QUASIIDENTITIES OF ABSOLUTELY FREE ALGEBRAS

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An algebra is said to be locally absolutely free if every finitely generated subalgebra of it is absolutely free. A. I. Mal'tsev [1] proved that the class of all locally absolutely free algebras with an adjoined unit algebra forms a quasivariety that does not have a finite basis of quasiidentities. It is not difficult to see that this quasivariety is generated by any absolutely free algebra, and so it is minimal.

In this paper we prove that the quasivariety generated by an absolutely free algebra of finite signature has a recursive independent basis of quasiidentities. This basis can be written out explicitly modulo a certain equivalence relation on the set of terms. The exact statement of the theorem in the case of groupoids is given in Sec. 1, and in the general case in Sec. 4.

The terminology corresponds to that of [2, 3].

1. Quasiidentities of an Absolutely Free Groupoid: Definitions and Statement of the Theorem

We fix a countable set of variables $X=\{x, x_0, \ldots\}$ and denote by y, z, possibly with indices, the elements of this set. In the usual way we define terms in the variables from the set X (see [2]). The length $l(u)$ of the term u is the number of the symbols of operations that occur in writing the term u.

By induction we define the set R of regular terms, as follows:

1) $x \in R$,

2) if $u \in R$, $l(u)=k$, then $x_{k+1} \cdot u,\ u \cdot x_{k+1} \in R$.

Obviously, the subterm of given nonzero length of a regular term is defined uniquely and is itself a regular term.

For any terms u, v we denote by uS_v^0 and uS_v^1, respectively, the terms $v \cdot u$ and $u \cdot v$, that is, on the set of all terms, S_v^0 is the operator of left multiplication, and S_v^1 is the operator of right multiplication. We observe that the operators S_v^i generate a free subgroup with respect to composition. If we fix an arbitrary occurrence of a proper subterm v in the term u, then u can be represented uniquely in the form:

$$vS_{v_0}^{i_0} S_{v_1}^{i_1} \ldots S_{v_{k-1}}^{i_{k-1}},\ k>0,$$

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for some terms $v_0, v_1, \ldots, v_{k-1}$ and numbers $i_0, i_1, \ldots, i_{k-1} \in \{0,1\}$. In particular, any regular term u of nonzero length can be represented uniquely in the form

$$x S^{i_0}_{x_0} S^{i_1}_{x_1} \ldots S^{i_{k-1}}_{x_{k-1}}.$$

Henceforth, if we do not say otherwise, we assume that $k>0$, $n>0$, $x_{-1}=x$; i, j, possibly with indices, are elements of the set $\{0,1\}$, and $u S^{i-1}_{v} = u$. Suppose that $u = x S^{i_0}_{x_0} S^{i_1}_{x_1} \ldots S^{i_{k-1}}_{x_{k-1}} \in R$. We put $u_0 = u$, $u_1 = x S^{i_1}_{x_0} \ldots S^{i_{k-1}}_{x_{k-2}} S^{i_0}_{x_{k-1}}$, $u_n = (u_{n-1})_1$.

On the set R of regular terms we define the relation $\sim$, putting $u \sim v$ if $v = u_m$ for a suitable m. Obviously, $\sim$ is an equivalence relation on R.

On every equivalence class with respect to $\sim$ we define an order $\leqslant$. Suppose that $u \sim v$ and

$$u = x S^{i_0}_{x_0} S^{i_1}_{x_1} \ldots S^{i_{k-1}}_{x_{k-1}}, \quad v = x S^{j_0}_{x_0} S^{j_1}_{x_1} \ldots S^{j_{k-1}}_{x_{k-1}}.$$

We put $u \leqslant v$ if $(i_0, i_1, \ldots, i_{k-1}) \leqslant (j_0, j_1, \ldots, j_{k-1})$ with respect to lexicographic order. From the definitions it follows that the equivalence classes with respect to $\sim$ are finite and the relation $\leqslant$ is a linear order on every class. Consequently, in every equivalence class there is a unique minimal element, which we shall call the minimal term.

We put $r_0 = 1$, $r_n = (2 \cdot 3 \cdot \ldots \cdot p_n)^n$, where p_n is the n-th prime number, and distinguish the subset R^* of R by putting $u \in R^*$ if and only if u is the minimal term and $l(u) = r_n$ for some n. From the definitions it follows immediately that R^* is a recursive set of terms.

