CHARACTERIZATION OF ω-STABLE THEORIES OF BOUNDED DIMENSION

T. A. Nurmagambetov UDC 510.67

INTRODUCTION

If two models are elementarily imbedded in each other, then we call them mutually elementarily imbeddable. It is clear that models can be mutually elementarily imbeddable, but nonisomorphic. The concept of the theory of bounded dimension, introduced by Shelah [1], plays an important role in solving the problem of the number of nonisomorphic models of the theories. The basic result of this paper is the following

THEOREM 1. Let T be an arbitrary ω-stable theory. Then the following two conditions are equivalent:

1) the theory T is a theory of bounded dimension;

2) if two models of T are mutually elementarily imbeddable, then they are isomorphic.

On the basis of a result from [2] and the proof of Theorem 1 in Sec. 2 is proved

THEOREM 2. Let T be an arbitrary ω-stable theory. Then the following two conditions are equivalent:

1) for any model $\mathcal{M}$ of T, $\bar{a} \in \mathcal{M}$ if $p \in S(\bar{a})$ is a multidimensional strongly regular type, then $dim(p, \mathcal{M}) \geq \omega$;

2) if two countable models of T are mutually elementarily imbeddable, then they are isomorphic.

Let us remark that condition 1 of Theorem 2 is precisely property (*) from [2].

In Sec. 1 we will give the necessary preliminary information, and in Sec. 2 the part $1 \Rightarrow 2$ of Theorem 1 is proved; in Sec. 3 we prove the following

Proposition 3. If a ω-stable theory has a multidimensional strongly regular type, then in each cardinality $\lambda > \omega$ there exist at least two nonisomorphic mutually elementarily imbeddable models.

It is clear that from Proposition 3 immediately follows the validity of the part $\neg 1 \Rightarrow \neg 2$ of Theorem 1.

From Theorem 1 and Proposition 3 we obtain

COROLLARY 4. If in some cardinality $\lambda > \omega$, their isomorphism follows from the mutually elementary imbeddability of any two models of a ω-stable theory T, then this property holds for the theory in each infinite cardinality.

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It is easy to see that from the satisfaction of this property in the cardinality ω, the satisfiability in uncountable cardinalities does not follow. A counterexample is, for example, any countable categorical multidimensional theory.

1. PRELIMINARY INFORMATION

Throughout the entire paper, T will denote a ω-stable theory. We will assume the reader is acquainted with the basic concepts, notation and assertions of stability theory. One can find them, for example, in [1-4]. Let us present here only the ones frequently used in this paper.

We will denote the models of T by the letters $\mathcal{M}$, $\mathcal{N}$, $\mathcal{A}$, $\mathcal{B}$, ..., sometimes with upper and lower indices. Their basic sets are, respectively $|\mathcal{M}|$, $|\mathcal{N}|$, $|\mathcal{A}|$, $|\mathcal{B}|$, ... We will denote sets by X, Y, A, B etc. The symbol $t(\bar{a}\,|\,X)$ denotes the type of the tuple $\bar{a}$ over X. $\bar{s}t(\bar{a}\,|\,X)$ is a strong type of tuple $\bar{a}$ over X. If $X=\emptyset$, then we will write for short $t(\bar{a})$ and $st(\bar{a})$. If the type of the tuple $\bar{a}$ over X forks (does not fork) over $Y\subseteq X$, then we sill briefly write this as $\bar{a}\not\downarrow X\,(\bar{a}\downarrow X)$ or as $t(\bar{a}|X)\not\downarrow Y\;(t(\bar{a}|X)\downarrow Y)$. If $Y=\emptyset$ then we will omit it in this notation. One denotes the orthogonality of two types P, q usually by $p\perp q$, and their nonorthogonality by $p\not\perp q$ or $p\sim q$.

Definition 1.1. a) Let P be a stationary nonalgebraic type from $S_1(X)$, $\varphi=\varphi(x,\bar{a})$. Then the pair (p,φ) is called strongly regular if for any $Y\supset X$ and $q\in S_1(Y)$ for $\varphi\in q$ either q is a nonforking extension of p or $q\perp p$. b) A type $p\in S_1(X)$ is called strongly regular if the pair (p,φ) is strongly regular for some formula $\varphi=\varphi(x,\bar{a})\in p$.

If $p\in S_1(\bar{a})$ and $t(\bar{a})=t(\bar{b})$, then the type $p_{\bar{b}}=\{\varphi(x,\bar{b}) : \varphi(x,\bar{a})\in p\}\in S_1(\bar{b})$ is called a copy of type p. A stationary type $p\in S_1(X)$ has a unique extension $q\in S_1(Y)$, $Y\supset X$ such that $q\downarrow X$, which one denotes usually as $q=p(Y)$.

Definition 1.2. A strongly regular type $p\in S_1(\bar{a})$ is called multidimensional if there exists a tuple $\bar{b}$ satisfying the properties:

1) $st(\bar{a})=st(\bar{b})$,

2) $\bar{a}\downarrow\bar{b}$,

3) $p\perp p_{\bar{b}}$.

If there exist a model $\mathcal{M}$ of T, a tuple $\bar{a}\in\mathcal{M}$ and a multidimensional strongly regular type $p\in S_1(\bar{a})$, then T is called a multidimensional theory. If the theory T is not multidimensional, then we will call it a nonmultidimensional theory (or a theory of bounded dimension).

