This completes the proof of the theorem.

LITERATURE CITED

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QUASIVARIETIES OF ALGEBRAS WITH DEFINABLE PRINCIPLE CONGRUENCES

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The main result of this article is the theorem on finite presentability of a quasivariety of algebras of finite signature with definable principle congruences in which the class of subdirectly (or finitely subdirectly) irreducible algebras is finitely axiomatizble; it is an improvement of McKenzie's well-known theorem [5]. The article also contains a characterization of (locally finite) quasivarieties with definable principle congruences along with various examples.

The terminology conforms to [1], [2], and [6].

1. DEFINITIONS AND AUXILIARY RESULTS

Let σ be an arbitrary finite functional signature. Henceforth we assume that all algebras and classes of algebras have the given signature σ.

For an arbitrary algebra A in a quasivariety $\mathcal{K}$ and an arbitrary set $H \subseteq A \times A$, let $\mathrm{Con}_{\mathcal{K}} A$ be the lattice of all $\mathcal{K}$-congruences on A, i.e., $\mathrm{Con}_{\mathcal{K}} A = \{\theta \in \mathrm{Con} A \mid A/\theta \in \mathcal{K}\}$, and $\theta_{\mathcal{K}}{}^A(H)$ the smallest $\mathcal{K}$-congruence containing H. In particular, $\theta_{\mathcal{K}}^A(a, b)$ denotes the principle $\mathcal{K}$-congruence on A generated by the set $\{(a, b)\}$. In the case of varieties, the subscript $\mathcal{K}$ in the expression $\theta_{\mathcal{K}}^A(a, b)$ and other similar situations is omitted. We say that in a quasivariety $\mathcal{K}$ <u>principle $\mathcal{K}$-congruences are (formula) definable</u> or $\mathcal{K}$ <u>has definable principle $\mathcal{K}$-congruences</u>, if there exists a first-order formula $\varphi(x, y, u, v)$ such that for all $A \in \mathcal{K}$ and a, b, c, d, $\in$ A the equivalence

$$(a, b) \in \theta_{\mathcal{K}}^A(c, d) \Leftrightarrow A \models \varphi(a, b, c, d)$$

holds. In this case we also say that $\varphi(x, y, u, v)$ defines principle $\mathcal{K}$-congruences.

The main tool for study of such quasivarieties is generalized Mal'tsev's lemma [2] along with its various modifications (cf. [3] and [9]). Let $\mathcal{K} = \mathrm{Mod}(I(\mathcal{K}) \cup \Sigma \cup E(\sigma))$, were $I(\mathcal{K})$ is the set of all identities valid in $\mathcal{K}$; let Σ be a fixed set of quasiidentities which are not identities and $E(\sigma)$ the set of equality axioms (without x = x) (cf. [1]). We define the set Γ of Σ-<u>congruence schemes</u> by induction:

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a) the formula $(x = y \vee \{x, y\} = \{u, v\})$ is a Σ-congruence scheme of height 0;

b) if $\underset{i \leq k}{\&} p_i(\bar{x}) = q_i(\bar{x}) \to p(\bar{x}) = q(\bar{x})$ is a quasiidentity in $\Sigma \cup E(\sigma)$, $\varphi_0(x, y, u, v), \ldots, \varphi_k(x, y, u, v)$ are Σ-congruence schemes of height $\leq n - 1$, then the formula

$$\exists \bar{z} \left[\{x, y\} = \{p(\bar{z}), q(\bar{z})\} \,\&\, \underset{i \leq k}{\&} \varphi_i(p_i(\bar{z}), q_i(\bar{z}), u, v)\right]$$

is a Σ-congruence scheme of height n.

LEMMA 1. For all $A \in \mathcal{R}$ and a, b, c, d, $\in$ A the equivalence

$$(a, b) \in \theta_{\mathcal{R}}^{A}(c, d) \Leftrightarrow A \models \bigvee \{\varphi(a, b, c, d) \mid \varphi \in \Gamma\}.$$

holds. Note that if $\mathcal{R}$ is a variety, then $\Sigma = \emptyset$ and the notion of a Σ-congruence scheme coincides with the notion of Mal'tsev's congruence scheme (cf. [13]).

LEMMA 2. For all $A \in \mathcal{R}$ and $H \subseteq A^2$ the equality

$$\theta_{\mathcal{R}}(H) = \bigcup_{n < \omega} H_n^A(\Sigma),$$

holds, where

$$H_0^A(\Sigma) = H \cup \{(a, a) \mid a \in A\} \quad \text{and } (a, b) \in H_n^A(\Sigma) \;, n > 0$$

if and only if either $(a, b) \in H_{n-1}^A(\Sigma)$ has a quasiidentity $\Sigma \cup E(\sigma)$

$$\underset{i \leq k}{\&} p_i(\bar{x}) = q_i(\bar{x}) \to p(\bar{x}) = q(\bar{x}),$$

such that

$$(p_i(\bar{c}), q_i(\bar{c})) \in H_{n-1}^A(\Sigma), \quad i \leq k,$$
$$(p(\bar{c}), q(\bar{c})) = (a, b)$$

for some $\bar{c} \in A$.