We can now state the main result in the case of groupoids.

THEOREM 1. The quasivariety generated by an absolutely free groupoid of countable rank has the following recursive independent basis of quasiidentities (quantifiers are omitted):

$$K_0 \leftrightharpoons x \cdot x_0 = y \cdot y_0 \to x = y;$$

$$K_0' \leftrightharpoons x_0 \cdot x = y_0 \cdot y \to x = y;$$

$$K_1 \leftrightharpoons x \cdot x_0 = x \to y = z;$$

$$K_1' \leftrightharpoons x_0 \cdot x = x \to y = z;$$

$$K_u \leftrightharpoons u(x, x_0, \ldots, x_{r_n - 1}) = x \to u'(x, x_0, \ldots, x_{r_{n-1} - 1}) = x,$$

where $u \in R^*$, $l(u) = r_n$, and u' is the subterm of u of length r_{n-1}.

2. Auxiliary Assertions

Let G be an arbitrary groupoid.

LEMMA 1. If $v = v(x, x_0, \dots, x_{m-1})$ is a term of nonzero length and the equation $v = x$ is solvable in $\mathfrak{G}$, then there is a regular term u of nonzero length such that the equation $u = x$ is solvable in $\mathfrak{G}$.

Proof. We write the term v in the form

$$xS^{i_0}_{v_0} S^{i_1}_{v_1} \dots S^{i_{k-1}}_{v_{k-1}}.$$

Then for u we can take the term

$$x = S^{i_0}_{x_0} S^{i_1}_{x_1} \dots S^{i_{k-1}}_{x_{k-1}}.$$

LEMMA 2. If $v = v(x, x_0, \dots, x_{r_n-1})$ is a regular term and the equation $v = x$ is solvable in $\mathfrak{G}$, then there is a term $u = u(x, x_0, \dots, x_{r_n-1}) \in R^*$ such that the equation $u = x$ is solvable in $\mathfrak{G}$.

Proof. Let $m = r_n$ and

$$v = xS^{i_0}_{x_0} S^{i_1}_{x_1} \dots S^{i_{m-1}}_{x_{m-1}}.$$

Suppose that

$$a = aS^{i_0}_{a_0} S^{i_1}_{a_1} \dots S^{i_{m-1}}_{a_{m-1}}$$

for some elements $a, a_0, \dots, a_{m-1} \in \mathfrak{G}$. Then for any $k \leq m$ we have

$$aS^{i_0}_{a_0} S^{i_1}_{a_1} \dots S^{i_{k-1}}_{a_{k-1}} = aS^{i_0}_{a_0} S^{i_1}_{a_1} \dots S^{i_{k-1}}_{a_{k-1}} \dots S^{i_{m-1}}_{a_{m-1}} S^{i_0}_{a_0} \dots S^{i_{k-1}}_{a_{k-1}}.$$

Consequently, the equation $v_k = x$ is solvable in $\mathfrak{G}$. The assertion of the lemma now follows from the definition of the set R^*.

For any term u, where

$$u = xS^{i_0}_{x_0} S^{i_1}_{x_1} \dots S^{i_{k-1}}_{x_{k-1}},$$

we put $u^0 = x$, $u^1 = u$ and

$$u^{n+1} = u^n S^{i_0}_{x_{nk}} S^{i_1}_{x_{nk+1}} \dots S^{i_{k-1}}_{x_{(n+1)k-1}}.$$

Obviously, $u^n \in R$ for all n.

LEMMA 3. If the equation $v = x$ is solvable in $\mathfrak{G}$, where $v = v(x, x_0, \dots, x_{m-1})$ is a term of nonzero length, then there is a term $w \in R^*$ such that the equation $w = x$ is solvable in $\mathfrak{G}$.

Proof. Suppose that the equation $v = x$ is solvable in $\mathfrak{G}$. Then, by Lemma 1, there is a regular term w of length k such that the equation $w = x$ is solvable in $\mathfrak{G}$. Obviously, the equation $w^n = x$ is then solvable for any n. We choose a number s such that $k | r_s$ and put $n = r_s/k$. Then $\ell(w^n) = r_s$ and by Lemma 2 there is a term u such that $u \in R^*$ and the equation $u = x$ is solvable in G. This proves the lemma.

