GROUPS WITH FINITELY IMBEDDED INVOLUTION

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The author has been aware, already in the seventies, that in his theorem in [4] the periodicity condition is unessential if all the subgroups of the form gr (i, i^g), $g\in G$ are finite, where G is a group and i is an involution in it with a finite $C_G(i)$. Further, reading again [4], the author has reached the conclusion that the basic idea of the proof in [4] (the construction of a periodic 2-complete Abelian subgroup A and the proof of the finiteness of $|A:A_g|$, $g\in G$) does not depend on the condition of the finiteness of $C_G(i)$ (the notations used here are taken from [4]). Subsequent reflection by the author in this direction has led to the necessity of introducing the concept of a finitely imbedded involution, the essence of which consists in the following: let G be a group, let i be some involution in it, and let $\mathcal{L}_i=\{i^g\mid g\in G\}$. The involution i is said to be finitely imbedded in G if for any element g from G the intersection $(\mathcal{L}_i\cdot\mathcal{L}_i)\cap gC_G(i)$ is finite, where $\mathcal{L}_i\cdot\mathcal{L}_i=\{i^{g_1}\cdot i^{g_2}\mid g_1,g_2\in G\}$.

We give the simplest examples of groups with a finitely imbedded involution.

1. If in the group there exists an involution G with a finite $C_G(i)$, then i is a finitely imbedded involution in G.

2. If in some group G the involution i is contained in a finite normal subgroup from G, then i is a finitely imbedded involution in G.

3. Let G be a Frobenius group with a periodic kernel and an infinite noninvariant factor H, containing the involution i. Then i is a finitely imbedded involution in G.

4. Let

$$B_1, B_2, \dots, B_n, \dots$$

be an infinite sequence of finite groups, in which only a finite number of groups is of even order and let $B_n \leftthreetimes (i_n)$ be a subgroup of the holomorph $Hol(B_n)$ (see [5]), where i_n is an involution inducing in B_n an automorphism of order two (n = 1, 2, ...). We consider the group $G=B\leftthreetimes(i)$, where B is a direct product of the form

$$B=B_1\times B_2\times\dots\times B_n\times\dots,$$

i is an involution of the form $i=(i_1, i_2, \dots, i_n, \dots)$. It is easy to show that i is a finitely imbedded involution in G.

<u>FUNDAMENTAL THEOREM.</u> Let G be a group, let i be its finitely imbedded involution, let $\mathcal{L}_i=\{i^g\mid g\in G\}$, $B=\mathrm{gr}(\{i^g\mid g\in G\})$, $R=\mathrm{gr}(\mathcal{L}_i\cdot\mathcal{L}_i)$, let Z be the subgroup generated

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by all 2-elements from R , and assume that the pair (G,i) satisfies condition *: the subgroups of the form gr (i, i^g) $(g \in G)$ are finite.

Then B, R, Z are normal subgroups in G and one of the following statements holds:

1) B is a finite subgroup;

2) the subgroup B is locally finite, $B = R \leftthreetimes (i)$ and Z is a finite extension of the full Abelian 2-subgroup A_2 with the minimality condition and, moreover, $ici = c^{-1} (c \in A_2)$.

COROLLARY 1. If the group G has an involution i with finite $C_G(i)$ and the pair (G, i) satisfies condition *, then G is locally finite.

Proof. The involution i is finitely imbedded in G (Example 1) and since G is a finite extension of a locally finite subgroup $B = \mathrm{gr}(\{i^g \mid g \in G\})$ (the fundamental theorem), then G is also locally finite [5, Theorem 23.1].

COROLLARY 2 [4]. If a periodic group G has an involution i with a finite $C_G(i)$, then G is locally finite.

Since in a periodic group any two involutions generate a finite subgroup, Corollary 2 follows from Corollary 1 and conversely.

COROLLARY 3. If in the group G there exists a finitely imbedded involution i and the pair (G, i) satisfies condition *, then gr $(i^g \mid g \in G)$ is a periodic subgroup.

If the involution i is not finitely imbedded in the group G, then, in general, the assertion of Corollary 3 is false. We give an example confirming this satement.

Let $A = \mathrm{gr}(b, c)$, where $b^p = c^p = d$, be a torsion-free group and let $A/(d)$ be the Novikov—Adyan group of prime order p (see [7]). We consider the group $T = A \wr (x) = (A \times A) \leftthreetimes (x)$, where x is an involution. Let us take the element $s = (d, d^{-1}) \in A \times A$. Obviously, $s \in Z(A \times A)$ and $s^x = s^{-1}$. We introduce the following notations: $G = T/(s)$, $i = x(s)$, $\mathfrak{M}$ is the set of strictly real elements of finite orders from G relative to i. It is easy to show that $G = C_G(i)\mathfrak{M}$ and for any $g \in G$ the subgroup gr (i, i^g) is finite. However, the group G does not possess a periodic part.

This same example and the fundamental theorem show that if the group G and some of its involution i satisfy condition *, while $C_G(i)$ possesses a finite periodic part, then the involution i need not be finitely imbedded in G.

COROLLARY 4. Let G be a simple group with involutions, let i be some involution in G, satisfying condition *, and let $\mathcal{L}_i = \{i^g \mid g \in G\}$. The group G is finite if and only if for all $g \in G$ the intersection $\mathcal{L}_i^2 \cap g C_G(i)$, where $\mathcal{L}_i^2 = \mathcal{L}_i \cdot \mathcal{L}_i$, is finite.

COROLLARY 5. Let G be a periodic simple group with involutions, let i be some involution in G, and let $\mathcal{L}_i = \{i^g \mid g \in G\}$. The group G is finite if and only if for all $g \in G$ the intersection $(\mathcal{L}_i \cdot \mathcal{L}_i) \cap g C_G(i)$ is finite.

Since in any periodic group with involutions any two involutions generate a finite subgroup, from the fundamental theorem there follow the following statements:

THEOREM 1. Let G be a periodic group, let i be its finitely imbedded involution, let $\mathcal{X}_i = \{i^g \mid g \in G\}$, $B = \mathrm{gr}\,(\{i^g \mid g \in G\})$, $R = \mathrm{gr}\,(\mathcal{X}_i \cdot \mathcal{X}_i)$, and let Z be the subgroup generated by all 2-elements from R.

Then B, R, Z are normal subgroups in G and one of the following statments holds:

1) B is a finite subgroup;

2) the subgroup B is locally finite, $B = R \leftthreetimes (i)$, and Z is a finite extension of the complete Abelian 2-subgroup A_2 with the minimality condition and, moreover, $ici = c^{-1}$, $c \in A_2$.

THEOREM 2. Let G be a periodic group, let H be a subgroup in it, containing the involution i, and let (G, H) be a Frobenius pair. The group G is a Frobenius group with complement H if and only if i is a finitely imbedded involution in G.

The restriction in Theorem 2, namely that i is a finitely imbedded involution in G, is essential. Indeed, assume, for example, that $L = B(r, n)$ is a free periodic group of odd period $n \geqslant 665$ and with number of generators $r \geqslant 2$ (see [7]). The group L possesses an automorphism φ of order two, transforming the free generators into the inverses, and, therefore, in the holomorph $\mathrm{Hol}(L)$ there exists a subgroup $G = L \leftthreetimes (i)$, where i is an involution inducing the automorphism φ in L. Making use of the abstract properties of the group $L = B(n, r)$ (see [7]), it is easy to show that $(G, C_G(i))$ is a Frobenius pair and $G = \mathrm{gr}\,(i^G)$. If the involution i would be finitely imbedded in G, then, according to the fundamental theorem, G would be a finite group, in spite of the fact that $L = B(r, n)$ is infinite (see [7]). Consequently, the restriction in Theorem 2, that the involution i is finitely imbedded in G, cannot be omitted.