Henceforth, we will usually omit the symbol A in $\theta_{\mathcal{R}}^A(H)$ and $H_n^A(\Sigma)$.

The article will often make use of the following known (cf. [9] and [14]).

LEMMA 3. Let $A \in \mathcal{R}$, a, b, c, d, $\in$ A and $\theta \in \mathrm{Con}_{\mathcal{R}} A$. Then

$$(a, b) \in \theta_{\mathcal{R}}(c, d) \vee_{\mathcal{R}} \theta \Leftrightarrow (a/\theta, b/\theta) \in \theta_{\mathcal{R}}^{A/\theta}(c/\theta, d/\theta),$$

where $\Lambda_{\mathcal{R}}$ is the sum in the lattice $\mathrm{Con}_{\mathcal{R}} A$.

The compactness theorem and Lemma 1 immediately imply

LEMMA 4. Each formula defining principle $\mathcal{R}$-congruences in a quasivariety $\mathcal{R}$ is logically equivalent in $\mathcal{R}$ to some finite disjunction of Σ-congruence schemes.

2. FINITELY PRESENTED QUASIVARIETIES OF ALGEBRAS WITH DEFINABLE PRINCIPLE CONGRUENCES

Let $\underline{V}(\mathcal{R})$ be the smallest variety of algebras containing a class $\mathcal{R}$.

LEMMA 5. If $\mathcal{N}$ and $\mathcal{K}$ are quasivarieties such that $\underline{V}(\mathcal{N}) = \underline{V}(\mathcal{K})$ and for each algebra $A \in \mathcal{N} \cap \mathcal{K}$ and all elements a, $b \in A$ the equality $\theta_{\mathcal{K}}(a,b) = \theta_{\mathcal{N}}(a,b)$ holds, then $\mathcal{N} = \mathcal{K}$.

Proof. It is known ([1], [2]) that each quasivariety $\mathcal{N}$ of algebras is determined by its finitely presented algebras and each finitely presented algebra in $\mathcal{N}$ is isomorphic to some factor algebra of an $\mathcal{N}$-free algebra of finite rank over a compact $\mathcal{N}$-congruence. Since $\underline{V}(\mathcal{N}) = \underline{V}(\mathcal{K})$, the free algebras in $\mathcal{N}$ and $\mathcal{K}$ coincide. Therefore, to prove the lemma it suffices to show that on an arbitrary $\mathcal{N}$-free algebra F each compact $\mathcal{N}$-congruence is an $\mathcal{N}$-congruence, i.e., $\bigvee^{\mathcal{N}}_{i \in n} \theta_{\mathcal{N}}(a_i, b_i) \in Con_{\mathcal{K}} A$. We will prove it by induction on n. By hypothesis, $\theta_{\mathcal{N}}(a,b) = \theta_{\mathcal{K}}(a,b)$. Let $\bigvee^{\mathcal{N}}_{i<n} \theta_{\mathcal{N}}(a_i,b_i) = \bigvee^{\mathcal{K}}_{i<n} \theta_{\mathcal{K}}(a_i,b_i) = \theta$ and $(c,d) \in \theta_{\mathcal{N}}(a,b) \vee_{\mathcal{N}} \theta$. Then, by Lemma 3, $(c/\theta, d/\theta) \in \theta_{\mathcal{N}}(a/\theta, b/\theta)$. But $F/\theta \in \mathcal{N} \cap \mathcal{K}$, therefore, $(c/\theta, d/\theta) \in \theta_{\mathcal{K}}(a/\theta, b/\theta)$. Applying Lemma 3 again, we obtain $(c, d) \in \theta_{\mathcal{K}}(a,b) \vee_{\mathcal{K}} \theta$, i.e., $\theta_{\mathcal{N}}(a,b) \vee_{\mathcal{N}} \theta \subseteq \theta_{\mathcal{K}}(a,b) \vee_{\mathcal{K}} \theta$. The reverse inclusion is proved similarly.

We will say that a quasivariety $\mathcal{K}$ has a finite basis of quasiidentities in a variety $\mathcal{N}$ if there exists a finite set of quasiidentities Σ such that $A \in \mathcal{K} \Rightarrow A \in \mathcal{N} \cap Mod(\Sigma)$.