Let us now present some familiar facts, which one can find, for example, in [2, 5, 6].

LEMMA 1.1. Let T be a theory of bounded dimension, $p\in S_1(\bar{a})$ be a strongly regular type. If $t(\bar{a})=t(\bar{b})$ and $p\sim p_{\bar{b}}$, then for any model $\mathcal{M}$, containing $\bar{a}$ and $\bar{b}$, we have the equality $dim(p,\mathcal{M})=dim(p_{\bar{b}},\mathcal{M})$.

LEMMA 1.2. Let the type $p \in S_1(\bar{a})$ be strongly regular. Then for any tuple $\bar{b}$ and for any model $\mathcal{M}$, containing $\bar{a}$ and $\bar{b}$, if $\alpha \geq \omega$, then

$$dim(p, \mathcal{M}) = \alpha \iff dim(p_{\bar{b}}, \mathcal{M}) = \alpha.$$

LEMMA 1.3. Let $\mathcal{M}_0$ be a simple model of a theory T. The type of p from $S_1(\mathcal{M}_0)$ is definable over $\bar{a} \in |\mathcal{M}_0|$ and $dim(p \upharpoonright \bar{a}, \mathcal{M}_0) = \omega$. Then $\mathcal{M}_0 \cong \mathcal{M}_0(p)$, where $\mathcal{M}_0(p)$ is a model simple over $\mathcal{M}_0 \cup \{c\}$ $t(c \mid \mathcal{M}_0) = p$.

LEMMA 1.4. Let $p \in S_1(\bar{a})$ be a strongly regular type, $\bar{a} \in \mathcal{M} \prec \mathcal{N}$ Then $dim(p, \mathcal{N}) = dim(p, \mathcal{M}) + dim(p(\mathcal{M}), \mathcal{N})$.

LEMMA 1.5. Let $p \in S_1(\bar{a})$ be a multidimensional strongly regular type, $\bar{a} \downarrow \mathcal{M}$, then for all $q \in S(\mathcal{M})$ $p \perp q$.

The results of the article were partially announced in [7].

2. THEORIES OF BOUNDED DIMENSION

In this section we prove the part $1 \Rightarrow 2$ of Theorem 1.

Let T be a ω-stable theory of bounded dimension, $\mathcal{M}_0$ be a fixed simple model of T. It is not hard to show that there exist $\mu \leq \omega$, $n_i \in \omega \setminus \{0\}$ for each $i < \mu$, types $p^{ij} \in S_1(\mathcal{M}_0)$ and tuples $\bar{a}_{ij} \in \mathcal{M}_0$ for all $i < \mu$ and $j < n_i$ possessing the following properties:

1) the type p^{ij} is strongly regular;

2) the type p^{ij} is definable over $\bar{a}_{ij}$;

3) $t(\bar{a}_{ik}) = t(\bar{a}_{i0})$ for all $i < \mu$ and for any $k < n_i$;

4) for any $\bar{b} \in \mathcal{M}_0$ and $i < \mu$ if $t(\bar{b}) = t(\bar{a}_{i0})$, then there exists $k < n_i$ such that $p \sim p_{\bar{b}}$, where $p = p^{ik} \upharpoonright \bar{a}_{ik}$;

5) the family $\{p^{ij} : i < \mu, j < n_i\}$ is a maximal set of pairwise orthogonal strongly regular types from $S_1(\mathcal{M}_0)$.

In order to "mutlistage" indices for any $p \in S_1(\mathcal{M})$ and $\bar{a} \in \mathcal{M}$, we will denote the type $p \upharpoonright \bar{a}$ by $[\bar{a}]$. We will call $\{[\bar{a}_{ij}] : i<\mu, j<n_i\}$ the set of types the canonical basis of T. Let us denote by Ω the set of pairs $\{(i,j) : i<\mu, j<n_i\}$ and by $\Omega_0 = \{(i,j) \in \Omega : dim([\bar{a}_{ij}], \mathcal{M}_0) = 0\}$, $\Omega_1 = \{(i,j) \in \Omega : dim([\bar{a}_{ij}], \mathcal{M}_0) = \omega\}$. One can choose the tuples $\bar{a}_{ij}$ so that $\Omega = \Omega_0 \cup \Omega_1$. Let I^0_{ij} - be an independent set of realizations of type p^{ij}, where $|I^0_{ij}| = 0$, if $(i,j) \in \Omega_0$ and $|I^0_{ij}| = \omega$ if $(i,j) \in \Omega_1$. Let us consider a fixed model $\mathcal{M}_0'$ simple over $\mathcal{M}_0 \cup \bigcup \{I^0_{ij} : (i,j) \in \Omega\}$. By Lemma 1.3 we have $\mathcal{M}_0' \cong \mathcal{M}_0$. Therefore, one can consider that for any model $\mathcal{M}$ we have $\mathcal{M}_0 \prec \mathcal{M}_0' \prec \mathcal{M}$.

Now let $\mathcal{M}$ and $\mathcal{N}$ be arbitrary models of the theory T, $\mathcal{M}_0 \prec \mathcal{M}_0' \prec \mathcal{M}$, $\mathcal{M}_0 \prec \mathcal{M}_0' \prec \mathcal{N}$ and there exist elementary imbeddings $f : \mathcal{M} \to \mathcal{N}$ and $g : \mathcal{N} \to \mathcal{M}$.