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The notations used in this paper are basically standard [5, 6].

1. FIRST FUNDAMENTAL LEMMA

LEMMA 1. Let G be a group with involutions. The following assertions hold:

1) if an involution in G is finitely imbedded in G, then it is finitely imbedded in any subgroup from G containing this involution;

2) an involution in G, conjugate to a finitely imbedded involution in G, is also finitely imbedded in G;

3) if k is an involution in G and $|G : C_G(k)|$ is finite, then k is a finitely imbedded involution in G.

Proof. All the assertions of the lemma follow directly from the definition of a finitely imbedded involution in a group.

Let G be a group with involutions, let i be its finitely imbedded involution, $\mathcal{L}_i = \{ i^g \mid g \in G \}$, satisfying condition *: $|G : C_G(i)|$ is infinite and all subgroups of the form gr (i, k), $k \in \mathcal{L}_i$, are finite.

LEMMA 2. For any involution $k \in \mathcal{L}_i$ the following assertions hold:

1) if H is a subgroup in G, containing k, and $|H : C_H(k)|$ is finite, then $\mathcal{L}_i^2 \cap H$ is finite;

2) if t is an involution from $\mathcal{L}_i$, then all the involutions from gr $(k, t) \setminus Z($ $(k, t))$ are finitely imbedded in G and conjugate with i in G;

3) the set of subgroups of the form Z gr $(k, t))(t \in \mathcal{L}_i)$ is finite.

The proof of all the assertions of the lemma can be obtained easily by using the definition of a finitely imbedded involution and the known properties of dihedral groups [9].

LEMMA 3. Let H be a periodic 2-complete Abelian subgroup in G (in particular, a periodic Abelian subgroup without involution), all elements of which are strictly real relative to i. Then the following asssertions hold:

1) a Sylow 2-subgroup S from H is a complete subgroup with the minimality condition;

2) if k is an involution in $N_G(H) \cap \mathcal{L}_i$, then H has a subgroup V_k of finite index in H and all the elements in V_k are strictly real relative to k.

Proof. Let Z be the lower layer of the subgroup S. Obviously, $Z < C_G(i)$ and $iZ \subset \mathcal{L}_i$. If the subgroup Z were infinite, then also the intersection $\mathcal{L}_i^2 \cap C_G(i)$ would be infinite in spite of Lemma 2. Consequently, Z is a finite subgroup. But then S satisfies the minimality condition [10]. Assertion 1 is proved. Since all the elements of $D = H \cap C_G(k)$ are strictly real relative to i, it follows, obviously, that $D \subset \mathcal{L}^2 \cap C_G(k)$. On the basis of Lemmas 1, 2, we conclude that D is a finite subgroup. By statement 2 from [4] we have $H = V_k D$, where V_k is a subgroup in H and all the elements in V_k are strictly real relative to k. Assertion 2 is proved and, at the same time, the lemma is proved.

Let T be a subgroup of G, containing i, let V be a normal subgroup of T, being the finite extension of a complete Abelian 2-subgroup A, not necessarily different from the identity subgroup and, moreover, all elements from A are strictly real relative to i, $\mathfrak{A} = T \cap \mathcal{L}_i$, $\bar{T} = T/V$, $\bar{i} = iV$, $\bar{\mathfrak{A}} = \mathfrak{A}V/V$.

LEMMA 4. For any $\bar{g} \in \bar{T}$, $k \in \mathfrak{A}$ the intersection $\bar{\mathfrak{A}}^2 \cap \bar{g} C_{\bar{T}}(kV)$ is finite and, in particular, kV is a finitely imbedded involution in $\bar{T}$.

Proof. First we consider the case when $V = H$. In this case, in view of statement 2 from [4] and the completeness of the subgroup V, all the elements from V are strictly real relative to any involution from $\mathfrak{A}$. Further, all involutions of the form $h\kappa$ $(h \in V)$ are conjugate with κ with the aid of elements from V. From here it follows that

$$C_{\bar{T}}(\kappa V) = C_T(\kappa)V/V. \tag{1}$$

Consequently, any element of the form jtV from $\bar{\mathfrak{A}}^2 \cap \bar{g} C_{\bar{T}}(\kappa V)$, where $j, t \in \mathfrak{A}$, has a representation $jtV = \bar{g}\bar{\tau}$, where $\bar{\tau} \in C_{\bar{T}}(\kappa V)$ and τ is the preimage of the element $\bar{\tau}$ in $C_T(\kappa)$. Passing in this equality to preimages, we obtain $jt = g\tau d$, where g is some preimage of the element $\bar{g}$ in T, $d \in V$. Futher $jtd^{-1} = g\tau$. Since $t \in \mathfrak{A}$, it follows, according to what has been proved above that $td^{-1} \in \mathfrak{A}$. But then $jtd^{-1} \in \mathfrak{A}^2 \cap g C_T(\kappa)$. By Lemma 1, κ is a finitely imbedded involution in T and, therefore, the intersection $\bar{\mathfrak{A}}^2 \cap \bar{g} C_{\bar{T}}(\kappa)$ is finite. From here, in view of the equality (1) and the arbitrariness of the selection of the element jtV from $\bar{\mathfrak{A}}^2 \cap \bar{g} C_{\bar{T}}(\kappa V)$, the set $\bar{\mathfrak{A}}^2 \cap \bar{g} C_{\bar{T}}(\kappa V)$ is finite. Thus, the assertion of the lemma is valid if $V = H$. Regarding the case when $V \neq H$ and $|V : H|$ is finite, this can be easily reduced, with the aid of Lemmas 1, 2, to the case that has been already considered. The lemma is proved.

Remark 1. Everywhere in the subsequent lemmas it will be assumed that G is a group, i is a finitely imbedded involution in it, satisfying condition * and $|\mathfrak{L}_i| = \infty$.

FIRST FUNDAMENTAL LEMMA. The group G has an infinite periodic 2-complete Abelian subgroup, all elements of which are strictly real relative to i.

Proof. If G has an infinite complete Abelian 2-subgroup, all elements of which are strictly real relative to i, then the assertion of the lemma holds. In connection with this, for the convenience of the subsequent arguments (within Sec. 1), we make the following

Remark 2. The group G does not have an infinite complete Abelian 2-subgroup, all elements of which are strictly real relative to i.

LEMMA 5. If S is a 2-subgroup in G, containing i, then $S \cap \mathfrak{L}_i$ is finite.

Proof. We assume that the lemma does not hold, i.e., the set $\mathfrak{G} = S \cap \mathfrak{L}_i$ is infinite. On the basis of Lemmas 2, 4 we conclude that for some infinite subset $\mathfrak{G}_1$ from $\mathfrak{G}$ the intersection $\bigcap\limits_{b \in \mathfrak{G}_1} (ib)$ has an involution t_1. We consider $S_1 = N_S((t_1))$. By the known properties of dihedral groups, we have $\{i, \mathfrak{G}_1\} \subset S_1$. Applying again Lemmas 2, 4 to the subgroup S_1 and to its quotient group $S_1/(t_1)$, we prove that for some infinite subset $\mathfrak{G}_2$ from $\mathfrak{G}_1$ the intersection $\bigcap\limits_{b \in \mathfrak{G}_2} (ib)$ has an element t_2 of order 4. We reason in a similar manner regarding the subgroup $S_2 = N_{S_1}((t_2))$ and its quotient group $S_2/(t_2)$, etc. With the aid of these arguments we construct a strictly increasing chain of subgroups

$$(t_1) < (t_2) < \ldots < (t_n) < \ldots,$$

which does not break at a finite index and, moreover

$$t_n^i = t_n^{-1}, \qquad n = 1, 2, \ldots$$

Consequently, the union of this chain is a quasicyclic subgroup, all elements of which are strictly real relative to i, despite Remark 2. The obtained contradiction means that $\mathcal{S} \cap \mathcal{Y}$ is finite and the lemma is proved.