LEMMA 6. If a quasivariety $\mathcal{K}$ has definable principle $\mathcal{K}$-congruences, then $\mathcal{K}$ is finitely based in the variety $\underline{V}(\mathcal{K})$.

Proof. Suppose that $\varphi(x, y, u, v)$ defines principle $\mathcal{K}$-congruences and Σ is a basis of quasiidentities of the quasivariety $\mathcal{K}$ in $\underline{V}(\mathcal{K})$. By Lemma 4, we may assume that $\varphi(x, y, u, v)$ is a finite disjunction of Σ-congruence schemes. We denote by Σ_0 the set of all quasiidentities over which the Σ-congruence schemes occurring in the expression of the formula φ are constructed. Obviously, Σ_0 is a finite set and $\Sigma_0 \subseteq \Sigma$. Put $\mathcal{N} = Mod(I(\mathcal{N}) \cup \Sigma_0 \cup E(\sigma))$. The lemma will be proved if we show that $\mathcal{N} = \mathcal{K}$. Since $\underline{V}(\mathcal{N}) = \underline{V}(\mathcal{K})$, we know that by Lemma 5 it suffices to show that for each algebra $A \in \mathcal{N} \cap \mathcal{K}$ and all a, $b \in A$ we have $\theta_{\mathcal{N}}(a,b) = \theta_{\mathcal{K}}(a,b)$ Since $\Sigma_0 \subseteq \Sigma$, $\theta_{\mathcal{K}}(a,b) \supseteq \theta_{\mathcal{N}}(a,b)$. Conversely, let $(c, d) \in \theta_{\mathcal{K}}(a, b)$, then $A \models \varphi(c,d,a,b)$. Therefore, by the choice of Σ_0, $A \models \psi(c, d, a, b)$, where ψ is some Σ_0-congruence scheme. Thus, $(c, d) \in \theta_{\mathcal{N}}(a, b)$, i.e., $\theta_{\mathcal{K}}(a,b) \subseteq \theta_{\mathcal{N}}(a,b)$.

We say that an algebra $A \in \mathcal{K}$ is (finitely) subdirectly $\mathcal{K}$-irreducible if the intersection of each (finite) set of nonzero $\mathcal{K}$-congruences on A is not a zero congruence. McKenzie [5] has proved that if a finitely generated quasivariety $\mathcal{K}$ with definable principle $\mathcal{K}$-congruences is a variety, then $\mathcal{K}$ is finitely based. Lemma 6 along with ideas from Burris' work [7] allow us to strengthen this theorem.

THEOREM 1. A quasivariety of algebras $\mathcal{K}$ with definable principle $\mathcal{K}$-congruences has a finite basis of quasiidentities if and only if the class of subdirectly (or finitely subdirectly) $\mathcal{K}$-irreducible algebras is finitely axiomatizable.

Sufficiency. Suppose $\mathcal{K} = Mod(I(\mathcal{K}) \cup \Sigma \cup E(\sigma))$ and $\varphi(x, y, u, v)$ defines principle $\mathcal{K}$-congruences. By Lemma 6, we may assume that Σ is a finite set. Let ϕ_0 be a proposition such that if ϕ_0 is true on an algebra A, then for all elements a, $b \in A$ the set $\theta_{ab} = \{\langle c,d \rangle \in$

$A^2|A \models \varphi(x,y,a,b)$ is the smallest congruence containing the pair (a, b) and $A/\theta_{ab} \models \Sigma$. For instance,

$$\forall uv\left[\forall x\, \varphi(x,x,u,v) \,\&\, \varphi(u,v,u,v) \,\&\, \mathop{\&}_{\kappa \in \Sigma \cup E(\sigma)} \varphi_{\kappa}(u,v)\right],$$

is such a proposition, where $\varphi_{\kappa}(u, v)$ is

$$\forall \bar{x} \left[\mathop{\&}_{i \le n} \varphi(p_i(\bar{x}), q_i(\bar{x}), u, v) \longrightarrow \varphi(p(\bar{x}), q(\bar{x}), u, v)\right]$$

for $k \in \Sigma \cup E(\sigma)$ having the form $\forall \bar{x} \left(\mathop{\&}_{i \le n} p(\bar{x}) = q_i(\bar{x}) \to p(\bar{x}) = q(\bar{x})\right)$.