LEMMA 2.1. Let $f(\bar{a}_{ij}) = \bar{b}_{ij}$, $g(\bar{a}_{ij}) = \bar{c}_{ij}$ for all $(i,j) \in \Omega$. Then $dim([\bar{a}_{ij}], \mathcal{M}) =$

$dim([\bar{b}_{ij}], \mathcal{N})$ and $dim([\bar{a}_{ij}], \mathcal{N}) = dim([\bar{c}_{ij}], \mathcal{M})$.

Proof. Due to the maximality of the set $\{p^{ij}: (i,j) \in \Omega\}$, there exist permutations σ and τ on the set Ω such that $[\bar{b}_{ij}] \sim [\bar{a}_{\sigma(i,j)}]$ and $[\bar{c}_{ij}] \sim [\bar{a}_{\tau(i,j)}]$ for each pair $(i,j) \in \Omega$ where due to condition 4 of the definition of a canonical basis $\sigma(i,j), \tau(i,j) \in \{(i,0), \ldots, (i, n_i - 1)\}$ for each pair $(i,j) \in \Omega$. By Lemma 1.1, we have that $dim([\bar{b}_{ij}], \mathcal{N}) = dim([\bar{a}_{\sigma(i,j)}], \mathcal{N})$ and $dim([\bar{c}_{ij}], \mathcal{M}) = dim([\bar{a}_{\tau(i,j)}], \mathcal{M})$ for all $(i,j) \in \Omega$. It is dvident $dim([\bar{a}_{ij}], \mathcal{M}) \leq dim([\bar{b}_{ij}], \mathcal{N})$, $dim([\bar{a}_{ij}], \mathcal{N}) \leq dim([\bar{c}_{ij}], \mathcal{M})$ for all $(i,j) \in \Omega$, since f and g are elementary imbeddings of the model $\mathcal{M}$ in $\mathcal{N}$ and $\mathcal{N}$ in $\mathcal{M}$, respectively.

In this manner, we have the following relations: $dim([\bar{a}_{ij}], \mathcal{M}) \leq dim([\bar{b}_{ij}], \mathcal{N}) = dim([\bar{a}_{\sigma(i,j)}], \mathcal{N}) \leq dim([\bar{c}_{\sigma(i,j)}], \mathcal{M}) = dim([\bar{a}_{\tau\sigma(i,j)}], \mathcal{M})$. Let us denote by ε the product of the permutations $\tau\sigma$, then we obtain $dim([\bar{a}_{ij}], \mathcal{M}) \leq dim([\bar{a}_{\varepsilon(i,j)}], \mathcal{M})$ for any pair $(i,j) \in \Omega$. It is clear that for any pair $(i,j) \in \Omega$ there is a k such $\varepsilon^k(i,j) = (i,j)$ and $dim([\bar{a}_{ij}], \mathcal{M}) \leq dim([\bar{a}_{\varepsilon(i,j)}], \mathcal{M}) \leq dim([\bar{a}_{\varepsilon^2(i,j)}], \mathcal{M}) \leq \ldots \leq dim([\bar{a}_{ij}], \mathcal{M})$. In other words, $dim([\bar{a}_{ij}], \mathcal{M}) = dim([\bar{a}_{\varepsilon(i,j)}], \mathcal{M})$, consequently, $dim([\bar{a}_{ij}], \mathcal{M}) \leq dim([\bar{b}_{ij}], \mathcal{N}) \leq dim([\bar{a}_{\varepsilon(i,j)}], \mathcal{M}) \leq dim([\bar{a}_{ij}], \mathcal{M})$, therefore $dim([\bar{a}_{ij}], \mathcal{M}) = dim([\bar{b}_{ij}], \mathcal{N})$. Similarly $dim([\bar{a}_{ij}], \mathcal{N}) = dim([\bar{c}_{ij}], \mathcal{M})$. The lemma is proved.

Let us return to our models $\mathcal{M}$ and $\mathcal{N}$. Let $I'_{ij} \subset \mathcal{M}$ be a basis of type $\bar{p}^{ij} = p^{ij}(\mathcal{M}'_0)$, $I_{ij} = I^0_{ij} \cup I'_{ij}$ for all $(i,j) \in \Omega$. Due to the bounded dimension and ω-stability of the theory, and also the properties of strongly-regular types, we have

LEMMA 2.2. a) The model $\mathcal{M}$ is simple and minimal over $\mathcal{M}'_0 \cup \bigcup\{I'_{ij} : (i,j) \in \Omega\}$;

b) the set I_{ij} is a basis of type p^{ij};

c) the model $\mathcal{M}$, is simple and minimal over $\mathcal{M}_0 \cup \bigcup\{I_{ij} : (i,j) \in \Omega\}$.

