigma^\tau$ is said to be coreduced if it does not contain nonzero, homogeneous, completely decomposable direct summands of type σ.

Corollaries 4.2 and 1.2 show that a group of class $\mathcal{F}_\sigma^\tau$ is coreduced if and only if the dimension of its (τ − σ)-adic invariant coincides with its rank.

The servant subgroups of a torsion-free group form a lattice relative to the intersection $A\cap B$ and the servant hull $A\vee B=\langle A+B\rangle_*$ of the sum.

THEOREM 4.6. Let $\mathcal{P}(A)$ be the lattice of the servant subgroups of a coreduced group $A\in\mathcal{F}_0^\tau$ of rank n with a τ-adic invariant [U]; let L(U) be the lattice of the subspaces over Q of the vector space $U\subset Q_\tau$. Then there exists a lattice antiisomorphism $\Phi: L(U)\to\mathcal{P}(A)$ and, moreover, if $V\in L(U)$, σ is the type of V, k = n − dim V, then

1) rank $\Phi(V)=k$;

2) $\Phi(V)\in\mathcal{F}_0^\sigma$, $A/\Phi(V)\in\mathcal{F}_0^{\tau-\sigma}$.

3) the σ-adic invariant of the group $\Phi(V)\in\mathcal{F}_0^\sigma$ coincides with $[U^c]$, where the descent is carried out from type τ to type σ;

4) the (τ − σ)-adic invariant of the group $A/\Phi(V)\in\mathcal{F}_0^{\tau-\sigma}$ coincides with $[V_1]$, where $V_1\subset Q_{\tau-\sigma}$ is uniquely determined in terms of V and an arbitrary τ-adic number α of type σ according to T24 by the equality $V=\alpha V_1$.

Proof. We consider a basis $\alpha_1, \ldots, \alpha_n$ of the space U such that $\alpha_{k+1},\ldots,\alpha_n$ is a basis of V. To this basis there corresponds a m.l.i.s. $x_1, \ldots, x_n$ of the group A such that $\alpha_1x_1+\cdots+\alpha_nx_n=0$ is a generating relation. We set $\Phi(V)=B$, where B is the servant hull of the elements $x_1, \ldots, x_k$ in A. By Theorem 4.1, $B\in\mathcal{F}_0^\sigma$, $U^c=\langle\alpha_1^c,\ldots,\alpha_k^c\rangle$ defines a σ-adic invariant of the group B, $A/B\in\mathcal{F}_0^{\tau-\sigma}$ and $V_1=\langle\alpha_{k+1}/\alpha,\ldots,\alpha_n/\alpha\rangle\subset Q_{\tau-\sigma}$ defines a (τ − σ)-adic invariant of the group A/B.

Let $\beta_1, \ldots, \beta_n$ be another basis of U such that $\beta_{z+1}, \ldots, \beta_n$ is a basis of V. Then the bases are connected by the transition matrix

$$M\begin{pmatrix}\beta_1\\ \cdots\\ \beta_n\end{pmatrix} = \begin{pmatrix}\alpha_1\\ \cdots\\ \alpha_n\end{pmatrix},$$

where $M \in GL(n, Q)$ and has the form

$$M = \left(\begin{array}{c|c} * & * \\ \hline \underbrace{0}_{k} & * \end{array}\right)\Big\} \, n-k \, .$$

We interchange the m.l.i.s. of the group A with the aid of M: $(y_1, \ldots, y_n) = (x_1, \ldots, x_n)M$. In the group A, $\beta_1 y_1 + \ldots + \beta_n y_n = 0$ is a generating relation and, from the form of the matrix M it is easy to determine that the servant hull of the elements $y_1, \ldots, y_k$ coincides with B, i.e., the construction of $\Phi(V)$ does not depend on the selection of the basis.

For the proof of the equalities $\Phi(V+V') = \Phi(V) \cap \Phi(V')$, $\Phi(V \cap V') = \Phi(V) \vee \Phi(V')$, $V, V' \in L(U)$ one has to take a basis of U, containing bases of V, V' and $V \cap V'$, a m.l.i.s. of the group A, corresponding to the given basis, and to make use of the definition of $\Phi(V)$. Finally, taking into account that Φ is a one-to-one mapping, we obtain that Φ is a lattice antiisomorphism.