A function $T(x)$ is called a translation if there is a regular term $u(x, x_0, \ldots, x_{m-1})$ (possibly of zero length) such that $T(x) = u(x, a_0, a_1, \ldots, a_{m-1})$ for some $a_0, a_1, \ldots, a_{m-1} \in G$.

LEMMA 4 (A. I. Malt'tsev [3]). Let $\theta = \mathrm{con}(a, b)$ be a principal congruence on a groupoid G, $\Gamma^0 = \{(d,d) : d \in G\}$, $\Gamma^1 = \Gamma = \{(T(c), T(d)) : \{c, d\} = \{a, b\},\ T(x)$ is a translation$\}$, $\Gamma^{n+1} = \Gamma^n \circ \Gamma$. Then $\theta = \bigcup\{\Gamma^m : m = 0, 1, 2, \ldots\}$.

From now on, F is an absolutely free groupoid of countable rank with free generators $a = a_{-1}, a_0, a_1, \ldots$; $w(c)$ is the set of all proper subwords of the word c.

COROLLARY 1. Let $a \in w(b)$, $\theta = \mathrm{con}(a, b)$. Then

1) if $a \notin w(c) \cup \{c\}$, then $c\theta = \{c\}$;

2) if $(c, d) \in \theta$, then $\ell(c) = \ell(d) \bmod \ell(b)$.

Proof. Part 1) follows immediately from Lemma 4. It is sufficient to prove part 2) for $(c, d) \in \Gamma$. In this case, by Lemma 4, there is a translation $T(x)$ such that $\{c, d\} = \{T(a), T(b)\}$. Hence $\ell(T(b)) = \ell(T(a)) + \ell(b)$, as required.

LEMMA 5 (A. I. Mal'tsev [1]). A groupoid G is locally absolutely free if and only if the following quasiidentities are true in it:

$$K_0 \rightleftharpoons x \cdot x_0 = y \cdot y_0 \to x = y;$$

$$K_0' \rightleftharpoons x_0 x = y_0 y \to x = y;$$

$$K_f \rightleftharpoons f(x, x_0, \ldots, x_{m-1}) = x \to y = z,$$

where f is a term of nonzero length.

In particular, the class of locally absolutely free groupoids is a quasivariety.

LEMMA 6. Let $a \in w(b)$, $\theta = \mathrm{con}(a, b) \in \mathrm{Con}\, F$. If $(hg, pt) \in \theta$, then $(h, p), (g, t) \in \theta$, that is, $F/\theta \models K_0 \,\&\, K_0'$.

Proof. Suppose that $(hg, pt) \in \theta$; then by Lemma 4, there is a number s such that $(hg, pt) \in \Gamma^s$. We proceed by induction on s.

If $s = 0$, then $hg = pt$ and, by Lemma 5, $h = p$, $g = t$.

Suppose that $s = 1$; then $(hg, pt) \in \Gamma$. Hence there is a translation $T(x)$ such that $hg = T(c)$, $pt = T(d)$, where $\{c, d\} = \{a, b\}$. Obviously, $T(x)$ is not the identity, so it can be represented in the form $T_1(x)T_2$ or $T_2T_1(x)$, where $T_1(x)$ is a translation, and $T_2 \in F$. Suppose that $T(x) = T_1(x)T_2$. Then $hg = T_1(c)T_2$, $pt = T_1 d\, T_2$ and, by Lemma 5, $h = T_1(c)$, $g = T_2$, $p = T_1(d)$ and $t = T_2$, that is, $(h, p) \in \theta$, $(g, t) \in \theta$. The case when $T(x) = T_2 T_1(x)$ is considered similarly.

Suppose that $s > 0$ and $(hg, pt) \in \Gamma^{s+1}$. Hence, either there are elements $u, v \in F$ such that $hg \Gamma^s uv \Gamma pt$, or there is an element $c \in \{a, a_0, \ldots\}$ such that $hg \Gamma^s c \Gamma pt$. In the first case, by the inductive hypothesis, $(h,u),(g,v),(u,p),(v,t)\in\theta$. Hence, $(h,p)\in\theta$ and $(g,t)\in\theta$. In the second case, by Corollary 1, $c=a$, that is, $hg\Gamma^{s}\bar{a}\Gamma pt$. But, by Lemma 4, $a\Gamma u$ is possible only if $u=b$. Consequently, $hg\Gamma^{s-1}b\Gamma a\Gamma b=pt$, that is, $hg\Gamma^{s-1}pt$, and by the inductive hypothesis $(h,p),(g,t)\in\theta$. This proves the lemma.