Let $\mathcal{R}_{FSI}$ be the class of all finitely subdirectly $\mathcal{R}$-irreducible algebras and suppose that $\mathcal{R}_{FSI} = \mathrm{Mod}(\phi_1)$ for some proposition ϕ_1. It is easily seen that $A \in \mathcal{R}_{FSI}$ if and only if $A \in \mathcal{R}$ and a proposition ϕ_2 of the form

$$\forall xyuv\left[x \ne y \,\&\, u \ne v \longrightarrow \exists zw\, (z \ne w \,\&\, \varphi(z,w,x,y) \,\&\, \varphi(z,w,u,v))\right]$$

is true on A. Therefore,

$$I(\mathcal{R}) \cup \Sigma \cup E(\sigma) \vdash \left[\phi_0 \,\&\, (\phi_2 \to \phi_1)\right] \vee \left[\forall xy\, (x = y)\right].$$

By the compactness theorem, there exists a finite set $I \subseteq I(\mathcal{R})$ such that

$$I \cup \Sigma \cup E(\sigma) \vdash \left[\phi_0 \,\&\, (\phi_2 \to \phi_1)\right] \vee \left[\forall xy\, (x = y)\right].$$

Let $\mathcal{N} = \mathrm{Mod}(I \cup \Sigma \cup E(\sigma))$ and $A \in \mathcal{N}_{FSI}$. Obviously, $\mathcal{R} \subseteq \mathcal{N}$. Since $A \models \phi_0$, we have $A \models \phi_2$, therefore $A \models \phi_1$, i.e., $A \in \mathcal{R}_{FSI}$. Thus, $\mathcal{N}_{FSI} \subseteq \mathcal{R}_{FSI}$ and $\mathcal{N} \subseteq \mathcal{R}$. If we consider the class $\mathcal{R}_{SI}$ of subdirectly $\mathcal{R}$-irreducible algebras, then instead of ϕ_2 we have to take a formula ϕ_2' of the form

$$\exists xy\, \forall uv\left[x \ne y \,\&\, (u \ne v \longrightarrow \varphi(x,y,u,v))\right].$$

<u>Necessity.</u> Let $\mathcal{R} = \mathrm{Mod}(K)$ for some finite set K of quasiidentities. Then $\mathcal{R}_{FSI} = \mathrm{Mod}(K \cup \{\phi_2\})$ and $\mathcal{R}_{SI} = \mathrm{Mod}(K \cup \{\phi_2'\})$.

<u>COROLLARY.</u> Each finitely generated quasivariety $\mathcal{R}$ of algebras with definable principle $\mathcal{R}$-congruences has a finite basis of quasiidentities.

This corollary has been also proved independently by Gorbunov.

3. CHARACTERIZATION OF QUASIVARIETIES OF ALGEBRAS WITH DEFINABLE PRINCIPLE CONGRUENCES

Let $\mathcal{R} = \mathrm{Mod}(I(\mathcal{R}) \cup \Sigma \cup E(\sigma))$ be an arbitrary quasivariety of algebras and $A \in \mathcal{R}$. We say that an $\mathcal{R}$-congruence θ on A is Σ_n-<u>permutable</u>, $n > 0$, if for each congruence $\theta' \in \mathrm{Con}_{\mathcal{R}} A$ the equality $\theta \vee_{\mathcal{R}} \theta' = H_{n-1}(\Sigma)$ holds, where $H = \theta \cup \theta'$. A quasivariety $\mathcal{R}$ is said to be Σ_n -<u>permutable</u> if on each algebra in $\mathcal{R}$ each $\mathcal{R}$-congruence is Σ_n-permutable.

<u>Remark.</u> In the case of varieties we may assume that $\Sigma = \emptyset$. It is easily seen that a $\emptyset_n$-permutable variety is m-permutable, where $m \le 2^{n-1}$ and, conversely, each m-permutable variety is $\emptyset_n$-permutable, where $n = [\log_2(m-1)] + 2$.

We say that in a class of algebras $\mathcal{N} \subseteq \mathcal{K}$ principle $\mathcal{K}$-congruences are strictly definable if there exists a finite disjunction of Σ-congruence schemes $\varphi(x, y, u, v)$, such that for all $A \in \mathcal{N}$ and $a, b, c, d \in A$ the equivalence

$$(a,b) \in \theta_{\mathcal{K}}(c,d) \Leftrightarrow A \models \varphi(a,b,c,d).$$

holds.

Note that if $\mathcal{N} = \mathcal{K}$, then the notions of definability and strict definability of principle $\mathcal{K}$-congruences coincide (see Lemma 4).

THEOREM 2. A quasivariety of algebras $\mathcal{K}$ has definable principle $\mathcal{K}$-congruences if and only if $\mathcal{K}$ is finitely based in $\underline{V}(\mathcal{K})$, and in an $\mathcal{K}$-free algebra F of countable rank principle $\mathcal{K}$-congruences are strictly definable and Σ_n-permutable for some $n > 0$ and some finite basis Σ of the quasivariety $\mathcal{K}$ in $\underline{V}(\mathcal{K})$.