COROLLARY 4.7. The set of the types of the nonzero elements of the coreduced group $A \in \mathcal{F}_0^{\tau}$ of rank n with a τ-adic invariant [U] coincides with the set of types of the subspaces U of dimension $n - 1$. The set of the cotypes, i.e., the types of the torsion-free quotient groups of rank 1, coincides with the set $\{\tau - \text{type}(\alpha) \mid \alpha \in U, \alpha \neq 0\}$.

COROLLLARY 4.8 [1]. Let A be an arbitrary Abelian group, having nontrivial servant subgroups. Each proper servant subgroup of the group A is free if and only if $A \in \mathcal{F}_0^{\tau}$ for some type τ and the $\mathcal{T}$-adic invariant of the group A is determined by the space of τ-adic numbers, in which each nonzero number is invertible.

Proof. In [1] it has been shown that each servant free group is a torsion-free group of finite rank and belongs to the class $\mathcal{F}_0^{\tau}$ for some type τ. The invertibility of any nonzero element of U, where [U] is a τ-adic invariant of the group A, is equivalent to the fact that each nonzero subspace of U has zero type, while this, according to Theorem 4.6, is equivalent to the fact that each proper servant subgroup of the initial group belongs to the class $\mathcal{F}_0^{0}$, i.e., it is free. The corollary is proved.

We recall that for a group $A \in \mathcal{F}_{\sigma}^{\tau}$ we have the equalities $IT(A) = \sigma$, $OT(A) = \tau$.

COROLLARY 4.9. Let B and C be arbitrary nonzero servant subgroups of a group $A \in \mathcal{F}$. Then:

1) if the rank of B is greater than or equal to 2, then IT(B) = IT(A);

2) $OT(B) \leq OT(A)$;

3) if the rank of B is equal to 1, then $IT(A) \leq IT(B) = OT(B) \leq OT(A)$;

4) if $B \cap C \neq 0$ then $OT(B \cap C) = OT(B) \wedge OT(C)$;

5) If $B \cap C = 0$ then $OT(B) \wedge OT(C) = IT(A)$.

Proof. Parts 1, 2, 3 follow directly from Theorem 4.1 and Lemma 1.2, while parts 4 and 5 follow from Theorem 4.6, T15, and Lemma 1.2.

COROLLARY 4.10. Let $A \in \mathcal{F}$ and let τ be an arbitrary type, satisfying the inequalities $IT(A) < \tau \leq OT(A)$. Then there exists a smallest servant subgroup B of the group A with the property $OT(B) \geq \tau$.

Proof. The set $\{G \subset A \mid OT(G) \geq \tau\}$ of the servant subgroups is nonempty since it contains the group A and, by Corollary 4.9, parts 4 and 5, it is closed with respect to intersections; therefore, it contains a least element.

The next statement is a consequence of Theorem 4.6 or of 4.1 and can be generalized in an obvious manner to an arbitrary class $\mathcal{F}_{\sigma}^{\tau}$; moreover, as shown in Corollary 5.5, also the converse statement holds.

COROLLARY 4.11. If a group $A \in \mathcal{F}_0^{\tau}$, $\tau > 0$, with a τ-adic invariant [U] decomposes into the quasidirect sum of nonzero groups, then there exist nonzero types σ and φ for which $\sigma \vee \varphi = \tau$, $\sigma \wedge \varphi = 0$ and $U = V \oplus W \subset Q_{\sigma} \oplus Q_{\varphi} = Q_{\tau}$(T1), where V and W coincide with the space U, descended to the types σ and φ, respectively.

5. Groups of Quasihomomorphisms of Groups of Class $\mathcal{F}$

The next lemma reduces the problem of the description of the group $Hom(A, B)$, $A \in \mathcal{F}_{\sigma}^{\tau}$, $B \in \mathcal{F}_{\psi}^{\varphi}$, to the case when $\sigma \leq \psi$ and $\tau \leq \varphi$.

LEMMA 5.1. Let $A \in \mathcal{F}_{\sigma}^{\tau}$, $B \in \mathcal{F}_{\psi}^{\varphi}$.

1. If $\sigma \vee \psi \neq \psi$ then either $Hom(A, B) = 0$ or in B there exists a servant subgroup C of rank 1 of type $\lambda \geq \sigma$ and $Hom(A, B) = Hom(A, C)$.