LEMMA 6. In G there exists a subgroup K, with a finite 2-subgroup P, normal in it, possessing the following properties:

1) $i \in K$ and $\mathfrak{A} = K \cap \mathcal{B}_i$ is an infinite set;

2) if $j \in \mathfrak{A}$, then $|K : C_K(j)|$ is infinite;

3) if P_1 is a finite (i)-invariant 2-subgroup in K with an infinite intersection $N_K(P_1) \cap \mathfrak{A}$ and $P \leq P_1$, then gr $(P, i) \cap \mathfrak{A} =$ gr $(P_1, i) \cap \mathfrak{A}$.

Proof. Let Q be a finite (i)-invariant 2-subgroup, not necessarily distinct from the identity subgroup, having an infinite intersection $N_G(Q_1) \cap \mathcal{B}_i = \mathfrak{A}_1$. If some involution t from $\mathfrak{A}_1$ would determine a finite class of conjugate involutions in $K_1 = N_G(Q_1)$, i.e., $|K_1 : C_{K_1}(t)|$ would be finite, then, obviously, for some infinite sequence of distinct involutions from $\mathfrak{A}_1$: $j_1, j_2, \ldots, j_n, \ldots$, we would obtain an infinite sequence of distinct elements

$$j_1 j_2 = t_1, \ldots, j_1 j_n = t_n, \ldots,$$

where $t_n \in C_{K_1}(t)$, $n = 1, 2, \ldots$, in spite of Lemma 4. Consequently, statements 1, 2 are valid for K_1. If, in addition, for K_1 and its subgroup Q_1 statement 3 holds, then, setting $K = K_1$, $P = Q_1$, we obtain the subgroup mentioned in the lemma.

Assume that Q_2 is a finite (i)-invariant 2-subgroup in K_1, containing Q_1, and that the intersection $\mathcal{B}_i \cap N_{K_1}(Q_2)$ is infinite; moreover,

$$(Q_1, i) \cap \mathcal{B}_i \neq \text{gr}\,(Q_2, i) \cap \mathcal{B}_i .$$

Regarding the subgroups $K_2 = N_{K_1}(Q_2)$, Q_2 we reason as at the consideration of the pair (K_1, Q_1). With the aid of these arguments we construct a strictly increasing chain of finite invariant 2-subgroups:

$$Q_1 < Q_2 < \ldots < Q_n < \ldots, \tag{2}$$

to which there corresponds a strictly increasing chain of subsets

$$\text{gr}\,(Q_1, i) \cap \mathcal{B}_i \subset \text{gr}\,(Q_2, i) \cap \mathcal{B}_i \subset \ldots \subset \text{gr}(Q_n, i) \cap \mathcal{B}_i \subset \ldots$$

According to Lemma 5, the chain (2) breaks at a finite index n, i.e., the subgroups $K = K_n$, $P = Q_n$ possess the properties mentioned in the lemma. The lemma is proved.

Let K, P be the subgroups from Lemma 6. We introduce the notations:

$$\mathfrak{A} = K \cap \mathcal{B}_i, \quad V = K/P, \quad \mathcal{Y} = \mathfrak{A}P/P, \quad \bar{i} = iP.$$

LEMMA 7. The number of elements of the form $\bar{i}t$, where $t \in \mathcal{Y}$, having even order, is finite.

Proof. We assume that $\mathcal{Y}$ has an infinite subset $\mathcal{P}$ such that all elements of the form $\bar{i}t$, $t \in \mathcal{P}$, have even order. By Lemma 4, we shall assume, without loss of generality, that $\bigcap_{t\in\mathfrak{P}} \bar{i}t$ has an involution j. By the properties of dehidral groups we have $\{\bar{i},\mathfrak{P}\}\subset C_V(j)$ and, by Lemma 2, $j\bar{i}\in C_V(j)\cap\mathfrak{G}$. If X and P_1 are the complete preimages of the subgroups $C_V(j)$ and (j) in K, respectively, then, obviously,

$$P(i)\cap\mathfrak{D}_i \neq P_1(i)\cap\mathfrak{D}_i,$$

where $P<P_1$, and $X\cap\mathfrak{D}_i$ is an infinite set. However, this is not possible in view of the definition of the subgroups K and P and in view of Lemma 6. Consequently, V may have only a finite number of elements of the form $\bar{i}t$ of even order, where $t\in\mathfrak{G}$. The lemma is proved.

We proceed directly to the proof of the fundamental lemma.

In view of Lemmas 6, 7, in the subgroup V the set $\mathfrak{M}$, of strictly real elements of odd order relative to i, is infinite. We fix some element $a\neq 1$ from $\mathfrak{M}$. We consider the elements of the form ba^2b, $b\in\mathfrak{M}$. By the definition of the elements of $\mathfrak{M}$, we have $b=\bar{i}t_b$, $a^2=s\bar{i}$ and $ba^2b=\bar{i}t_b s t_b$, where $t_b s t_b\in\mathfrak{G}$. From here, in view of Lemma 7, the number of elements of the form ba^2b, $b\in\mathfrak{M}$, having even order, is finite. Therefore, $\mathfrak{M}$ has an infinite subset $\mathfrak{N}$ such that all elements of the form $ba^2b\,(b\in\mathfrak{N})$ have odd order, i.e., $|ba^2b|=2q_b-1\ (b\in\mathfrak{N})$.

We consider elements of the form

$$r_b=(ba^2b)^{q_b}b^{-1}a^{-1},\qquad b\in\mathfrak{N}. \tag{3}$$

We prove that $r_b=r_b^{\bar{i}}$. Indeed, $1=(ba^2b)^{2q_b-1}\Rightarrow(ba^2b)^{q_b}=(b^{-1}a^{-2}b^{-1})^{q_b-1}=(b^{-1}a^{-2}b^{-1})^{q_b}(ba^2b)=(b^{-1}a^{-2}b^{-1})^{q_b}ba^2b\Rightarrow r_b=(ba^2b)^{q_b}b^{-1}a^{-1}=(b^{-1}a^{-2}b^{-1})^{q_b}ba=r_b^{\bar{i}}$, since a, b are strictly real relative to $\bar{i}$.

We represent equality (3) in the form

$$r_b a=[(ba^2b)^{q_b}\bar{i}](\bar{i}b^{-1}),\qquad b\in\mathfrak{N}. \tag{4}$$

As shown above, $\bar{i}b^{-1}$, $(ba^2b)^{q_b}\bar{i}\in\mathfrak{G}$. From here, by Lemma 4, the set $\mathfrak{M}_a=\{r_b\mid b\in\mathfrak{N}\}$ is finite. But then $\mathfrak{N}$ possesses an infinite sequence of distinct elements

$$b_1,\ b_2,\ \dots,\ b_n,\ \dots \tag{5}$$

such that

$$r=r_{b_1}=r_{b_2}=\dots=r_{b_n}=\dots$$

We denote: $s_n=(b_n a^2 b_n)^{q_{b_n}}\bar{i}$, $t_n=\bar{i}b_n^{-1}$, $n=1,2,\dots$. In accordance with the sequence (5) and taking into account the introduced notations, we rewrite the equalities (4) in the form

$$ra=s_n t_n,\quad n=1,2,\dots, \tag{6}$$

where s_n, $t_n\in\mathfrak{G}$.