Necessity. By Lemmas 4 and 6, it suffices to find a number $n > 0$ such that for all elements a, $b \in F$ and each $\mathcal{K}$-congruence θ on F the equality $\theta V_{\mathcal{K}} \theta_{\mathcal{K}}(a,b) = H_{n-1}(\Sigma)$ holds, where Σ is some finite basis of quasiidentities of $\mathcal{K}$ in $\underline{V}(\mathcal{K})$ and $H = \theta \cup \theta_{\mathcal{K}}(a, b)$. Let $(c, d) \in \theta_{\mathcal{K}}(a,b) V_{\mathcal{K}} \theta$; then, by Lemma 3, $(c/\theta, d/\theta) \in \theta_{\mathcal{K}}(a/\theta, b/\theta)$. Since $\mathcal{K}$ has definable principle $\mathcal{K}$-congruences, we have $(c/\theta, d/\theta) \in H_{n-1}^{F/\theta}(\Sigma)$ for some $n > 0$, where $H = \{(a/\theta, b/\theta)\}$. We will show that $(c, d) \in H_{3n-1}(\Sigma)$. If $(c/\theta, d/\theta) \in H_0^{F/\theta}(\Sigma)$, then $(c, d) \in H_2(\Sigma)$. Suppose that for all $k < n - 1$ the implication

$$(c/\theta, d/\theta) \in H_{k-1}^{F/\theta}(\Sigma) \Rightarrow (c,d) \in H_{3k-1}(\Sigma)$$

holds. Suppose $(c/\theta, d/\theta) \in H_{n-1}^{F/\theta}(\Sigma)$. Then in $\Sigma \cup E(\sigma)$ there exists a quasiidentity $\underset{i \le m}{\&} p_i(\bar{x}) = q_i(\bar{x}) \to p(\bar{x}) = q(\bar{x})$, such that $(p_i(\bar{a})/\theta, q_i(\bar{a})/\theta) \in H_{n-2}^{F/\theta}(\Sigma)$, $i \le m$, and $(c/\theta, d/\theta) = (p(\bar{a})/\theta, q(\bar{a})/\theta)$ for some $\bar{a} \in F$. By the induction hypothesis, $(p_i(\bar{a}), q_i(\bar{a})) \in H_{3(n-1)-1}(\Sigma)$. Therefore, $(p(\bar{a}), q(\bar{a})) \in H_{3(n-1)}(\Sigma)$; hence $(c, d) \in H_{3(n-1)+2}(\Sigma)$, i.e., $(c, d) \in H_{3n-1}(\Sigma)$.

Sufficiency. Suppose that $A \in \mathcal{K}$, $a,b,c,d \in A$, $(a,b) \in \theta_{\mathcal{K}}(c, d)$ and a formula $\psi(x, y, u, v)$ strictly defines principle $\mathcal{K}$-congruences in F. Also, let m be the maximal height of Σ-congruence schemes occurring in the expression of ψ. We will show that $A \models \varphi(a, b, c, d)$ for some Σ-congruence scheme φ of height $\le m + n - 1$. Let h be a homomorphism of the algebra F onto A such that $x' = h^{-1}a$, $y' = h^{-1}b$, $u' = h^{-1}c$, $v' = h^{-1}d$. By Lemma 3, $(x', y') \in \theta_{\mathcal{K}}(u',v') V_{\mathcal{K}} \ker h$. Since the congruence $\theta_{\mathcal{K}}(u', v')$ is Σ_n-permutable, $\theta_{\mathcal{K}}(u',v') V_{\mathcal{K}} \ker h = H_{n-1}(\Sigma)$, where $H = \theta_{\mathcal{K}}(u', v') \cup \ker h$. Let $(s, t) \in H_0(\Sigma)$, then $A \models \varphi(hs, ht, c, d)$, where φ is a Σ-congruence scheme of height $\le m$. Suppose that for all $k < n - 1$ and $(s, t) \in H_k(\Sigma)$ there exists a Σ-congruence scheme φ of height $\le k + m$ such that $A \models \varphi(hs, ht, c, d)$. By Lemma 2, in $\Sigma \cup E(\sigma)$ there exists a quasiidentity $\underset{i \le l}{\&} p_i(\bar{x}) = q_i(\bar{x}) \to p(\bar{x}) = q(\bar{x})$, such that $(p_i(\bar{a}), q_i(\bar{a})) \in H_{n-2}(\Sigma)$ and $(x', y') = (p(\bar{a}), q(\bar{a}))$ for $\bar{a} \in F$. Hence

$$A \models \exists \bar{z} \Big[\underset{i \le l}{\&} \varphi_i(p_i(\bar{z}), q_i(\bar{z}), c, d) \,\&\, \{a,b\} = \{p(\bar{z}), q(\bar{z})\} \Big],$$

where φ_i are Σ-congruence schemes of height $\leq m + n - 2$. Therefore, $A \models \varphi(a, b, c, d)$, where φ is a Σ-congruence scheme of height $\leq m + n - 2$. Since $|\Sigma| < \omega$, the set of all Σ-congruence schemes of height $\leq m + n - 2$ is finite. It remains to note that the disjunction of these formulas defines principle $\mathcal{R}$-congruences.