Suppose that $\theta\in\mathrm{Con}\,F$. We say that an element $c\in F$ is <u>θ-incontractible</u> if for any d of the class $c\theta$ we have $\ell(c)\leq\ell(d)$.

<u>LEMMA 7.</u> Let $\theta=\mathrm{con}(a,aS_{a_0}^{i_0}S_{a_1}^{i_1}\ldots S_{a_{k-1}}^{i_{k-1}})$. If c is θ-incontractible, $c\in\omega(d)$ and $(c,d)\in\theta$, then there are numbers s and n such that $s<k$ and

$$c=aS_{a_0}^{i_0}S_{a_1}^{i_1}\ldots S_{a_{s-1}}^{i_{s-1}},$$
$$d=c(S_{a_s}^{i_s}S_{a_{s+1}}^{i_{s+1}}\ldots S_{a_{k-1}}^{i_{k-1}}S_{a_0}^{i_0}\ldots S_{a_{s-1}}^{i_{s-1}})^n.$$

<u>Proof.</u> Suppose that $c\in\omega(d)$. Then d can be represented in the form

$$cS_{c_0}^{j_0}S_{c_1}^{j_1}\ldots S_{c_{\ell-1}}^{j_{\ell-1}},\ \ell>0,$$

for some $c_0,c_1,\ldots,c_{\ell-1}\in F$; $j_0,j_1,\ldots,j_{\ell-1}$. We carry out the proof of the lemma by induction on the length of the word c.

Suppose that $\ell(c)=0$. Then, by Corollary 1, we obtain $c=a$. Thus,

$$c=a\theta aS_{c_0}^{j_0}S_{c_1}^{j_1}\ldots S_{c_{\ell-1}}^{j_{\ell-1}}.$$

But

$$a\theta aS_{a_0}^{i_0}S_{a_1}^{i_1}\ldots S_{a_{k-1}}^{i_{k-1}}.$$

Consequently,

$$aS_{a_0}^{i_0}S_{a_1}^{i_1}\ldots S_{a_{k-1}}^{i_{k-1}}\theta aS_{c_0}^{j_0}S_{c_1}^{j_1}\ldots S_{c_{\ell-1}}^{j_{\ell-1}}.$$

By Lemma 6 and Corollary 1 we obtain $a_p=c_r, i_p=i_r$ for $p\equiv r\bmod k$ that is,

$$d=a(S_{a_0}^{i_0}S_{a_1}^{i_1}\ldots S_{a_{k-1}}^{i_{k-1}})^n,$$

where $n=\ell/k$.

Suppose that $\ell(c)>0$. Then $c=c_1'c_2'$, $d=d_1d_2$ and, by Lemma 6, $(c_1',d_1),(c_2',d_2)\in\theta$. Since $c\in\omega(d)$, we have $c\in\omega(d_1)\cup\{d_1\}$ or $c\in\omega(d_2)\cup\{d_2\}$. Suppose that $c\in\omega(d_1)\cup\{d_1\}$. Then $c_1'\in\omega(d_1)$ and, by the inductive hypothesis, there are numbers s and n such that $s<k$ and

$$c_1'=aS_{a_0}^{i_0}S_{a_1}^{i_1}\ldots S_{a_{s-1}}^{i_{s-1}},$$

$$d_1 = c_1'(S_{a_s}^{i_s} S_{a_{s+1}}^{i_{s+1}} \dots S_{a_{k-1}}^{i_{k-1}} S_{a_0}^{i_0} \dots S_{a_{s-1}}^{i_{s-1}})^n.$$

Since c is a subword of the word d_1 (possibly $c = d_1$) and $c_1 = c_1' c_2'$ we have $c_2' = a_s$ and $c = c_1' S_{a_s}^{i_s}$. By Corollary 1, we obtain $d_2 = a_s$ and $d = d_1 S_{a_s}^{i_s}$. Consequently,

$$c_1 = c_1' S_{a_s}^{i_s} = a S_{a_0}^{i_0} S_{a_1}^{i_1} \dots S_{a_{s-1}}^{i_{s-1}} S_{a_s}^{i_s},$$

$$d = c_1'(S_{a_s}^{i_s} \dots S_{a_{k-1}}^{i_{k-1}} S_{a_0}^{i_0} \dots S_{a_{s-1}}^{i_{s-1}})^n S_{a_s}^{i_s} = c(S_{a_{s+1}}^{i_{s+1}} \dots S_{a_{k-1}}^{i_{k-1}} S_{a_0}^{i_0} \dots S_{a_s}^{i_s})^n.$$

The case $c \in \omega(d_2) \cup \{d_2\}$ is considered similarly.