Let us denote by $\mathcal{N}_0$, $\mathcal{N}'_0$, q^{ij}, $\bar{q}^{ij}$, respectively, the images of $\mathcal{M}_0$, $\mathcal{M}'_0$, p^{ij}, $\bar{p}^{ij}$ under the elementary imbedding f. We have the following

LEMMA 2.3. For all $(i,j) \in \Omega$ we have the equation

$$dim(p^{ij}, \mathcal{M}) = dim(q^{ij}, \mathcal{N}).$$

Proof. According to Lemma 2.1, we have the equations $dim([\bar{a}_{ij}], \mathcal{M}) = dim([\bar{b}_{ij}], \mathcal{N})$. On the other hand, due to imbeddability, the inequality $dim(p^{ij}, \mathcal{M}) \leq dim(q^{ij}, \mathcal{N})$ is satisfied. Furthermore, for any $(i,j) \in \Omega_1$ we have $dim(p^{ij}, \mathcal{M}) \geq \omega$. By Lemma 1.4, we have $dim(p^{ij}, \mathcal{M}) = dim(p^{ij}, \mathcal{M}_0) + dim(\bar{p}^{ij}, \mathcal{M})$. Due to this very fact $dim([\bar{a}_{ij}], \mathcal{M}) = dim([\bar{a}_{ij}], \mathcal{M}_0) + dim(p^{ij}, \mathcal{M})$ and $dim([\bar{b}_{ij}], \mathcal{N}) = dim([\bar{b}_{ij}], \mathcal{N}_0) + dim(q^{ij}, \mathcal{N})$ If $(i,j) \in \Omega_0$, then $dim([\bar{a}_{ij}], \mathcal{M}_0) = dim([\bar{b}_{ij}], \mathcal{N}_0) = 0$, consequently, $dim(p^{ij}, \mathcal{M}) = dim([\bar{a}_{ij}], \mathcal{M}) = dim([\bar{b}_{ij}], \mathcal{N}) = dim(q^{ij}, \mathcal{N})$. If $(i,j) \in \Omega_1$ and $dim([\bar{a}_{ij}], \mathcal{M}) > \omega$, then similarly to what preceded $dim(p^{ij}, \mathcal{M}) = dim(q^{ij}, \mathcal{N})$. Now let $(i,j) \in \Omega_1$ and $dim([\bar{a}_{ij}], \mathcal{M}) = \omega$. Then $\omega \leq dim(p^{ij}, \mathcal{M}) \leq dim(q^{ij}, \mathcal{N})$. But $dim(q^{ij}, \mathcal{N})$ can not be uncountable, since $dim([\bar{b}_{ij}], \mathcal{N}) = dim([\bar{a}_{ij}], \mathcal{M}) = \omega$ by Lemma 2.1. Lemma 2.3 is proved.

Now let us choose in model $\mathcal{N}$ bases $\mathcal{J}_{ij}$ of the types q^{ij}. Due to Lemma 2.3, the cardinality $\mathcal{J}_{ij}$ equals the cardinality I_{ij} for all $(i,j)\in\Omega$. Arbitrarily (but bijectively!) mapping the basis I_{ij} on $\mathcal{J}_{ij}$ for all pairs $(i,j)\in\Omega$, we obtain an elementary mapping f' of the set $\mathcal{M}_0\cup\bigcup\{I_{ij}:(i,j)\in\Omega\}$ on $\mathcal{N}_0\cup\bigcup\{\mathcal{J}_{ij}:(i,j)\in\Omega\}$ continuing $f\upharpoonright\mathcal{M}_0$. By Lemma 2.2, the models $\mathcal{M}$ and $\mathcal{N}$ are simple and minimal over $\mathcal{M}_0\cup\bigcup\{I_{ij}:(i,j)\in\Omega\}$ and $\mathcal{N}_0\cup\bigcup\{\mathcal{J}_{ij}:(i,j)\in\Omega\}$ respectively. Consequently, there exists an elementary imbedding $F:\mathcal{M}\to\mathcal{N}$ continuing f', which indeed will be an imbedding "on" due to the minimality of $\mathcal{N}$ over $\mathcal{N}_0\cup\bigcup\{\mathcal{J}_{ij}:(i,j)\in\Omega\}$ therefore it is an isomorphism.

3. MULTIDIMENSIONAL THEORIES

The fundamental goal of this section is to prove Proposition 3, from which follows the part 2 $\Longrightarrow$ 1 of Theorem 1.

LEMMA 3.1. Let $\bar{a}_0,\bar{a}_1,\dots$ be an infinite indiscernible sequence, $\mathcal{M}_0'$ be a model simple over $\{\bar{a}_i : 1\le i<\omega\}$, and $\mathcal{M}_0$ be a simple model over $\mathcal{M}_0'\cup\bar{a}_0$. Then the model $\mathcal{M}_0$ is simple over $\{\bar{a}_i : i<\omega\}$.

Proof. Since the model $\mathcal{M}_0$ is countable, then it is sufficient to show that it is atomic over $\{\bar{a}_i : i<\omega\}$. In its turn, for this, it is sufficient by Theorem IV.3.2 from [1] to prove that $\mathcal{M}_0'\cup\bar{a}_0$ is atomic over $\{\bar{a}_i : i<\omega\}$. Let $\bar{m}\in\mathcal{M}_0'$, since $\mathcal{M}_0'$ is atomic over $\{\bar{a}_i : 1\le i<\omega\}$, there exists a formula $\varphi(\bar{x},\bar{a}_1,\dots,\bar{a}_k)$ isolating $t(\bar{m}\,|\,\{\bar{a}_i : 1\le i<\omega\})$. Let us show that the formula $\psi(\bar{x},\bar{y})=\varphi(\bar{x},\bar{a}_1,\dots,\bar{a}_k)\wedge\bar{y}=\bar{a}_0$ isolates the type $t(\bar{m}{}^\wedge\bar{a}_0\,|\,\{\bar{a}_i : i<\omega\})$. Let us assume that this is not so. Then there is a formula $\theta(\bar{x},\bar{y})=\theta(\bar{x},\bar{y},\bar{a}_1,\dots,\bar{a}_\ell,\bar{a}_0)$, $\ell\ge k$, such that

$$\models\exists\bar{x}\exists\bar{y}\,[\psi(\bar{x},\bar{y})\wedge\theta(\bar{x},\bar{y})]$$

and

$$\models\exists\bar{x}\exists\bar{y}\,[\psi(\bar{x},\bar{y})\wedge\neg\theta(\bar{x},\bar{y})].$$