The following question is raised in [8] (also see [12]): is it true that if in (the replica class of) a quasivariety $\mathcal{R}$ the principle $\mathcal{R}$-congruences are (strictly) definable, then the variety $\underline{V}(\mathcal{R})$ has definable principle congruences? This question has been solved in the affirmative there under condition that $\underline{V}(\mathcal{R})$ has P_0-projective principle congruences which, according to [10], is equivalent to permutability of the variety $\underline{V}(\mathcal{R})$. Moreover (cf. [12]), the question is solved in the affirmative if the variety $\underline{V}(\mathcal{R})$ is n-permutable or the principle congruences on each algebra in $\mathcal{R}$ are 3-permutable. Theorem 2 unifies and improves these statements.

<u>COROLLARY.</u> In the variety of algebras $\underline{V}(\mathcal{R})$ the principle congruences are definable if an only if in (the replica class of) a quasivariety $\mathcal{R}$ the principal congruences are (strictly) definable amd n-permutable for some $n > 0$.

4. LOCALLY FINITE QUASIVARIETIES OF ALGEBRAS WITH DEFINABLE PRINCIPLE CONGRUENCES

We say that a quasivariety $\mathcal{R}$ <u>possesses the property of n-extension of principle $\mathcal{R}$-congruences</u> if for each algebra $A \in \mathcal{R}$ and any elements a, b, c, d, $\in A$ there exists an n-generated subalgebra $B \leq A$ containing these elements such that

$$(a, b) \in \theta_{\mathcal{R}}^{A}(c, d) \Longleftrightarrow (a, b) \in \theta_{\mathcal{R}}^{B}(c, d).$$

<u>Proposition 1.</u> A locally finite quasivariety of algebras $\mathcal{R}$ has definable $\mathcal{R}$-congruences if and only if for some $n > 0$ the quasivariety $\mathcal{R}$ possesses the property of n-extension of principle $\mathcal{R}$-congruences.

The proof is standard (cf. [11]).

Let $\mathcal{R}$ be a finitely generated quasivariety of algebras and $\mathcal{R} = \underline{SP}(A_1, \ldots, A_n)$. Then each finite algebra $A \in \mathcal{R}$ can be written in the form $A \leq A_1^{k_1} \times \ldots \times A_n^{k_n}$ for some $k_i \in \omega$. The <u>degree</u> of an algebra $A \in \mathcal{R}$ is defined as the number

$$d(A) = \min\{\max\{k_1, \ldots, k_n \mid A \leq A_1^{k_1} \times \ldots \times A_n^{k_n}\}\},$$

where the minimum is taken over all representations of the algebra A. We say that a principle $\mathcal{R}$-congruence $\theta_{\mathcal{R}}^{A}(a, b)$ on an algebra $A \in \mathcal{R}$ is <u>reducible</u> if for each pair $(c, d) \in \theta_{\mathcal{R}}^{A}(a, b)$ there exists a subalgebra $B \leq A$ such that a, b, c, d, $\in B$, $(c, d) \in \theta_{\mathcal{R}}^{B}(a, b)$ and $d(B) < d(A)$.

<u>THEOREM 3.</u> A finitely generated quasivariety of algebras $\mathcal{R} = \underline{SP}(A_1, \ldots, A_n)$ has definable principle $\mathcal{R}$-congruences if and only if there exists a number $N > 0$ such that on each finite algebra in $\mathcal{R}$ whose degree exceeds N each principle $\mathcal{R}$-congruence is reducible.

Necessity. By Proposition 1, the quasivariety $\mathscr{K}$ possesses the property of n-extension of principle $\mathscr{K}$-congruences. Since $\mathscr{K}$ is locally finite, the set of n-generated algebras in $\mathscr{K}$ is finite and all such algebras are finite. It is easily seen that the number equal to the maximum degree of n-generated algebras in $\mathscr{K}$ satisfies the conditions of the theorem.