<u>LEMMA 8.</u> The quasivariety of locally absolutely free groupoids is generated by any locally absolutely free groupoid of it, and so it is minimal.

The proof is obvious, since by [4] an absolutely free groupoid of countable rank can be embedded in an absolutely free groupoid of rank 1.

3. Proof of Theorem 1

Let

$$\Sigma = \{K_0, K_0', K_1, K_1'\} \cup \{K_u : u \in R^*\}.$$

By Lemma 8 the quasivariety of locally absolutely free groupoids is generated by an absolutely free groupoid of countable rank and, by Lemma 5, all the quasiidentities of the set Σ are true on it. The fact that Σ is recursive follows from the definition of the set R^*. We show that Σ is a basis of the quasiidentities of the given quasivariety, that is, if all the quasiidentities of G are true on Σ, then G is a locally absolutely free groupoid. Let G' be the subgroupoid of G generated by the elements $d = d_{-1}, d_0, \dots, d_{\ell-1}$. From the set of generators we reject those elements that can be expressed termwise in terms of others. Let $d, d_0, \dots, d_{m-1}$ be the remaining reduced system of generators of G'. We need to prove that the equality $f(d, d_0, \dots, d_{m-1}) = g(d, d_0, \dots, d_{m-1})$ where $f(x, x_0, \dots, x_{m-1})$, $g(x, x_0, \dots, x_{m-1})$ are terms, possibly with fictitious variables, is true if and only if f and g coincide graphically. We shall carry out induction on the minimal length of the words f and g.

Suppose that $\ell(g) = 0$, that is, $g = d_{s-1}$ for some $s \leq m$. Since d_{s-1} cannot be expressed termwise in terms of the other generators, d_{s-1} does not occur fictitiously in the word f. If $\ell(f) = 0$, then f is d_{s-1}. We therefore suppose that $\ell(f) > 0$. Then, by Lemma 3, there is a term $u \in R^*$ of length r_n such that the equation $u = x$ is solvable in G'. But the quasiidentity G' is true in K_u, so the equation $u' = x$ is solvable in it, where u' is a subterm of u of length r_{n-1}. Also, by Lemma 2, there is a term v of length r_{n-1} such that $v \in R^*$ and the equation $v = x$ is solvable in G'. But the quasiidentity G' is true on K_v. Hence the equation $v' = x$ is solvable in G', where $v' \in R$ and $\ell(v') = r_{n-2}$. Thus, applying Lemma 2 successively and taking account of the truth of the quasiidentities

K_ω, $\omega \in R^*$, we find that one of the equations $xx_0 = x$ or $x_0 x = x$ is solvable in G'. Now, taking the quasiidentities K_1 and K_1' into account, we see that G' is trivial. Consequently, f and g coincide graphically.

Suppose that $\min(\ell(f), \ell(g)) > 0$. Then, since the quasiidentities K_0 and K_0' are true, the relation $f = g$ splits into relations with smaller lengths.

To complete the proof of Theorem 1 we need to show that Σ is an independent basis of quasiidentities. We recall that a system of quasiidentities Σ is said to be independent if for any quasiidentity $K \in \Sigma$ there is a groupoid G such that all the quasiidentities of the set $\Sigma \setminus \{K\}$ are true in G, while K is false.

We carry out the proof as follows: with each quasiidentity K of Σ we associate a congruence θ on an absolutely free groupoid F of countable rank and show that all the quasiidentities of $\Sigma \setminus \{K\}$ are true on F/θ, while K is false.

1. Let $K = K_0$, $\theta = \mathrm{con}(aa, a_0 a)$. Obviously, F/θ is false on K_0. We show that the quasiidentities a) K_0' and b) K_1, K_1', K_u are true on F/θ for any $u \in R^*$.

a) Suppose that $(cc_0, dd_0) \in \theta$. Then, by Lemma 4, there is a number m such that $cc_0 \Gamma^m dd_0$. We proceed by induction on m.