Since $i t_1 = s_1^{-1}$ and the element $\bar{t}_1$ has odd order, it follows that for some element $x \in (s_1^{-1})$ we have $t_1^x = \bar{i}$. We transform equalities (6) with the aid of the element x :

$$a_i = (xa)^x = s_n^x i_n^x, \quad n = 1, 2, \ldots$$

Obviously, $t_n^x \in N_V((d_1))$, $n = 1, 2, \ldots$, and, therefore, $N_V((d_1)) \cap \mathfrak{G}$ is an infinite set. If the element d_1 were of even order, then, taking into account $d_1 = s_1^x \bar{i}$, where $s_1^x, \bar{i} \in \mathfrak{G}$, and Lemma 2, we would easily arrive at a contradiction with the definition of the subgroup V and with Lemma 6. Consequently, d_1 is an element of odd order and, obviously $d_1 \neq 1$.

Let $K_1 = N_V((d_1))$, $\mathfrak{A}_1 = K_1 \cap \mathfrak{G}$, $\mathfrak{G}_1 = \mathfrak{A}_1(d_1)/(d_1)$ and $V_1 = K_1/(d_1)$. According to what has been shown above, $\mathfrak{A}_1$ is an indefinite set. Further, applying to the triple $(V_1, \mathfrak{G}_1, \bar{i}(d_1))$ the same arguments as in the consideration of the triple $(V, \mathfrak{G}, i)$, we prove the existence in V_1 of an element $\bar{d}_2 \neq 1$ of odd order, strictly real relative to $\bar{i}(d_1)$, with infinite intersection $N_{V_1}((\bar{d}_2)) \cap \mathfrak{G}_1$. We introduce the following notations: d_2 is the preimage of $\bar{d}_2$ in K_1, $D_1 = (d_1)$, $D_2 = \text{gr}(d_1, d_2)$, $K_2 = N_{K_1}(D_2)$, $\mathfrak{A}_2 = K_2 \cap \mathfrak{G}_1$, $V_2 = K_2/D_2$. Relative to the triple $\mathfrak{G}_2 = \mathfrak{A}_2 D_2/D_2$ we reason in the same way as at the consideration of the triple $(V_2, \mathfrak{G}_2, \bar{i}D_2)$, etc. With the aid of such arguments we construct a strictly increasing chain of finite subgroups of odd order $(V_1, \mathfrak{G}_1, \bar{i}D_1)$

$$D_1 < D_2 < \ldots < D_n < \ldots$$

Its union D is infinite and, by statement 4 of [4], D is a periodic Abelian subgroup without involution, all elements of which are strictly real relative to $\bar{i}$. But then, obviously, also G has an infinite periodic Abelian subgroup without involutions, all elements of which are strictly real relative to i. The first fundamental lemma is proved.

2. SECOND FUNDAMENTAL LEMMA

By Zorn's lemma, $B = \text{gr}(\{i^g \mid g \in G\})$ possesses a complete Abelian 2-subgroup A_2 with rank of maximal cardinality, all of whose elements are strictly real relative to i. By Lemma 3, the rank of the subgroup A_2 is finite. Making use again of Zorn's lemma, we enclose A_2 in a maximal periodic 2-complete Abelian subgroup A, all of whose elements are strictly real relative to i. by the first fundamental lemma, A is an infinite group. We denote by L_g the subgroup from A, generated by all the elements from A that are strictly real relative to $i^{g^{-1}} = g i g^{-1}$, $g \in G$.

<u>SECOND FUNDAMENTAL LEMMA.</u> The index $|A : L_g|$ is finite.

Before proceeding directly to the proof of the fundamental lemma, we establish some facts, auxiliary for this purpose.

<u>LEMMA 8.</u> Let $\mathfrak{M}$ be a set of one of the following types:

1) the set of the elements from A, generating some quasicyclic 2-subgroup in A;

2) some infinite set of elements of odd order from A.

If for some infinite subset $\mathcal{Y}$ from $\mathfrak{M}$ and some element $g \in G$, the intersection

$$\bigcap_{a \in \mathcal{Y}} (ig^{-1}ia^2g) = (t),$$

where t is a 2-element such that all quotient groups of the form $(ig^{-1}ia^2g)/(t)$, $a \in \mathcal{Y}$, have odd order, then $\mathfrak{M}$ is a set of type 2 and for some infinite subset $\mathcal{P}$ from $\mathcal{Y}$ we have the inclusions

$$a^2 b^{-2} \in L_g \ (a, b \in \mathcal{P}).$$

<u>Proof.</u> By the properties of dihedral groups, we have $i, g^{-1}ia^2g \in N_G((t))$. Obviously, $g^{-1}b^{-2}a^2g \in N_G((t))$, $b, a \in \mathcal{Y}$, and all such elements generate an infinite Abelian subgroup Q from $N_G((t))$; moreover, all elements from Q are strictly real relative to $g^{-1}ia^2g$, $a \in \mathcal{Y}$.

Let d be some element from $\mathcal{Y}$ and let $k = g^{-1}id^2g$. Since $k^{-1} = g^{-1}d^{-2}ig$, $iai = a^{-1}$ and $g^{-1}a^{-2}d^2g \in Q$, $a \in \mathcal{Y}$, we have $g^{-1}ia^2g = g^{-1}a^{-2}d^2gk^{-1} \in H = Q \leftthreetimes (k)$, $a \in \mathcal{Y}$. According to the definition of the set $\mathcal{Y}$ the subgroup Q is either a quasicyclic 2-subgroup or a subgroup without involutions. But then, as it is known, all involutions of the form $g^{-1}ia^2g$, $a \in \mathcal{Y}$, are conjugate with k in H, i.e.,

$$g^{-1}ia^2g = c_a^{-1}kc_a, \quad a \in \mathcal{Y},$$

where $c_a \in Q$.

We introduce the following notations: $q = ik$, $\ell_a = c_a i c_a^{-1} k$, $T = N_G((t))/(t)$, $\bar{Q} = Q(t)/(t)$, $\bar{i} = i(t)$, $\bar{k} = k(t)$, $\bar{q} = q(t)$, $\bar{c}_a = c_a(t)$, $\bar{\ell}_a = \ell_a(t)$, $a \in \mathcal{Y}$. Obviously, $|(\ell_a)| = |(ig^{-1}ia^2g)|$, $a \in \mathcal{Y}$, and, thus, the orders of the elements $\bar{q}$, $\bar{\ell}_a \ (a \in \mathcal{Y})$ are odd and, therefore, $\bar{\ell}_a = s_a^{-2} = \bar{c}_a \bar{i} \bar{c}_a \bar{k}$, $s_a \in (\bar{\ell}_a)$, $\bar{q} = x^{-2} = \bar{i}\bar{k}$, $x \in (\bar{q})$. From here it follows that

$$x\bar{i}x^{-1} = k, \quad s_a \bar{c}_a \bar{i} \bar{c}_a^{-1} s_a^{-1} = \bar{k}.$$

Making equal the left-hand sides of these equalities, we obtain that $x^{-1}s_a\bar{c}_a = z_a \in C_T(\bar{i})$ or

$$s_a = xz_a\bar{c}_a^{-1} \quad (a \in \mathcal{Y}). \tag{7}$$

Since $\bar{k}s_a\bar{k} = s_a^{-1}$, $\bar{k}\bar{c}_a\bar{k} = \bar{c}_a^{-1}$, we have

$$\bar{k}s_a\bar{k} = \bar{k}(xz_a)\bar{k}\bar{k}\bar{c}_a^{-1}\bar{k} = \bar{k}(xz_a)\bar{k}\bar{c}_a = \bar{c}_a(xz_a)^{-1}$$

or

$$(\bar{k}\bar{c}_a)^{-1}(xz_a)\bar{k}\bar{c}_a = (xz_a)^{-1} \ (a \in \mathcal{Y}). \tag{8}$$

We rewrite the equalities (7) in the form

$$c_a^{-1}s_a^{-1} = z_a^{-1}x^{-1}, \quad a \in \mathcal{Y}.$$

In these equalities the element x is fixed, the elements c_a, s_a are strictly real relative to the involution i and have odd orders, while the elements of the form $z_a^{-1} \in C_T(\bar{i})$,