Sufficiency. Since $\mathscr{K}$ is locally finite, the set of all algebras whose degree is less than N + 1 is finite. Therefore, one can choose a finite set Γ_0 of Σ-congruence schemes, where Σ is a basis of quasiidentities of $\mathscr{K}$ in $\underline{V}(\mathscr{K})$, in such a way that for each algebra A whose degree $\le N$ and arbitrary elements a, b, c, d $\in$ A there exists $\varphi \in \Gamma_0$ such that

$$(a, b) \in \theta_{\mathscr{K}}(c, d) \Longleftrightarrow A \models \varphi(a, b, c, d).$$

We will show that the formula $\gamma(x, y, u, v) \rightleftharpoons \bigvee\{\varphi(x, y, u, v) \mid \varphi \in \Gamma_0\}$ defines principle $\mathscr{K}$-congruences. Let $A \in \mathscr{K}$, a, b, c, d $\in$ A and $(a, b) \in \theta_{\mathscr{K}}(c, d)$. By Lemma 2, we may assume that the algebra A is finite. We perform induction on the degree d(A). If $d(A) \le N$, then the definition of γ implies that $A \models \gamma(a, b, c, d)$. If $d(A) > N$, then by the conditions of the theorem there exists a subalgebra $B \le A$ such that $(a, b) \in \theta^{B}_{\mathscr{K}}(c, d)$ and $d(B) < d(A)$. Next, by the induction hypothesis, $B \models \gamma(a, b, c, d)$; hence $A \models \gamma(a, b, c, d)$.

A quasivariety of algebras $\mathscr{K}$ is said to be directly representable (cf. [5]) if there exists a finite set $\mathscr{M} \subset \mathscr{K}$ of finite algebras such that each finite algebra in $\mathscr{K}$ is isomorphic to a direct product of algebras in $\mathscr{M}$.

It has been proved in [5] that each directly representable variety has definable principle congruences. Theorem 3 allows us to improve this statement.

COROLLARY. Each subquasivariety $\mathscr{K}$ of a directly representable quasivariety has definable principle $\mathscr{K}$-congruences.

Indeed, let $\mathscr{K}$ be a subquasivariety of a directly representable quasivariety $\mathscr{H}$ and $\mathscr{H} = \underline{P}(A_1, \ldots, A_n)$. Note that if $B^m \times C^k \in \mathscr{K}$, k, m > 0, then $B^i \times C^j \in \mathscr{K}$ for all i, j > 0. Let $N = \max\{|A_1|^4, \ldots, |A_n|^4\}$. If $A = A_1^{k_1} \times \ldots \times A_n^{k_n}$ and, without loss of generality, $d(A) = k_1 > N$, then there exist numbers $i \ne j$ such that $e(i) = e(j)$ for all $e \in \{a, b, c, d\}$. Therefore, $(a, b) \in \theta^{B}_{\mathscr{K}}(c, d)$, $B \cong A_1^{k_1 - 1} \times \ldots \times A_n^{k_n}$ and $d(B) < d(A)$. Thus, each principle $\mathscr{K}$-congruence of the algebra A whose degree >N is reducible. Hence, by Theorem 3, $\mathscr{K}$ has definable principle $\mathscr{K}$-congruences.

Note that, unlike varieties, not each subquasivariety of a directly representable quasivariety is directly representable.

Example 1. Let $L_n = \langle\{0, 1/n, \ldots, [(n-1)/n], 1\}; \vee, \wedge, \to, '\rangle$ be the Lukasiewicz algebra (cf. [15]), where $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x' = 1 - x$, $x \to y = \min \times \{1, x' + y\}$. The variety $\underline{V}(L_n)$ is discriminatory; therefore, it is directly representable. Let n = 2k, where k > 1 and k is an odd number. Then $L_2 < L_{2k}$ and $L_k < L_{2k}$. Since the quasiidentity L_k is true in L_k and false in L_2, $\underline{Q}(L_2 \times L_k) \ne \underline{V}(L_2, L_k)$ and $Q(L_2 \times L_K)$ is not directly representable.

An algebra $A \in \mathcal{R}$ is said to be $\mathcal{R}$-simple if the set of $\mathcal{R}$-congruences on A has exactly two elements. If each nontrivial subalgebra of an algebra $A \in \mathcal{R}$ is $\mathcal{R}$-simple, then A is said to be hereditarily $\mathcal{R}$-simple. A quasivariety $\mathcal{R}$ is said to be distributive if for each algebra $A \in \mathcal{R}$ the lattice $\mathrm{Con}_{\mathcal{R}} A$ is distributive.

Proposition 2. Let $\mathcal{R}$ be a locally finite distributive quasivariety of algebras in which each finite subdirectly $\mathcal{R}$-irreducible algebra is hereditarily $\mathcal{R}$-simple. Then $\mathcal{R}$ has definable principle $\mathcal{R}$-congruences.