2. If $\tau \wedge \varphi \neq \tau$ then either $Hom(A, B) = 0$ or in A there exists a servant subgroup C such that $A/C \in \mathcal{F}_{\sigma}^{\lambda}$, where $\lambda \leq \varphi$, $Hom(A, B) = Hom(A/C, B)$.

Proof. 1. Let $f: A \to B$ be a nonzero homomorphism and let C be the servant hull of Im f in B. Then the type of any element of the group C is greater than or equal to IT(A) and IT(B) and, therefore, $IT(C) \geq \sigma \vee \psi > \psi$. By Corollary 4.9, a strict inequality is possible only if the rank of C is equal to 1. By Part 5 of the same corollary, there exists only one servant subgroup of rank 1, whose type is greater than or equal to $\sigma \vee \psi$; therefore, the image of any homomorphism lies in C and Hom(A, B) = Hom(A, C).

2. By virtue of the proved part 1, we assume that $\sigma \leq \psi$. Let $f: A \to B$ be a nonzero homomorphism. By Theorem 4.1, we have $Ker f \in \mathcal{F}_{\sigma}^{\sigma+\mu}$ for some type $\mu \leq \tau - \sigma$ and $A/Ker f \in \mathcal{F}_{\sigma}^{\tau-\mu}$. Since we have the imbedding $A/Ker f \to B$ it follows that $\tau - \mu = OT(A/Ker f) \leq OT(B) = \varphi$. Then $\tau - \mu \leq \varphi \wedge \tau$ and $\mu \geq \tau - \varphi \wedge \tau$. From the relations $\varphi \wedge \tau < \tau$ and $\sigma \leq \psi \leq \psi$ there follows that $\tau \geq \sigma + (\tau - \varphi \wedge \tau) > \sigma$. By Corollary 4.10, in A there exists a smallest servant subgroup C with the property $OT(C) \geq$

$\sigma+(\tau-\varphi\wedge\tau)$. Since $\sigma+\mu\geq\sigma+(\tau-\varphi\wedge\tau)$ we have $Ker f\supset C$ and f passes through the canonical homomorphism A → A/C. Thus, $Hom(A,B)=Hom(A/C,B)$, $A/C\in\mathcal{F}_{\sigma}^{\lambda}$, where $\lambda=\tau-(OT(C)-\sigma)\leq\tau-(\tau-\varphi\wedge\tau)\leq\varphi\wedge\tau\leq\varphi$.

Definition 5.2. Let U and V be subspaces of the algebra Q_τ, finite-dimensional over Q. We denote $U:V=\{\alpha\in Q_\tau \mid \alpha V\subset U\}$.

If V is a subspace of zero type, then U:V is a finite-dimensional subspace of Q_τ, V:V is closed with respect to multiplication, contains the identity, and it is a subalgebra of the algebra Q_τ, finite-dimensional over Q.

THEOREM 5.3. Assume that the types σ,τ,ψ,φ satisfy the inequalities $\sigma\leq\tau$, $\psi\leq\varphi$, $\sigma\leq\psi$, $\tau\leq\varphi$. The groups $A\in\mathcal{F}_{\sigma}^{\tau}$, $B\in\mathcal{F}_{\psi}^{\varphi}$ are groups of ranks n and m, defined by the invariants [U] and [V], respectively. Then $QHom(A,B)\cong Q^{m(n-k)}\oplus(U^c:V^c)$, where the descent of the subspaces U and V is carried out to the type $\tau-\tau\wedge\psi=\tau\vee\psi-\psi$, $k=dim(U^c)$. In particular, if k = 0, i.e., $\tau\leq\psi$, then $QHom(A,B)\cong Q^{mn}$ and any mapping of an arbitrary m.l.i.s. of the group A into the divisible hull of the group B can be extended to a quasihomomorphism A → B. If k = n, then $QHom(A,B)\cong U^c:V^c$.

Proof. In A and in B we fix m.l.i.s. such that $\alpha_1x_1+\dots+\alpha_nx_n=0$ is a generating relation with $(\tau-\sigma)$-adic coefficients in A, $\beta_1y_1+\dots+\beta_my_m=0$ is a generating relation with $(\varphi-\psi)$-adic coefficients in B, $U=\langle\alpha_1,\dots,\alpha_n\rangle$, $V=\langle\beta_1,\dots,\beta_m\rangle$.

