MAL'TSEV CONDITIONS AND REPRESENTABILITY OF MANIFOLDS

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Let $\mathcal{T}_S(\mathfrak{U})$ be the aggregate of all strict Mal'tsev conditions that are satisfiable in a given manifold $\mathfrak{U}$ of algebras. From Taylor's theorem it follows that if $\mathfrak{U}$ is representable in some manifold $\mathfrak{V}$ in Jónsson's sense [1], then $\mathcal{T}_S(\mathfrak{U})\subseteq\mathcal{T}_S(\mathfrak{V})$. The converse is true for a finitely based manifold $\mathfrak{U}$. In this paper we show that in the general case the inclusion $\mathcal{T}_S(\mathfrak{U})\subseteq\mathcal{T}_S(\mathfrak{V})$ does not imply that $\mathfrak{U}$ is representable in $\mathfrak{V}$. We also give a characterization of the property $S_{\delta\Sigma}$ of a class of manifolds of being the union of intersections of strong Mal'tsev classes, which is a revised version of Theorem 3.8 of [4].

We investigate the representability of manifolds in Post manifolds of infinite order. We prove that if a manifold $\mathfrak{U}$ has an algebra of infinite cardinality α, then it representable in a Post manifold of order α and above. This enables us to establish that all Post manifolds of infinite order have the same Mal'tsev theory $\mathcal{T}_S(\mathcal{P}_\omega)$, which is the only complete Mal'tsev theory and contains all nontrivial S-theories. From this it follows, in particular, that all Post manifolds of infinite order are arithmetic and each of them has isomorphic free algebras of finite rank. We also show that Post manifolds of order greater than or equal to 2^ω do not have a finite basis for their identities.

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1. Characterization of $S_{\delta\Sigma}$-Classes

Let θ be the conjunction of finitely many equalities of terms of functional symbols $f_1,\ldots,f_m$ and objective variables $x_1,\ldots,x_n$. According to Taylor [2], a strict Mal'tsev condition is a formula

$$(\exists f_1)\ldots(\exists f_m)(\forall x_1)\ldots(\forall x_n)\theta \tag{1}$$

of a second-order language with specialized quantors $\exists f_1,\ldots,\exists f_m$. Formula (1) is said to be satisfiable in a manifold $\mathfrak{V}$ of algebras of signature Ω if there are terms $\bar{f}_1,\ldots,\bar{f}_m$ of signature Ω in the variables $x_1,\ldots,x_n$ such that

$$\mathfrak{V}\vDash(\forall x_1)\ldots(\forall x_n)\bar{\theta},$$

where the formula $\bar{\theta}$ is obtained from θ by replacing each symbol f_i by $\bar{f}_i$.

Let us agree to denote the aggregate of all strict Mal'tsev conditions of n-ary functional symbols $f_1^{(n)},f_2^{(n)},\ldots(n=1,2,\ldots)$ and objective variables $x_1,x_2,\ldots$ that are satisfiable in

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a given manifold $\mathcal{V}$ by $\mathcal{T}_S(\mathcal{V})$ and to call it the Mal'tsev theory (or breifly $\mathcal{S}$-theory) of the manifold $\mathcal{V}$.

In the class of all manifolds of algebras the relation

$$\mathcal{U} \equiv \mathcal{V} \Leftrightarrow \mathcal{T}_S(\mathcal{U}) = \mathcal{T}_S(\mathcal{V})$$

is an equivalence relation. We shall call mixed classes with respect to this equivalence Mal'tsev fibers, and the relation $\equiv$ an $\mathcal{S}$-equivalence.

Let K be a class of manifolds of algebras. It is called a strong Mal'tsev class, or $\mathcal{S}$-class, if there is a strict Mal'tsev condition (1) such that the manifold $\mathcal{V}$ belongs to K if and only if condition (1) is satisfiable in $\mathcal{V}$. The class K is simply called a Mal'tsev class, or $\mathcal{M}$-class, if there are $\mathcal{S}$-classes $K_1 \subseteq K_2 \subseteq \ldots$ such that $K = \bigcup K_i \; (i < \omega)$.