In other words,

$$\models\exists\bar{x}\,[\varphi(\bar{x},\bar{a}_1,\dots,\bar{a}_k)\wedge\theta(\bar{x},\bar{a}_0,\bar{a}_1,\dots,\bar{a}_\ell,\bar{a}_0)]$$

and

$$\models\exists\bar{x}\,[\varphi(\bar{x},\bar{a}_1,\dots,\bar{a}_k)\wedge\neg\theta(\bar{x},\bar{a}_0,\bar{a}_1,\dots,\bar{a}_\ell,\bar{a}_0)].$$

Since we took an indiscernible sequence, then

$$\models\exists\bar{x}\,[\varphi(\bar{x},\bar{a}_1,\dots,\bar{a}_k)\wedge\theta(\bar{x},\bar{a}_{\ell+1},\bar{a}_1,\dots,\bar{a}_\ell,\bar{a}_{\ell+1})]$$

and

$$\models\exists\bar{x}\,[\varphi(\bar{x},\bar{a}_1,\dots,\bar{a}_k)\wedge\neg\theta(\bar{x},\bar{a}_{\ell+1},\bar{a}_1,\dots,\bar{a}_\ell,\bar{a}_{\ell+1})].$$

But this contradicts the fact that the formula $\varphi(\bar{x},\bar{a}_1,\dots,\bar{a}_k)$ isolates the type $t(\bar{m}\,|\,\{\bar{a}_i : 1\le i<\omega\})$. The lemma is proved.

Now let $p\in S(\bar{a})$ be a stationary strongly regular multidimensional type. Let us fix it until the end of this section. By the definition of multidimensionality, one can choose a sequence $\{\bar{a}_n : n < \omega\}$ satisfying the following properties:

1) $st(\bar{a}_n) = st(\bar{a})$;

2) $\{\bar{a}_n : n < \omega\}$ is an independent set;

3) $p_{\bar{a}_i} \perp p_{\bar{a}_j}$ for all $i < j < \omega$.

It is clear that $\{\bar{a}_n : n < \omega\}$ is an indiscernible set. Let $\mathcal{M}_0'$ be a model simple over $\{\bar{a}_i : 1 \leq i < \omega\}$ and such that $\bar{a}_0 \downarrow \mathcal{M}_0'$. Now let $\mathcal{M}_0$ be a model simple over $\mathcal{M}_0' \cup \{\bar{a}_0\}$. By Lemma 3.1, $\mathcal{M}_0$ is a model simple over $\{\bar{a}_i : i < \omega\}$. Let λ be a fixed uncountable cardinal. Let us construct the models $\mathcal{M}^0$, $\mathcal{N}^0$ in the following manner. Let us choose a $\mathcal{J}_k$ -independent set of realizations of type $p_{\bar{a}_k}(\mathcal{M}_0)$ of cardinality λ for all $k < \omega$. The set I_k coincides with the set $\mathcal{J}_k$ for all $k \geq 1$, and with the set I_0 – some countable set of $\mathcal{J}_0$. Now $\mathcal{M}^0$ is a model simple over $\mathcal{M}_0 \cup \bigcup_{k<\omega} I_k$ and $\mathcal{N}^0$ is a model simple over $\mathcal{M}_0 \cup \bigcup_{k<\omega} \mathcal{J}_k$, where $\mathcal{M}^0 \prec \mathcal{N}^0$. It is clear that $\|\mathcal{M}^0\| = \|\mathcal{N}^0\| = \lambda$. In the same way it is easy to show that $dim(p_{\bar{a}_i}, \mathcal{N}^0) = \lambda$ for all $i < \omega$, $dim(p_{\bar{a}_i}, \mathcal{M}^0) = \lambda$ for all $i \geq 1$ but $dim(p_{\bar{a}_0}, \mathcal{M}^0) = \omega$. This follows from the pairwise orthogonality of the types $p_{\bar{a}_i}$ and $p_{\bar{a}_j}$ for $i < j < \omega$.