If $m = 0$, then $cc_0 = dd_0$ and, by Lemma 5, $c_0 = d_0$.

If $m = 1$ there is a translation $T(x)$ such that $\{cc_0, dd_0\} = \{T(aa), T(a_0 a)\}$. If $T(x)$ is the identity, then $\{cc_0, dd_0\} = \{aa, a_0 a\}$ and, by Lemma 5, $c_0 = d_0$. Suppose that $T(x)$ is not the identity; then $T(x)$ can be represented in the form $T_1(x)T_2$ or $T_2 T_1(x)$, where $T_1(x)$ is a translation, and $T_2 \in F$. In the first case we obtain $c_0 = d_0 = T_2$ and in the second case $\{c_0, d_0\} = \{T_1(aa), T_1(a_0 a)\}$, that is, $c_0 \theta d_0$.

Suppose that $cc_0 \Gamma^{m+1} dd_0$, $m > 0$. This means that there are elements b, $b_0 \in F$ such that $cc_0 \Gamma^m bb_0 \Gamma dd_0$. Then, by the inductive hypothesis, $(c_0, b_0), (b_0, d_0) \in \theta$. Consequently, $(c_0, d_0) \in \theta$.

b) Since c and $cc_0(c_0 c, u(c, c_0, \ldots))$ have different lengths, the assertion will be proved if we show that elements comparable with respect to θ have the same lengths. Obviously, it is sufficient to consider the case $(c, d) \in \Gamma$. If this is so, then $\{c, d\} = \{T(aa), T(a_0 a)\}$ for a suitable translation $T(x)$. From this it is clear that $\ell(c) = \ell(d)$.

2) The case $K = K_0'$, $\theta = \mathrm{con}(aa, aa_0)$ is considered similarly.

3) Let $K = K_1$, $\theta = \mathrm{con}(a, aa_0)$. Obviously, the quasiidentity K_1 is false on F/θ and, by Lemma 6, K_0 and K_0' are true. We show that the quasiidentities K_1', K_u are also true for any $u \in R^*$.

Suppose that $(c, c_0 c) \in \theta$. Clearly, the element c can be assumed to be θ-incontractible. Then, by Lemma 7, we obtain $c = a$, $c_0 c = a(S^1_{a_0})^m$ for some $m > 0$. Hence, by Lemma 5, $a = a_0$. This is impossible. Hence, $(c, c_0 c) \notin \theta$.

Suppose that $c\theta u(c,c_0,\ldots,c_{r_n-1})$, $u\in R^*$, where the element c is θ-incontractible. Then, by Lemma 7, we obtain $c=a$, $u(c,c_0,\ldots,c_{r_n-1})=a(S^1_{a_0})^{r_n}$. Hence, $u'(c,c_0,\ldots,c_{r_{n-1}-1})=a(S^1_{a_0})^{r_{n-1}}\theta a$. Thus, the identities K_1', K_u are true on F/θ for all $u\in R^*$.

4) The case $K=K_1'$, $\theta=\mathrm{con}(a,a_0a)$ is considered similarly.

5) Let $K=K_{u'}$, $\theta=\mathrm{con}(a,u(a,a_0,\ldots,a_{r_n-1}))$. By Corollary 1, any word not equal to a and of length less than r_n is incomparable with θ with respect respect to a. Consequently, the quasiidentity K_u is false on F/θ. The truth of the quasiidentities K_0 and K_0' follows from Lemma 6. We put

$$u(a,a_0,\ldots,a_{r_n-1})=aS^{i_0}_{a_0}S^{i_1}_{a_1}\ldots S^{i_{r_n-1}}_{a_{r_n-1}}.$$

Suppose that c is a θ-incontractible element of $c\theta cc_0$. Then, by Lemma 7, there are numbers $s>0$ and $m\geq 0$ such that

$$c=aS^{i_0}_{a_0}S^{i_1}_{a_1}\ldots S^{i_{m-1}}_{a_{m-1}},$$

$$cc_0=c(S^{i_m}_{a_m}\ldots S^{i_{r_n-1}}_{a_{r_n-1}}S^{i_0}_{a_0}\ldots S^{i_{m-1}}_{a_{m-1}})^s.$$