$\alpha \in \mathfrak{Y}$. From here and from Lemma 4 we can easily see that the number of the distinct elements of the form τ_α , $\alpha \in \mathfrak{Y}$, is finite, i.e., for some infinite subset $\mathfrak{P}$ from $\mathfrak{Y}$ we have

$$\tau = \tau_m = \tau_b , \quad m, b \in \mathfrak{P} .$$

With the aid of these equalities we rewrite the equalities (8) in the form

$$(\bar{\kappa}\bar{c}_a)^{-1}(x\tau)(\bar{\kappa}\bar{c}_a) = (x\tau)^{-1}, \quad a \in \mathfrak{P} . \tag{9}$$

If b is an element from $\mathfrak{P}$, then from (9) we obtain that

$$\bar{c}_b^{-1}\bar{c}_a \in C_T(x\tau), \quad a, b \in \mathfrak{P} . \tag{10}$$

We denote by $\bar{Q}_\tau$ the subgroup $\bar{Q}_\tau$ = gr $(\{\bar{c}_b^{-1}\bar{c}_a \mid a, b \in \mathfrak{P}\})$. From the inclusions (10) there follows that $x\tau \in C_T(\bar{Q}_\tau)$. But $C_T(\bar{Q}_\tau) \triangleleft N_T(\bar{Q}_\tau)$ and $\kappa \in N_T(\bar{Q}_\tau)$ and, therefore, $[x\tau, \bar{\kappa}] \in C_T(\bar{Q}_\tau)$. Further, $\bar{\kappa}x\bar{\kappa} = x^{-1}$, $\tau^{-1}\bar{i}\tau = \bar{i}$, $x^{-2} = \bar{i}\bar{\kappa}$ and, taking this into account, we write $[x\tau, \bar{\kappa}] = \tau^{-1}x^{-1}\bar{\kappa}x\tau\bar{\kappa} = \tau^{-1}x^{-2}\bar{\kappa}\tau\bar{\kappa} = \tau^{-1}\bar{i}\bar{\kappa}\bar{\kappa}\tau\bar{\kappa} = \bar{i}\bar{\kappa} = x^{-2} \in C_T(\bar{Q}_\tau)$. Since $\bar{\kappa} \in N_T(\bar{Q}_\tau)$, we have also $\bar{i} \in N_T(\bar{Q}_\tau)$ and, moreover, if h is an arbitrary element from $\bar{Q}_\tau$, then $\bar{i}h\bar{i} = h^{-1}$.

From what has been proved above there follows that $\mathfrak{N}$ cannot be a set of type 1. We assume that this is not so. Obviously, $\bar{Q}_\tau = \bar{Q}$ and $\bar{i} \in N_T(\bar{Q})$ By the definition of the elements of the form $c_a (a \in \mathfrak{Y})$, we have $g^{-1}ia^2g = c_a^{-1}\kappa c_a$ and $(ig^{-1}ia^2g)(t) = \bar{i}\bar{c}_a\bar{\kappa}\bar{c}_a$, and, moreover, $|\bar{i}\bar{c}_a^{-1}\bar{\kappa}\bar{c}_a| (a \in \mathfrak{Y})$ is odd. However $\bar{i}\bar{c}_a\kappa\bar{c}_a = \bar{i}\bar{c}_a^{-2}\bar{\kappa}$ and $\bar{c}_a \in \bar{Q} = \bar{Q}_\tau$, while, according to what has been proved above, we have $\bar{i} \in N_T(\bar{Q}_\tau)$. Consequently,

$$\bar{\ell}_a = \bar{c}_a\bar{i}\bar{c}_a^{-1}\bar{\kappa} = \bar{c}_a^2\bar{i}\bar{\kappa} = \bar{c}_a^2\bar{q} = \bar{q}\bar{c}_a^2 ,$$

where $\bar{q}$ is a fixed element of odd order. Since $\bar{Q}$ is a quasicyclic 2-subgroup, it follows, obviously, that for $\bar{\ell}_a \neq \bar{q}$ all the elements of the form ℓ_a , $a \in \mathfrak{Y}$, have even order, which contradicts the assumption regarding the fact that their orders are odd, made at the beginning of the proof of the lemma. Consequently, $\mathfrak{N}$ can be only a set of type 2.

If Q_τ is the preimage of $\bar{Q}_\tau$ without involution in G , then, obviously, $i \in N_G(Q_\tau)$. From the equalities $g^{-1}ia^2g = c_a^{-1}\kappa c_a$, $g^{-1}ib^2g = c_b^{-1}\kappa c_b$ we obtain $g^{-1}b^{-2}a^2g = c_b^{-2}c_a^2$, $a, b \in \mathfrak{P}$ Since c_a , $c_b \in Q$ and Q is an Abelian subgroup, not containing involutions, we have $(g^{-1}b^2a^2g) = (c_b^{-2}c_a^2) = (c_b^{-1}c_a) < Q_\tau < Q$ $ig^{-1}b^{-2}a^2gi = g^{-1}a^{-2}b^2g$. But then

$$i^{g^{-1}}(b^{-2}a^2)i^{g^{-1}} = (b^{-2}a^2)^{-1},$$

$a, b \in \mathfrak{P}$. The lemma is proved.

<u>LEMMA 9.</u> The subgroup A_2 is normal in G .

<u>Proof.</u> Since the rank of the subgroup A_2 is finite (Lemma 3), we shall prove the lemma by induction on the rank of the subgroup A_2 . Let H be an arbitrary but fixed quasicyclic subgroup in A_2 .

Applying Lemmas 4, 8, we construct in H for the element $g\in G$ a strictly decreasing chain of infinite subsets of its elements

$$\mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \ldots \supset \mathfrak{A}_n \supset \ldots \tag{11}$$

such that

$$(t_1) = \bigcap_{a\in\mathfrak{A}_1}(ig^{-1}ia^2g), \quad (t_2) = \bigcap_{a\in\mathfrak{A}_2}(ig^{-1}ia^2g),$$

$$\ldots, (t_n) = \bigcap_{a\in\mathfrak{A}_n}(ig^{-1}ia^2g), \ldots$$

and

$$(t_1) < (t_2) < \ldots < (t_n) < \ldots \tag{12}$$

We denote by V the union of the chain (12) and by κ the involution $\kappa = g^{-1}id^2g$, where d is some element from $\mathfrak{A}_1$. Since the chain (12) does not break at a finite index, it follows that V is a quasicyclic 2-subgroup. Further, the sets from (11) are infinite and belong to the quasicyclic subgroup H and, therefore,

$$H^g \wedge (\kappa) = \mathrm{gr}\,(\{g^{-1}ia^2g \mid a\in\mathfrak{A}_n\}),$$

$n = 1, 2, \ldots$ But then, obviously

$$V \lhd \mathrm{gr}\,(V, H^g, i, \kappa) = L.$$

If $V = H^g$, then $ig^{-1}cgi = g^{-1}c^{-1}g$ $(c\in H)$ and $i^{g^{-1}}ci^{g^{-1}} = c^{-1}$ for any c from H.