The sequence $\{\bar{a}_i : i < \omega\}$, the models $\mathcal{M}_0'$, $\mathcal{M}_0$, $\mathcal{M}^0$ and $\mathcal{N}^0$ we will also fix to the end of this section. Now let $\mathcal{A}$ be an arbitrary model of cardinality λ, containing $\{\bar{a}_i : i < \omega\}$. Let us define the model $\mathcal{A}(\lambda)$ in the following manner. Let $B_{\mathcal{A}}$ be the set of all $\bar{b} \in \mathcal{A}$ such that $t(\bar{b}) = t(\bar{a})$ and $p_{\bar{b}} \perp p_{\bar{a}_i}$ for all $i < \omega$. Let $B_{\mathcal{A}} = \{\bar{b}_\alpha : \alpha < \lambda\}$. For each $\alpha < \lambda$ let us take an independent set $\mathcal{K}_{\bar{b}_\alpha}^{\mathcal{A}}$ of realizations of type $p_{\bar{b}_\alpha}(\mathcal{A})$ of power λ, in order that $t(\mathcal{K}_{\bar{b}_\alpha}^{\mathcal{A}} \mid \mathcal{A} \cup \bigcup_{\beta<\alpha} \mathcal{K}_{\bar{b}_\beta}^{\mathcal{A}}) \downarrow \mathcal{A}$. After this, let us take as $\mathcal{A}(\lambda)$ a model simple over $\mathcal{A} \cup \bigcup_{\alpha<\lambda} \mathcal{K}_{\bar{b}_\alpha}^{\mathcal{A}}$. It is clear that the set $\bigcup_{\alpha<\lambda} \mathcal{K}_{\bar{b}_\alpha}^{\mathcal{A}}$ is a set independent over $\mathcal{A}$.

LEMMA 3.2. 1) For all $\alpha < \lambda$ $dim(p_{\bar{b}_\alpha}, \mathcal{A}(\lambda)) = \lambda$; 2) for all $i < \omega$ $dim(p_{a_i}, \mathcal{A}(\lambda)) = dim(p_{\bar{a}_i}, \mathcal{A})$.

Proof. The proof of point 1 immediately follows from the construction of the model $\mathcal{A}(\lambda)$. Let us prove point 2. Let $i < \omega$, since $p_{\bar{a}_i} \perp p_{\bar{b}_\alpha}$ for all $\alpha < \lambda$, then it is not hard to show that $dim(p_{\bar{a}_i}(\mathcal{A}), \mathcal{A}(\lambda)) = 0$ and then the assertion follows from the equation

$$dim(p_{\bar{a}_i}, \mathcal{A}(\lambda)) = dim(p_{\bar{a}_i}, \mathcal{A}) + dim(p_{\bar{a}_i}(\mathcal{A}), \mathcal{A}(\lambda)).$$

The lemma is proved.

Now let us move on to constructing the models $\mathcal{M}$ and $\mathcal{N}$ which will be mutually elementarily imbeddable, but isomorphic. The models $\mathcal{M}^0$ and $\mathcal{N}^0$ have already been constructed. Let $\mathcal{M}^n$ and $\mathcal{N}^n$ be defined. Then $\mathcal{M}^{n+1}$ is this $\mathcal{M}^n(\lambda)$,$\mathcal{N}^{n+1} = \mathcal{N}^n(\lambda)$. The models $\mathcal{M} = \bigcup_{n<\omega} \mathcal{M}^n(\lambda)$, $\mathcal{N} = \bigcup_{n<\omega} \mathcal{N}^n(\lambda)$.

LEMMA 3.3. 1) For all $\bar{b} \in B_{\mathcal{M}}$ we have the equation $dim(p_{\bar{b}}, \mathcal{M}) = \lambda$;

2) for all $\bar{c} \in B_{\mathcal{N}}$ we have the equation $dim(p_{\bar{c}}, \mathcal{N}) = \lambda$;

3) for all $i < \omega$ $dim(p_{\bar{a}_i}, \mathcal{M}) = dim(p_{\bar{a}_i}, \mathcal{M}^0)$, $dim(p_{\bar{a}_i}, \mathcal{N}) = dim(p_{\bar{a}_i}, \mathcal{N}^0)$.

It is easy to conduct the proof of induction on n, using Lemma 3.2.

LEMMA 3.4. The models $\mathcal{M}$ and $\mathcal{N}$ are nonisomorphic.

Proof. Let us assume that there exists an isomorphism $f: \mathcal{M} \to \mathcal{N}$. Let $\bar{b} = f(\bar{a}_0)$. It is clear, on the one hand, $dim(p_{\bar{b}}, \mathcal{N}) = dim(p_{\bar{a}_0}, \mathcal{M}) = \omega$. But, on the other hand, there can occur two possibilities: either $p_{\bar{b}} \perp p_{\bar{a}_i}$ or $i < \omega$ for some $p_{\bar{b}} \sim p_{\bar{a}_j}$ for $j < \omega$. If case occurs, then, by Lemma 3.3 (2), $dim(p_{\bar{b}}, \mathcal{N}) = \lambda$. Let $p_{\bar{a}_j} \sim p_{\bar{b}}$. Then, by Lemma 1.2, $dim(p_{\bar{b}}, \mathcal{N}) = dim(p_{\bar{a}_j}, \mathcal{N}) = \lambda$. A contradiction, consequently $\mathcal{M} \not\cong \mathcal{N}$.

Now let us move on to constructing the elementary imbeddings $f: \mathcal{M} \to \mathcal{N}$ and $g: \mathcal{N} \to \mathcal{M}$. At first let us construct the mapping f as a union of the mappings $f_n: \mathcal{M}^n \to \mathcal{N}^n$, $n < \omega$. The mapping $f_0: \mathcal{M}^0 \to \mathcal{N}^0$ is the identity mapping. Let the mappings $f_0 \subseteq f_1 \subset \ldots \subset f_{n-1}$ be constructed by us, where $f_i: \mathcal{M}^i \to \mathcal{N}^i$ for all $i < n$. Let $B_{\mathcal{M}^{n-1}} = \{\bar{b}_\alpha : \alpha < \lambda\}$. It is clear that $\{f_{n-1}(\bar{b}_\alpha) : \alpha < \lambda\} \subset B_{\mathcal{N}^{n-1}}$, since $id_{\mathcal{M}^0} = f_0 \subset f_{n-1}$, consequently, if $p_{\bar{b}} \perp p_{\bar{a}_i}$ for all $i < \omega$, then $p_{f_{n-1}(\bar{b})} \perp p_{\bar{a}_i}$ for all $i < \omega$.