By Lemma 5 and Corollary 1, we obtain $c=a_{m-1}$ for $m\neq 0$, $c=a_{r_n-1}$ for $m=0$ and $l(cc_0)=l(c)+1\equiv l(c)(\mathrm{mod}\, r_n)$. But the last equality is impossible. Hence, $(c,cc_0)\in\theta$. Similarly we obtain $(c,c_0c)\notin\theta$. Consequently, the quasiidentities K_1 and K_1' are true on F/θ. To complete the proof of Theorem 1 it remains to show that the quasiidentities K_v with $v\neq u$ are true. Suppose that for some elements $c_0,c_1,\ldots,c_{r_k-1}$, $c\in F$, and term $v(x,x_0,\ldots,x_{r_n-1})\in R^*$ we have $(c,v(c,c_0,\ldots,c_{r_k-1}))\in\theta$. We may assume that c is θ-incontractible. Then, by Lemma 7, there are numbers $s>0$ and $m\geq 0$ such that

$$c=aS^{i_0}_{a_0}S^{i_1}_{a_1}\ldots S^{i_{m-1}}_{a_{m-1}},$$

$$v(c,c_0,\ldots,c_{r_n-1})=c(S^{i_m}_{a_m}\ldots S^{i_{r_n-1}}_{a_{r_n-1}}S^{i_0}_{a_0}\ldots S^{i_{m-1}}_{a_{m-1}})^s.$$

By Lemma 5 and Corollary 1 we obtain $l(c_q)=0, 0\leq q\leq r_n-1$. Since $v\in R^*$ we have $l(v(c,c_0,\ldots,c_{r_k-1}))=l(c)+r_k$. At the same time, by Corollary 1, we have $l(v(c,c_0,\ldots,c_{r_k-1}))\equiv l(c)\,\mathrm{mod}\, r_n$. Consequently, $k\geq n$.

Suppose that $k=n$. Then $s=1$ and $v=u_m$. But $u,v\in R^*$ so $v=u$. This contradicts our choice of the term v.

Suppose that $k>n$. Then $s=r_k/r_n$ and the subword $v'(c,c_0,\ldots,c_{r_{k-1}-1})$ of the word $v(c,c_0,\ldots,c_{r_k-1})$ has the form

$$c(S^{i_m}_{a_m}\ldots S^{i_{r_n-1}}_{a_{r_n-1}}S^{i_0}_{a_0}\ldots S^{i_{m-1}}_{a_{m-1}})^{s_1},$$

where $s_1=r_{k-1}/r_n$. Obviously, $c\theta v'(c,c_0,\ldots,c_{r_{k-1}-1})$. Consequently, quasiidentities of the type K_v with $v\neq u$ are true on F/θ.

We have thus proved Theorem 1.

4. The General Case

Let $\sigma = \{\varphi_0, \varphi_1, \dots, \varphi_m\}$ be a finite functional signature, and suppose that φ_i, $0 \le i \le m$ are (n_i+1)-ary operations.

We define the set R of regular terms of the signature σ by induction as follows:

1) $x \in R$,

2) if the term $u(x, x_0, \dots, x_{\ell-1}) \in R$ then

$$uS_i^j \rightleftharpoons \varphi_i(y_0, y_1, \dots, y_{n_i}) \in R, \quad 0 \le i \le m, \quad 0 \le j \le n_i,$$

where $y_j = u$, $y_s = x_{\ell+s}$ for $s < j$, and $y_s = x_{\ell+s-1}$ for $s > j$. From the definition of R it follows that any regular term u of nonzero length can be represented in the form

$$xS_{i_0}^{j_0} S_{i_1}^{j_1} \dots S_{i_{k-1}}^{j_{k-1}}, \quad 0 \le i_s \le m, \quad 0 \le j_s \le n_{i_s}, \quad s < k,$$

for some $k > 0$. For any $u \in R$ we put $u_1 = xS_{i_1}^{j_1} S_{i_2}^{j_2} \dots S_{i_{k-1}}^{j_{k-1}} S_{i_0}^{j_0}$ where $u = xS_{i_0}^{j_0} S_{i_1}^{j_1} \dots S_{i_{k-1}}^{j_{k-1}}$, and $u_0 = u$, $u_n = (u_{n-1})_1$ for $n > 0$. We assume that $u \sim v$ if there is a number s such that $v = u_s$. Obviously, $\sim$ is an equivalence relation on R. On each equivalence class with respect to $\sim$ we define an order $\le$. Let

$$u = xS_{i_0}^{j_0} S_{i_1}^{j_1} \dots S_{i_{k-1}}^{j_{k-1}}, \qquad v = xS_{i'_0}^{j'_0} S_{i'_1}^{j'_1} \dots S_{i'_{k-1}}^{j'_{k-1}}, \quad u \sim v.$$