Proof. By Proposition 1, it suffices to show that for all $A \in \mathcal{R}$ and a, b, c, $d \in A$, if $(c, d) \in \theta^{A}_{\mathcal{R}}(a, b)$, then $(c, d) \in \theta^{B}_{\mathcal{R}}(a, b)$, where B is the subalgebra of A generated by the elements a, b, c, d. To this end, by virtue of Lemma 2 and the local finiteness of $\mathcal{R}$, it suffices to verify that for any finite algebras A, $B \in \mathcal{R}$, $A \leq B$, and each $\mathcal{R}$-congruence θ on A there exists an $\mathcal{R}$-congruence $\overline{\theta}$ on B such that $\theta = \overline{\theta} \cap A^2$. Without loss of generality, we may assume that $A \leq \prod_{i \leq n} B_i$, where B_i are finite, hereditarily $\mathcal{R}$-simple algebras. Then $A \leq_s \prod_{i \leq n} B_i'$, where $B_i' \leq B_i$. Since $\mathcal{R}$ is distributive, a routine verification of Lemma 11.6 [6, p. 177] for quasivarieties shows that $\theta = \theta_1' \times \ldots \times \theta_n' \cap A^2$, where $\theta_i' \in \mathrm{Con}_{\mathcal{R}} B_i'$. But the algebras B_i' and B_i are $\mathcal{R}$-simple; therefore, θ_i' is either zero or unity congruence. It is easily seen that for the role of $\overline{\theta}$ one may take $\overline{\theta}_i \times \ldots \times \overline{\theta}_n$, where $\theta_i' = \overline{\theta}_i \cap B_i'^2$.

Example 2. There exists a locally finite, not finitely generated quasivariety $\mathcal{R} = \underline{Q}(A)$ with definable principle $\mathcal{R}$-congruences.

Let be the set of integers, $\mathbb{Z}_0 = \{\mathbb{Z} \setminus \{0\}; \vee, \wedge, \to, '\}$ be the Sugihara algebra [16], where $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x' = -x$, $x \to y = x' \vee y$ for $x \leq y$ and $x \to y = x' \wedge y$ in the remaining cases. According to [9] and [14], $\mathcal{R}_{FSI} \subseteq \underline{SPu}(\mathbb{Z}_0)$. Since the proposition

$$\forall xy\big[(x \leq y \vee y \leq x) \& (x'' = x) \& (x \leq y \Rightarrow x \to y =$$
$$= x' \vee y) \& (y < x \Rightarrow x = y = x' \wedge y) \& (x' = x \Rightarrow x = y) \&$$
$$\& (x \leq y \to y' \leq x')\big],$$

is true in $\mathbb{Z}_0$, each algebra in $\underline{SPu}(\mathbb{Z}_0)$ is finitely subdirectly irreducible and $\mathcal{R}$-simple. Since $\underline{V}(\mathbb{Z}_0)$ is a locally finite distributive variety, we deduce by [9] that $\mathcal{R}$ is a distributive quasivariety. Therefore, by Proposition 2, $\mathcal{R}$ has definable principle $\mathcal{R}$-congruences. Obviously, $\underline{V}(\mathcal{R}) \neq \mathcal{R}$.

Note also that by Lemma 6 $\mathcal{R}$ is a finitely based quasivariety and the interval $[\mathcal{R}, \underline{V}(\mathcal{R})]$ is infinite and contains an infinite set of quasivarieties $\mathcal{R}'$ with definable principle $\mathcal{R}'$-congruences.

Example 3. There exists a quasivariety $\mathcal{R}$ such that $\underline{V}(\mathcal{R})$ has definable principle congruences while $\mathcal{R}$ does not.

Indeed, let A = ⟨{0, 1, 2}, f, g⟩ be a unary algebra, where f(1) = 0, g(1) = 2, and f(x) = g(x) = x for x ≠ 1, and let $\mathcal{K}$ = SP(A). It has been proved in [4] that $\mathcal{K}$ has no finite basis of quasiidentities in V($\mathcal{K}$). Therefore, by Lemma 6, principle $\mathcal{K}$-congruences in $\mathcal{K}$ are not definable. But V($\mathcal{K}$) possesses the property of 4-extension of principle congruences; therefore, by Proposition 1, V($\mathcal{K}$) has definable principle congruences.

Final Remarks. The results of this article were reported by the author at the "Algebra and Logic" seminar in Novosibirsk in April of 1988. Gorbunov informed the author that a theorem similar to Theorem 1 had been announced by Dziobiak and Chelakowski at the International Symposium on Universal Algebra in Turawa (Poland) in May of 1988. The author expresses his gratitude to Gorbunov for his attention and support during the completion of this work.

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