Let $V \neq H^g$, and we consider $\bar{L} = L/V$ with the involution $\bar{\kappa} = \kappa V$. All subgroups of the form gr $(\bar{\kappa}, \bar{\kappa}^s)$, $s\in\bar{L}$, are finite and, by Lemma 4, $\bar{\kappa}$ is a finitely imbedded involution in $\bar{L}$, i.e., the pair $(\bar{L}, \bar{\kappa})$ satisfies all the conditions of the theorem. Let $\bar{X}$ be a complete Abelian 2-subgroup, all elements of which are strictly real relative to $\bar{\kappa}$ and, moreover, its rank $r(\bar{X})$ is the largest of all the ranks of such subgroups. By statement 4 from [4], its complete preimage X in L is a complete Abelian 2-subgroup, all elements of which are strictly real relative to κ. Since $\kappa = g^{-1}d^{-1}idg$, it follows that all the elements from $(dg)X(dg)^{-1}$ are strictly real relative to i and, by the definition of the subgroup A_2, we have

$$r((dg)X(dg)^{-1}) \leq r(A_2).$$

Consequently, $r(\bar{X}) \leq r(A) - 1$. By the induction hypothesis, $\bar{X} \lhd \bar{L}$. But then, obviously, $H^gV/V \leq \bar{X}$ and $H^g < X \lhd L$, and since $i\in L$ and by Lemma 3 all the elements from X are strictly real relative to i, it follows that $ig^{-1}cgi = g^{-1}c^{-1}g$, $c\in H$, and $i^{g^{-1}}ci^{g^{-1}} = c^{-1}$ for any c from H. From here, in view of the arbitrariness of the section of the quasicyclic subgroup H from the complete Abelian 2-subgroup A_2, there follows that $gig^{-1}\in N_G(A_2)$ and each element from A_2 is strictly real relative to gig^{-1}. In particular, $A_2 \lhd$ gr $(\{i^g \mid g\in G\}) = B \lhd G$ and $A_2^g \lhd B$ and, moreover, all the elements from A_2^g are strictly real relative to i. From here, in view of the definition of the subgroup A_2 and statement

4 from [4], we obtain that $A_2 = A_2^g$, $g \in G$. Further, by Lemma 4, the involution G/A_2 is finitely imbedded in iA_2 and, again by the definition of the subgroup A_2 and statement 4 of [4], G/A_2 does not have an infinite complete Abelian 2-subgroup, all elements of which are strictly real relative to iA_2. The lemma is proved.

We proceed directly to the proof of the second fundamental lemma.

By Lemma 9 we have $A_2 \triangleleft G$ and, by Lemma 4, the pair $(G/A_2, iA_2)$ satisfies all the conditions of the lemma and, moreover, G/A_2 does not possess an infinite complete Abelian 2-subgroup, all elements of which are strictly real relative to iA. This circumstance allows us to make the following

<u>Remark 3.</u> Without loss of generality, we shall assume that $A_2 = 1$ and G does not possess an infinite complete Abelian 2-subgroup, all elements of which are strictly real relative to i.

We assume that $|A : L_g|$ is infinite. Based on this assumption and on the properties of Abelian groups [5], we prove easily that A possesses at least one of two sets of elements from A, one of which consists of representatives, taken oen each from each coset of A with respect to L_g, some quasicyclic subgroup from A/L_g, while the other one consists of representatives of distinct cosets, taken one each from each such class, being the generators of the cyclic factors of the direct decomposition of some infinite subgroup from A/L_g.

We denote by $\mathfrak{N}$ one of these sets. If in the set of elements of the form $ig^{-1}ia^2g$, $a \in \mathfrak{N}$, there exist infinitely many elements of even order, then, by Lemma 4, $\mathfrak{N}$ has an infinite subset $\mathfrak{G}_1$ such that $\bigcap_{a \in \mathfrak{G}_1} (ig^{-1}ia^2g) \ni t_1$, where t_1 is an involution. If in the quotient group $N_G((t_1))/(t_1)$ in the set of cosets $ig^{-1}ia^2g(t_1)$, $a \in \mathfrak{G}_1$, infinitely many elements are of even order, then, by Lemma 4, $\mathfrak{G}_1$ possesses an infinite subset $\mathfrak{G}_2$ such that $\bigcap_{a \in \mathfrak{G}_2} (ig^{-1}ia^2g) \ni t_2$, where t_2 is an element of order 4. We reason in a similar manner regarding $\mathfrak{G}_2$ and $N_G((t_2))$, etc. As a final result, we construct a strictly increasing chain of cyclic 2-subgroups

$$(t_1) < (t_2) < \ldots < (t_n) < \ldots \tag{13}$$

If the chain (13) would not break at a finite index, then its union would be a quasicyclic subgroup, all elements of which would be strictly real relative to i. But then we would obtain a contradiction with Remark 3. Consequently, the chain (13) breaks at a finite index and $\mathfrak{N}$ has an infinite subset $\mathfrak{G}$ such that $\bigcap_{a \in \mathfrak{G}} (ig^{-1}ia^2g) \ni t$, where t is a 2-element and the elements $ig^{-1}ia^2g(t)$, $a \in \mathfrak{G}$, from $N_G((t))/(t)$ have odd order. By Lemma 8, for some infinite subset $\mathfrak{P}$ of $\mathfrak{G}$ we have

$$a^2b^{-2} \in L_g, \quad a, b \in \mathfrak{P}. \tag{14}$$

But, as one can easily see, from the definition of the set $\mathfrak{N}$ there follows that the set of distinct cosets of the form $a^2b^{-2}L_g$, $a, b \in \mathfrak{P}$, is infinite in spite of the inclusions (14). Consequently, $|A : L_g|$ is finite and the second fundamental lemma is proved.

3. THE LOCAL FINITENESS OF THE SUBGROUP $\mathcal{B}=\mathrm{gr}(\{i^g \mid g\in G\})$ AND THE PROPERTIES OF THE SUBGROUP $R=\mathrm{gr}(\mathcal{L}_i^2)$

Taking into account Lemmas 4, 9 and Schmidt's theorem [5], we shall prove the local finiteness under the assumtion that Remark 3 holds for the pair (G, i).

We denote by R the set of all elements from $\mathcal{B}$ of the form $b=i_1 i_2 \ldots i_{2n}$, where i_s $(s=1,2,\ldots,2n)$ is an involution from $\mathcal{L}_i$. Obviously, R is a subgroup and $R \lhd G$. Since in A all the elements are strictly real relative to i, we have $A<R$.

LEMMA 10. If b is an element from R, then $|A:C_R(b)|$ is finite, $i\notin R$ and $\mathcal{B}=R\leftthreetimes(i)$.

Proof. First we consider the case when $b=g_1 i g_1^{-1} g_2 i g_2^{-1}$. By the second fundamental lemma, $|A:L_{g_1}|$, $|A:L_{g_2}|$ are finite and, in view of the definition of the subgroups L_{g_1}, L_{g_2}, we have, obviously, $b\in C_G(L_{g_1}\cap L_{g_2})$; by Poincaré's theorem [5, Exercise 2.4.8], $|A: L_{g_1}\cap L_{g_2}|$ is finite.

Assume now that $b=(i_1\cdot i_2)\ldots(i_{2n-1}\cdot i_{2n})$, where n is the number of parts in the representation of b. We shall prove the lemma by induction on the number n of pairs. For $n=1$ the lemma has been proved above. Let $n>1$ and $b=(i_1 i_2)c$, where $c=(i_3\cdot i_4)\ldots(i_{2n-1} i_{2n})$. By the induction hypothesis, $|A:X|<\infty$, where $X=A\cap C_R(c)$, while, according to what has been proved above, $|A:Z|<\infty$, where $Z=A\cap C_R(i_1 i_2)$. But then $X\cap Z<C_R(b)$ and $|A:X\cap Z|<\infty$. From here and from the definition of the subgroup A there follows that $i\notin R$ and, obviously, $\mathcal{B}=R\leftthreetimes(i)$. The lemma is proved.

Let j be an involution from R and $j\in\mathcal{L}_i^2=\mathcal{L}_i\mathcal{L}_i$, $\mathcal{J}=\{j^g \mid g\in G\}$. Obviously, $\mathcal{J}\subset\mathcal{L}_i^2=\mathcal{L}_i\mathcal{L}_i$.

LEMMA 11. The involution j is contained in a finite normal subgroup in G.