Let us map $\mathcal{M}^{n-1} \cup \bigcup\limits_{\alpha<\lambda} \mathcal{K}_{\bar{b}_\alpha}^{\mathcal{M}^{n-1}}$ on $f_{n-1}(\mathcal{M}^{n-1}) \cup \bigcup\limits_{\alpha<\lambda} \mathcal{K}_{f_{n-1}(\bar{b}_\alpha)}^{\mathcal{N}^{n-1}}$

so that the model $\mathcal{M}^{n-1}$ is mapped on $f_{n-1}(\mathcal{M}^{n-1})$ with the help of f_{n-1} and $\mathcal{K}_{\bar{b}_\alpha}^{\mathcal{M}^{n-1}}$ on $\mathcal{K}_{f_{n-1}(\bar{b}_\alpha)}^{\mathcal{N}^{n-1}}$ abritrarily, but one-to-one. It is easy to understand that $\mathcal{K}_{f_{n-1}(\bar{b}_\alpha)}^{\mathcal{N}^{n-1}}$ is an independent set of realizations of type $p_{f_{n-1}(\bar{b}_\alpha)}$ and $\bigcup\limits_{\alpha<\lambda} \mathcal{K}_{f_{n-1}(\bar{b}_\alpha)}^{\mathcal{N}^{n-1}}$ is an independent set over $f_{n-1}(\mathcal{M}^{n-1})$. Therefore, the mapping defined above is elementary. Now it is not hard to construct $f_n: \mathcal{M}^n \to \mathcal{N}' \prec \mathcal{N}^n$, where $\mathcal{N}'$ is a model simple over

$$f_{n-1}(\mathcal{M}^{n-1}) \cup \bigcup_{\alpha<\lambda} \mathcal{K}_{f_{n-1}(\bar{b}_\alpha)}^{\mathcal{N}^{n-1}}.$$

In this manner, the construction $f: \mathcal{M} \to \mathcal{N}$ is concluded, $f: \bigcup\limits_{n<\omega} f_n$.

In order to construct the elementary mapping $g: \mathcal{N} \to \mathcal{M}$, let us prove preliminarily the following lemma.

LEMMA 3.5. Let $\mathcal{M}$ and $\mathcal{N}$ be two models of cardinality λ, conaining $\{\bar{a}_i : i < \omega\}$, $h: \mathcal{N} \to \mathcal{M}$ be an elementary imbedding satisfying the next two conditions:

1) $h(\bar{a}_i) = \bar{a}_{i+1}$ for all $i < \omega$,

2) $\bar{a}_0 \downarrow h(\mathcal{N})$.

Then there exists $h' : \mathcal{N}(\lambda) \to \mathcal{M}(\lambda)$, continuing h and also satisfying the condition $\bar{a}_0 \downarrow h'(\mathcal{N}(\lambda))$.

Proof. Let $B_{\mathcal{N}}$ $\{\bar{c}_\alpha : \alpha < \lambda\}$. Let us denote $h(\bar{c}_\alpha)$ by $\bar{b}'_\alpha$. Let us show that $\{\bar{b}'_\alpha : \alpha < \lambda\} \subset B_{\mathcal{M}}$. Let $\alpha < \lambda$. Then $p_{\bar{c}_\alpha} \perp p_{\bar{a}_i}$, $i < \omega$, consequently, $p_{\bar{b}'_\alpha} \perp p_{\bar{a}_{i+1}}$ for all $i < \omega$. However $\{\bar{b}'_\alpha : \alpha < \lambda\} \subset h(\mathcal{N})$, and on the other hand, by condition 2 $\bar{a}_0 \downarrow h(\mathcal{N})$. Then by Lemma 1.5 we have the relation $p_{\bar{a}_0} \perp p_{\bar{b}'_\alpha}$ for all $\alpha < \lambda$. In this manner, $\{\bar{b}'_\alpha : \alpha < \lambda\} \subset B_{\mathcal{M}}$. Now let us one-to-one map $\mathcal{K}^{\mathcal{N}}_{\bar{c}_\alpha}$ on $\mathcal{K}^{\mathcal{M}}_{\bar{b}'_\alpha}$. It is easy to check that $\bigcup_{\alpha<\lambda} \mathcal{K}^{\mathcal{M}}_{\bar{b}'_\alpha}$ is an independent set over $h(\mathcal{N})$. Since $\mathcal{N}(\lambda)$ is simple over $\mathcal{N} \cup \bigcup_{\alpha<\lambda} \mathcal{K}^{\mathcal{N}}_{\bar{c}_\alpha}$, there exists an elementary mapping $h' : \mathcal{N}(\lambda) \to \mathcal{M}(\lambda)$ continuing the mapping h. It remains to show that $\bar{a}_0 \downarrow h'(\mathcal{N}(\lambda))$. It is clear that the model $h'(\mathcal{N}(\lambda))$ is simple over $h(\mathcal{N}) \cup \bigcup_{\alpha<\lambda} \mathcal{K}^{\mathcal{M}}_{\bar{b}'_\alpha}$, therefore it is sufficient to convince oneself that $\bar{a}_0 \downarrow_{h(\mathcal{N})} \mathcal{K}$, where $\mathcal{K} = \bigcup_{\alpha<\lambda} \mathcal{K}^{\mathcal{M}}_{\bar{b}'_\alpha}$. Indeed, it is easy to show that $t(\mathcal{K}|\mathcal{M}) \downarrow h(\mathcal{N})$, $h(\mathcal{N}) \prec \mathcal{M}$. On the other hand, $\bar{a}_0 \cup h(\mathcal{N}) \subset \mathcal{M}$, therefore $\mathcal{K} \downarrow_{h(\mathcal{N})} \bar{a}_0$. Now $\bar{a}_0 \downarrow_{h(\mathcal{N})} h'(\mathcal{N}(\lambda))$, by condition 2, $\bar{a}_0 \downarrow h(\mathcal{N})$, consequently $\bar{a}_0 \downarrow h'(\mathcal{N}(\lambda))$ Lemma 3.5 is proved.