A quasihomomorphism f:A → B determines uniquely an m × n matrix M with rational elements $(fx_1,\dots,fx_n)=(y_1,\dots,y_m)M$. Since $\sigma\leq\psi$, it follows that the condition "type$(fx_i)\geq\sigma$, i = 1, ..., n" is automatically satisfied. By the quasihomomorphism criterion in B_τ, we must also have $\alpha_1fx_1+\dots+\alpha_nfx_n=0$, which is equivalent to the condition

$$(y_1,\dots,y_m)M\begin{pmatrix}\alpha_1\\ \dots\\ \alpha_n\end{pmatrix}=0.$$

By Lemma 3.3, there exists a unique $(\tau-\tau\wedge\psi)$-adic number such that

$$M\begin{pmatrix}\alpha_1^c\\ \dots\\ \alpha_n^c\end{pmatrix}=\alpha\begin{pmatrix}\beta_1^c\\ \dots\\ \beta_m^c\end{pmatrix},$$

i.e., $\alpha V^c\subset U^c$. Thus, we have obtained a homomorphism $QHom(A,B)\to U^c:V^c$. The kernel of this homomorphism consists of those quasihomomorphisms f which correspond to the matrices $M\in Q^{m\times n}$ having the property

$$M\begin{pmatrix}\alpha_1^c\\ \dots\\ \alpha_n^c\end{pmatrix}=\begin{pmatrix}0\\ \dots\\ 0\end{pmatrix}.$$

Since the rank of the system of $(\tau-\tau\wedge\psi)$-adic numbers $\alpha_1c,\dots,\alpha_nc$ is equal to k, it follows that the dimension of the kernel is equal to m(n − k); this concludes the proof of the theroem.

COROLLARY 5.4 [15]. Let A be a coreduced group of class $\mathcal{F}_{\sigma}^{\tau}$, $\sigma < \tau$, with a $(\tau - \sigma)$-adic invariant [U]. Then the algebra Q End A of quasiendomorphisms is isomorphic to the subalgebra U:U of the algebra $Q_{\tau-\sigma}$.

Each subalgebra with identity, finite-dimensional over Q, of the algebra $Q_{\tau-\sigma}$ can be realized as an algebra of quasiendomorphisms of an appropriate coreduced group $A \in \mathcal{F}_{\sigma}^{\tau}$.

Proof. The mapping QHom (A, A) $\to$ U:U, constructed in Theorem 5.3, is bijective and preserves the multiplication since to the product of quasiendomorphisms there corresponds the product of matrices, to which there corresponds the product of $(\tau - \sigma)$-adic numbers.

If $\mathcal{U} \subset Q_{\tau-\sigma}$ is an arbitrary finite-dimensional subalgebra with identity, then for groups $A \in \mathcal{F}_{\sigma}^{\tau}$ with a $(\tau - \sigma)$-adic invariant [U] we have $Q\,End\,A \cong \mathcal{U}:\mathcal{U} \cong \mathcal{U}$.

COROLLARY 5.5. A coreduced group $A \in \mathcal{F}_{\sigma}^{\tau}$, $\sigma < \tau$, is quasidecomposable into a direct sum of nonzero groups if and only if there exists a nontrivial (different from 0 and 1) idempotent $\varepsilon \in Q_{\tau-\sigma}$ such that $\varepsilon\mathcal{U} \subset \mathcal{U}$.

Proof. The quasidecomposability of A is equivalent to the existence of a quasihomomorphism $f: A \to A$, different from zero and the identity, with the property $f \cdot f = f$; by Corollary 5.4, this is equivalent to the existence of an idempotent in $\mathcal{U}:\mathcal{U}$.

COROLLARY 5.6. If $A \doteq A_1 \oplus \ldots \oplus A_s$ is a decomposition into a quasidirect sum of strongly indecomposable groups of a coreduced group $A \in \mathcal{F}_{\sigma}^{\tau}$, $\sigma < \tau$, then

$$Q\,End\,A \cong Q\,End\,A_1 \oplus \ldots \oplus Q\,End\,A_s .$$

COROLLARY 5.7. Under the assumptions of Theorem 4.6, the subgroup $\Phi(V)$ is a completely characteristic subgroup of the group A if and only if V is a (U:U)-submodule of the (U:U)-module U.

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