We put $u \le v$ if $(i_0, j_0, i_1, j_1, \dots, i_{k-1}, j_{k-1}) \le (i'_0, j'_0, i'_1, j'_1, \dots, i'_{k-1}, j'_{k-1})$ with respect to lexicographic order. From the definitions it follows that in each equivalence class there is a unique minimal element, which we shall call the <u>minimal term.</u> We recall that the length $\ell(u)$ of a term u is the number of symbols of operations that occur in writing the term u. From the set R of all regular terms we pick out a subset R^*, putting $u \in R^*$ if and only if u is the minimal term and $\ell(u) = r_n$ for some n. We now state the main result.

<u>THEOREM.</u> The quasivariety generated by an absolutely free algebra of countable rank and signature σ has minimally the following recursive independent basis of quasiidentities:

$$K_0^{ij} \rightleftharpoons \varphi_i(x_0, x_1, \dots, x_{n_i}) = \varphi_i(y_0, y_1, \dots, y_{n_i}) \to x_j = y_j; \; 0 \le j \le n_i;$$

$$K_1^{ij} \rightleftharpoons \varphi_i(x_0, x_1, \dots, x_{n_i}) = x_j \to x = y; \quad 0 \le j \le n_i;$$

$$K_2^{ij} \rightleftharpoons \varphi_i(x_0, x_1, \dots, x_{n_i}) = \varphi_j(y_0, y_1, \dots, y_{n_j}) \to x = y; \; 0 \le j \le m;$$

$$K_u \rightleftharpoons u(x, x_0, \dots) = x \to u'(x, x_0, \dots) = x,$$

where $u \in R^*$, $\ell(u) = r_n$ for some n, and u' is a subterm of the term u of length r_{n-1}.

The proof is completely analogous to that of Theorem 1, except that we need to introduce the following obvious additions.

1. With the quasiidentity K_2^{ij} there is associated the congruence $\theta = \mathrm{con}(\varphi_i(a_0, a_1, \ldots, a_{n_i}), \varphi_j(a_0, a_1, \ldots, a_{n_j})) \in \mathrm{Con}\, F$, where F is an absolutely free algebra of countable rank and signature σ with generators $a, a_0, a_1, \ldots$

2. If θ is the congruence associated with the quasiidentity K_0^{ij} or K_1^{ij} or K_u, then we need to verify additionally that the quasiidentities K_2^{ij} are true on F/θ.

We note that this result for unars was obtained by Kartashov in [5].

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ONE SUPERINTUITIONISTIC CALCULUS OF PROPOSITIONS

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The construction of an undecidable superintuitionistic calculus of propositions in Popov's article in [1] is based on inference modeling in a system of Post productions. To this end, one assigns to a production system Π with an undecidable problem of inference from an initial word V_0 (in an alphabet σ) a superintuitionistic calculus $\tilde{\Sigma} = \tilde{\Sigma}(\Pi, V_0)$, and to an arbitrary word V in the alphabet σ and a set of numbers $\theta = (i_1, \ldots, i_\ell)$ of length $\ell \geq 2$ (where $i_1, \ldots, i_\ell \in \{1, \ldots, m\}$ and m is the number of productions in the system Π) some formula $U^*(\Pi, V_0, V, \theta)$.

The main theorem [1, p. 698, Theorem 6.1] says:

$$(\tilde{\Sigma}(\Pi, V_0) \vdash U^*(\Pi, V_0, V, \theta)) \Leftrightarrow$$

$$\Leftrightarrow \left(\left(\begin{array}{c}\text{the word } V \text{ can be inferred in the production system } \Pi \text{ from the word } V_0\\ \text{with a concluding sequence } (\Pi)_{i_\ell}, \ldots, (\Pi)_{i_1} \text{ of}\\ \text{applying productions in the inference}\end{array}\right)\right\} \quad (*)$$

(more exactly, this is asserted not for all V but only for words V in some decidable set $\mathfrak{M}_\theta^\sigma$ (defined in [1], p. 683) containing the set of words satisfying (*) (see [1, p. 671,

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