Now let us define the elementary mappings $g_n : \mathcal{N}^n \to \mathcal{M}^n$. At first let us construct $g_0 : \mathcal{N}^0 \to \mathcal{M}^0$. Let $g'_0 : \bar{a}_i \to \bar{a}_{i+1}$ for all $i < \omega$. The model $\mathcal{M}_0$ is simple over $\{\bar{a}_i : i < \omega\}$ and by Lemma 3.1, $\mathcal{M}'_0$ is simple over $\{\bar{a}_i : 1 \leq i < \omega\}$. Due to the uniqueness of the simple model, there exists an isomorphism $g''_0 : \mathcal{M}_0 \to \mathcal{M}'_0$ continuing g'_0 . By construction of the models $\mathcal{M}^0$ and $\mathcal{N}^0$, one can one-to-one map the set J_i on I_{i+1} . It is clear that there is an elementary imbedding g_0 of the model $\mathcal{N}^0$ in the model $\mathcal{M}^0$. In the same way as in Lemma 3.5, one can show that

$$\bigcup_{1 \leq i < \omega} I_i \downarrow_{\mathcal{M}'_0} \mathcal{M}_0 ,$$

consequently. $\bigcup_{1\leq i<\omega} I_i \downarrow_{\mathcal{M}'_0} \bar{a}_0$. Since $g_0(\mathcal{N}^0)$ is a model simple over $\mathcal{M}'_0 \cup \bigcup_{1\leq i<\omega} I_i$, then $\bar{a}_0 \downarrow_{\mathcal{M}'_0} g_0(\mathcal{N}^0)$. But by the primary construction $\bar{a}_0 \downarrow \mathcal{M}'_0$, consequently, $\bar{a}_0 \downarrow g_0(\mathcal{N}^0)$. Now it remains to use Lemma 3.5 in order to continue the elementary mapping $g_0 : \mathcal{N}^0 \to \mathcal{M}^0$ to $g_1 : \mathcal{N}^1 \to \mathcal{M}^1$ etc. Then it is understood that the mappings $f = \bigcup_{n<\omega} f_n$ and $g = \bigcup_{n<\omega} g_n$ will be elementary mappings of the model $\mathcal{M} = \bigcup_{n<\omega} \mathcal{M}^n$ to the model $\mathcal{N} = \bigcup_{n<\omega} \mathcal{N}^n$ and the model $\mathcal{N}$ to $\mathcal{M}$, respectively. Proposition 3, and consequently, Theorem 1 are proved.

Now let us briefly describe the proof of Theorem 2.

The proof from 1 to 2 is obtained from the results [2]. where it follows from condition 1, denoted there by (*), that all the countable moles of a theory T are almost homogeneous. On the other hand, in this same paper, a result of Pillay is mentioned that two almost homogeneous models realizing identical types, are isomorphic. It is clear that mutually elementarily imbeddable models realize identical sets of types.

Now let us point the way to the proof from 2 to 1. Let condition 1 not be satisfied, that is there exists a multidimensional strongly regular type $p \in S(\bar{a})$, $\bar{a}$ from some model $\mathcal{M}$ and $dim(p,\mathcal{M}) < \omega$.

Let us select a sequence $\{\bar{a}_i : i < \omega\}$ of the model $\mathcal{M}_0'$ and $\mathcal{M}_0$ in same way as above in the proof of the point $\neg 1 \Longrightarrow \neg 2$ at the start of Sec. 3. Everywhere further in the construction of the models $\mathcal{M}^n$ and $\mathcal{N}^n$ let us replace the cardinal λ by ω , and in the model $\mathcal{M}^0$ let us take the independent set I_0 of realizations of type $p_{\bar{a}_0}(\mathcal{M}_0)$ of cardinality 0, that is $I_0 = \emptyset$. The remaining arguments are similar. As a result, we obtain two countable models $\mathcal{M} = \bigcup_{n<\omega} \mathcal{M}^n$ and $\mathcal{N} = \bigcup_{n<\omega} \mathcal{N}^n$, which are mutually elementarily imbeddable but nonisomorphic.

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