Proof. If $\mathcal{J}$ is a finite set, then, by Dietzmann's lemma [10], j is contained in a finite normal subgroup from G. We assume that $\mathcal{J}$ is an infinite set. Let P be a finite (i)-invariant 2-subgroup from R and assume that the intersection $\mathfrak{A}=N_{\mathcal{B}}(P)\cap\mathcal{J}$ is infinite. We introduce the notations $V=N_{\mathcal{B}}(P)$, $\bar V=V/P$, $\bar i=iP$, $\bar{\mathfrak{A}}=\mathfrak{A}P/P$, $Q=R\cap V$, $\bar Q=Q/P$. By Lemma 10, $V=Q\leftthreetimes(i)$ and $\bar V=\bar Q\leftthreetimes(\bar i)$.

We consider the elements of the form $a_k=\bar i k$, where $k\in\bar{\mathfrak{A}}$. Since $\bar{\mathfrak{A}}<\bar Q$ and $\bar V=\bar Q\leftthreetimes(\bar i)$, it follows that all elements of the form a_k, $k\in\bar{\mathfrak{A}}$, have even order, i.e., (a_k) possesses an involution $\bar t_k$. We prove that

1) the set of involutions of the form $\bar t_k$, $k\in\bar{\mathfrak{A}}$, is finite and $|\bar V:C_{\bar V}(\bar i)|$ is infinite.

If the element a_k^2 has even order, then, as one can easily show, we have $\bar t_k=\bar i\bar j_k$, where $\bar j_k$ is the involution conjugate with $\bar i$ in gr $(\bar i, k)$ and j_k is its preimage from $\mathcal{L}_i$. If, however, $|a_k^2|$ is odd, then $\bar t_k=\bar i\bar s_k$, where $\bar s_k$ is the involution conjugate with k in gr $(\bar i, k)$ and s_k is its preimage from $\mathfrak{A}$. Obviously, $\bar t_k$, $\bar j_k$, $\bar s_k\in C_{\bar V}(\bar i)$. If X is the complete preimage of $C_{\bar V}(\bar i)$ in V, then i, j_k, $s_k\in X$ and $|X:C_X(i)|$ is finite. But then, by Lemma 2, the set of elements of the form j_k, s_k, $k\in\bar{\mathfrak{A}}$, is finite and this means, obviously, that also the set of elements of the form $\bar t_k$, $k\in\bar{\mathfrak{A}}$, is finite. Further,

$|V:X|$ is infinite since otherwise we would obtain a contradiction with Lemma 2 and the fact that the set $\mathfrak{A}$ is infinite. Statement 1) is proved. Now we prove that

2) in the set of elements of the form a_κ^2, $\kappa \in \overline{\mathfrak{A}}$, there are only a finite number of elements of odd order.

We assume that $\overline{\mathfrak{A}}$ has an infinite subset $\overline{\mathfrak{B}}$ such that all elements of the form a_κ^2, $\kappa \in \overline{\mathfrak{B}}$, have odd orders. In this case $\bar{t}_\kappa = \bar{i}\bar{s}_\kappa$, where $\bar{s}_\kappa \in \overline{\mathfrak{A}}$ and s_κ is the preimage of $\bar{s}_\kappa$ in $\mathfrak{A}$. Taking into account statement 1, we can assume, without loss of generality, that $\bar{s} = \bar{s}_\kappa = \bar{s}_{\kappa'}$, $\bar{t} = \bar{t}_\kappa = \bar{t}_{\kappa'}$, for any involutions κ, $\kappa' \in \overline{\mathfrak{B}}$. We consider the subgroup $\bar{T} = N_{\bar{V}}(\bar{t})$. By the properties of dihedral groups we have $\{\bar{i}, \overline{\mathfrak{B}}\} \subset \bar{T}$, and since $\overline{\mathfrak{B}}$ is an infinite set, it follows, by statement 1, that $|\bar{T}:C_{\bar{T}}(\bar{i})|$ is infinite and, by Lemma 4, that $\bar{i}$ is a finitely imbedded involution in $\bar{T}$. But then, by the first fundamental lemma and Remark 3, $\bar{T}$ has an infinite periodic Abelian subgroup $\bar{K}$ without involutions, all elements of which are strictly real relative to $\bar{i}$. We select the subgroup $\bar{K}$ in such a manner that its preimage K without involutions in V should belong to $C_B(P)$. Obviously, t, i, $s \in N_V(K)$, where t, s are preimages of the involutions $\bar{t}$, $\bar{s}$ in V and $\mathfrak{A}$, respectively. Obviously, all the elements from K are strictly real relative to i and gr $(P, t) < C_V(K)$. Since $s \in \mathfrak{A} < Q \leq R$, it follows by Lemma 10 that $|K:C_V(s) \cap K|$ is finite. From here and from the representation $t = hsi$, where $h \in P$, there follows that $C_V(s) \cap K < C_V(i)$ and $|K:C_V(i) \cap K|$ is finite in spite of the fact that K is infinite and that all elements from K are strictly real relative to i. The obtained contradiction concludes the proof of statement 2,

We proceed directly to the proof of the lemma.

Since $\mathfrak{J}$ is an infinite set, making use of statements 1, 2, we prove the existence in $\mathfrak{J}$ of an infinite subset $\mathfrak{A}_1$ such that $\bigcap_{\kappa \in \mathfrak{A}_1} (i\kappa) \ni t_1$, where t_1 is an involution and $t_1 \in R$. We consider $T_1 = N_B((t_1))$. By the known properties of the dihedral group we have $\{i, \mathfrak{A}_1\} \subset T_1 = Q_1 \leftthreetimes (i)$, where $Q_1 < R$. We introduce the following notations: $\bar{T}_1 = T_1/(t_1)$, $i_1 = i(t_1)$, $\overline{\mathfrak{A}}_1 = \mathfrak{A}_1(t_1)/(t_1)$, $\bar{Q}_1 = Q_1/(t_1)$. In view of the statements 1, 2, $\overline{\mathfrak{A}}_1$ has an infinite subset $\overline{\mathfrak{A}}_2$ such that $\bigcap_{\kappa \in \overline{\mathfrak{A}}_2} (i_1\kappa) \ni \bar{t}_2$, where $\bar{t}_2$ is an involution and $\bar{t}_2 \in \bar{Q}_1$. If t_2 is the preimage of $\bar{t}_2$ in Q_1 and $\mathfrak{A}_2$ is the preimage of $\overline{\mathfrak{A}}_2$ in $\mathfrak{A}_1$, then $(t_1) < (t_2)$ and $\{i, \mathfrak{A}_2\} \subset T_2 = N_{T_1}((t_2))$. We reason in a similar manner with respect to the triplet $(T_2, \mathfrak{A}_2, i)$, etc. As a result we construct a strictly increasing chain of cyclic 2-subgroups

$$(t_1) < (t_2) < \ldots < (t_n) < \ldots, \tag{15}$$

which does not break at a finite index. However, in this case we obtain a contradiction with Remark 3, since the union of the chain (15) would be a quasicyclic 2-subgroup, all elements of which are strictly relative to i. The obtained contradiction means that $\mathfrak{J}$ is a finite set and the lemma is proved.

LEMMA 12. The subgroup R has a finite subgroup Z such that $Z \lhd G$ and in the quotient group $\bar{B} = B/Z$ all the elements of the form $\bar{i}\kappa$, where $\bar{i} = iZ$, $\kappa \in \bar{\mathcal{B}}_i = \mathcal{B}_i Z/Z$, have odd order.

Proof. If for some involution $\kappa_1 \in \mathcal{B}_i$ the subgroup $(i\kappa_1)$ has an involution j_1, then, obviously, $j_1 \in \mathcal{B}_i^2 = \mathcal{B}_i \mathcal{B}_i$ and, by Lemma 11, the subgroup $Z_1 = \mathrm{gr}(\{j_1^g \mid g \in G\})$ is finite and $Z_1 < R$. By Lemma 4, in G/Z_1 the involution $i_1 = iZ_1$ is finitely imbedded. If for some involution $\bar{\kappa}_2 \in \mathcal{B}_i Z_1/Z_1$ the subgroup $(i_1 \bar{\kappa}_2)$ has an involution $\bar{j}_2$, then, obviously, $\bar{j}_2 \in (\mathcal{B}_i Z_1/Z_1)$ and, by Lemma 11, the subgroup $\bar{Z}_2 = \mathrm{gr}(\{\bar{j}_2^g \mid g \in G/Z_1\})$ is finite and $\bar{Z}_2 < R/Z_1$. If Z_2 is the complete preimage of the subgroup $\bar{Z}_2$ in G, then $Z_1 < Z \lhd G$. We reason in a similar manner with respect to the quotient group G/Z_2, etc. As a result of this we construct a strictly increasing chain of finite subgroups

$$(1) = Z_0 < Z_1 < Z_2 < \ldots < Z_n < \ldots , \tag{16}$$

where $Z_n \lhd G$, $n = 1, 2, \ldots$.

We show that the chain (16) breaks at a finite index. Let κ_n be the preimage of $\bar{\kappa}_n$ in $\mathcal{B}_i$, and let j_n be the preimage of $\bar{j}_n$ in $(i\kappa_n)$, $n = 1, 2, \ldots$. Obviously, ij_n is an involution in $\mathcal{B}_i$.

Let S_1 be an (i)-invariant Sylow 2-subgroup from Z_1. By Theorem 11.1.1 from [5], such a group exists in Z_1 and $S_1 \leftthreetimes (i) \setminus S_1$ has an involution, conjugate with ij_1. By similar considerations, Z_2 has an (i)-invariant Sylow 2-subgroup S_2 and, moreover, $S_1 < S_2$ and some involution from $C_2 \leftthreetimes (i) \setminus S_2$ is conjugate with ij_2. Reasoning in this manner, we construct in B a strictly increasing chain of finite (i)-invariant 2-subgroups:

$$S_1 < S_2 < \ldots < S_n < \ldots \tag{17}$$

Such that $S_n \leftthreetimes (i) \setminus S_n$ has an involution, conjugate to ij_n. If the chain (17) would not break and S is its union, then, obviously, the intersection $S \cap \mathcal{B}_i$ would be infinite, in spite of Remark 3 and Lemma 5. Consequently, the chain (17) breaks at a finite index and, at the same time, also the chain (16) breaks. The lemma is proved.

Remark 4. Based on Lemmas 4, 12, in the subsequent arguments we shall assume, without loss of generality, that all the elements of the form $ig^{-1}ig$, $g \in G$, have finite odd orders.

LEMMA 13. The subgroup B is locally finite and R does not contain involutions.

Proof. Let $t_1, t_2, \ldots, t_n$ be an arbitrary collection of involutions from $\mathcal{B}_i$. We show that the subgroup $K = \mathrm{gr}(t_1, t_2, \ldots, t_n, i)$ is finite. By the second fundamental lemma, A has a system of subgroups $L_1, L_2, \ldots, L_n$, such that all elements from L_s, $s = 1, 2, \ldots, n$, are strictly real relative to t_s and $|A : L_s|$, $s = 1, 2, \ldots, n$, is finite. By Poincarés theorem [5, Exercise 2.4.8], the subgroup

$$D = L_1 \cap L_2 \cap \ldots \cap L_n$$

has a finite index in A. We consider the subgroup $V = N_B(D)$. Obviously, $K < V$ and, by Lemma 10, $V = Q \leftthreetimes (i)$, where $Q < R$. If κ is an arbitrary involution from $V \cap \mathcal{B}_i$, then, by Remark 4, $|i\kappa|$ is finite and odd. By the properties of dihedral groups, i and κ are conjugate in V and, therefore, all the elements from D are strictly real relative to κ. But then, by Remark 3,

$$(\mathcal{L}_i \cap V)\mathcal{D} = \mathcal{L}_i \cap V = \{i^g \mid g \in V\} \tag{18}$$

and $C_{\bar{V}}(\bar{i}) = C_V(i)\mathcal{D}/\mathcal{D}$, where $\bar{V} = V/\mathcal{D}$, $\bar{i} = i\mathcal{D}$. From here, as one can easily see, there follows that $\bar{i}$ is a finitely imbedded involution in $\bar{V}$ and, in addition, condition * holds for the pair $(\bar{V}, \bar{i})$. If $|\bar{V} : C_{\bar{V}}(\bar{i})|$ would be finite, then, in view of equality (18), Dietzmann's lemma [10], and [5, Theorem 23.1.1], the subgroup K would be finite since $K = \mathrm{gr}\ (t_1, t_2, \ldots, t_n, i) < V$ and $t_s \in \{i^g \mid g \in V\}$, $s = 1, 2, \ldots, n$.

We assume that $|\bar{V} : C_{\bar{V}}(\bar{i})|$ is infinite. Since the pair $(\bar{V}, \bar{i})$ satisfies the conditions of the first fundamental lemma, it follows, by this lemma and by Remark 4, that $\bar{Q} = Q/\mathcal{D}$ possesses an infinite periodic Abelian subgroup T without involutions, all elements of which are strictly real relative to $\bar{i}$. Further, the subgroup $\bar{A} = A/\mathcal{D}$ is finite, $\bar{A} < \bar{Q}$, and, by Lemma 10, the index $|T : C_{\bar{V}}(\bar{A}) \cap T|$ is finite. From here it follows that $\bar{A}$ is contained in an infinite periodic Abelian subgroup $\bar{X}$ without involutions from $\bar{Q}$ and, moreover, all the elements from $\bar{X}$ are strictly real relative to $\bar{i}$. By statement 4 of [4], the complete preimage X of the subgroup $\bar{X}$ in $\mathcal{B}$ would be also a periodic Abelian subgroup without involutions, all elements of which are strictly real relative to i, and, moreover, $A < X$ and $A \neq X$. However, in this case we would obtain a contradiction with the definition of the subgroup A. Consequently, the index $|\bar{V} : C_{\bar{V}}(\bar{i})|$ is finite and, as shown above, the subgroup K is finite.

Thus, the local finiteness of the subgroup $\mathcal{B}$ is proved. Now we prove that R does not contain involutions. By Remark 4, for any $\kappa \in \mathcal{L}_i$ the element $i\kappa$ has odd order. From here and from the local finiteness of $\mathcal{B}$, by Glauberman's well-known Z^*-theorem [11], we obtain that $\mathcal{B} = O_{2'}(\mathcal{B}) \times C_{\mathcal{B}}(i)$. Further, if j is an involution from $C_{\mathcal{B}}(i) \cap \mathcal{L}_i$, then, by Remark 4, $ij = 1$ and $i = j$. But then, obviously, $R = O_{2'}(\mathcal{B})$. The lemma is proved.

<u>PROOF OF THE FUNDAMENTAL THEOREM.</u> If the index $|G : C_G(i)|$ is finite, then, by Dietzmann's lemma [10], the subgroup $\mathcal{B} = \mathrm{gr}\ (\{i^g \mid g \in G\})$ is finite. If, however, the index $|G : C_G(i)|$ is infinite, then the validity of the theorem follows from Remark 1 and Lemmas 4, 9, 12, 13. The theorem is proved.

<u>COROLLARY.</u> Let G be a group and let a be its involution. Then at least one of the following statements holds:

1) for some element $t \in G$ the subgroup gr (a, a^t) is an infinite dihedral group;

2) for some element $t \in G$ the intersection $tC_G(a) \cap (a^G)^2$ is infinite;

3) gr (a^G) is a periodic almost locally solvable subgroup.

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