If $\mathcal{R}$ is some property of classes of manifolds of algebras, let us agree to denote by $\mathcal{R}_\delta$ ($\mathcal{R}_\sigma$) the property of the class of manifolds of being the intersection (respectively, union) of a no more than countable set of classes that have the property $\mathcal{R}$. If we allow arbitrary intersections and unions of $\mathcal{R}$-classes in this definition, we obtain the definition of the properties $\mathcal{R}_\Delta$ and $\mathcal{R}_\Sigma$ respectively.

We gave the diagram of the properties $\mathcal{S}, \mathcal{M}, \mathcal{S}_\sigma \leftrightarrow \mathcal{M}_\sigma$, $\mathcal{S}_\delta$, $\mathcal{M}_\delta$, and $\mathcal{S}_{\sigma\delta} \leftrightarrow \mathcal{M}_{\sigma\delta}$ in [3]. In this paper we enlarge it with the properties $\mathcal{S}_{\delta\Sigma}$, $\mathcal{S}_{\sigma\Delta}$, and $\mathcal{J}$.

A characterization of the property $\mathcal{S}_{\delta\Sigma}$ was given by Baldwin and Berman ([4], Theorem 3.8).* We show that this theorem is true only in one direction, and we give a revised statement of this theorem, which we use to construct the diagram.

<u>LEMMA.</u> $\mathcal{U} \equiv \mathcal{V}$ if and only if for any $\mathcal{S}_\sigma$-class K we have

$$\mathcal{U} \in K \Leftrightarrow \mathcal{V} \in K. \tag{2}$$

In fact, suppose that $\mathcal{U} \equiv \mathcal{V}$ and that K is an $\mathcal{S}_\sigma$-class. Then K is representable as a union of a no more than countable set of $\mathcal{S}$-classes K_n, $n = 1, 2, \ldots$. If $\mathcal{U} \in K$, then $\mathcal{U} \in K_n$ for some n. Suppose that the class K_n is definable by the strict condition (1). Since this condition is satisfiable in $\mathcal{U}$ and $\mathcal{V} \equiv \mathcal{U}$, it is also satisfiable in $\mathcal{V}$. Consequently, $\mathcal{V} \in K_n \subseteq K$.

Conversely, if (2) is satisfied for the manifolds $\mathcal{U}$ and $\mathcal{V}$ for any $\mathcal{S}_\sigma$-class K, it is satisfied, in particular, for any $\mathcal{S}$-class, so $\mathcal{U} \equiv \mathcal{V}$.

<u>COROLLARY ([4, Theorem 3.8]).</u> Any $\mathcal{S}_{\delta\Sigma}$-class K is closed with respect to the $\mathcal{S}$-equivalence $\equiv$.

In fact, suppose that $\mathcal{U} \equiv \mathcal{V}$, $\mathcal{U} \in K$ and that K is representable in the form

$$K = \bigcup K_\lambda \; (\lambda \in \Lambda), \quad K_\lambda = \bigcap L_i^{(\lambda)} \; (i < \omega), \tag{3}$$

*In [4] the property $\mathcal{S}$ was denoted by MC.

where all the $L_i^{(\lambda)}$ are $\mathcal{S}$-classes. Then $\mathfrak{U} \in L_1^{(\lambda)}, L_2^{(\lambda)}, \ldots$ for some λ. By the lemma, we also have $\mathfrak{V} \in L_1^{(\lambda)}, L_2^{(\lambda)}, \ldots$ for this λ. Consequently, $\mathfrak{V} \in K_\lambda \subseteq K$.

Despite the assertion of Theorem 3.8 of [4], we show that the property $S_{\delta\Sigma}$ does not follow from the fact that the class K of manifolds is closed with respect to $\mathcal{S}$-equivalence. For this we use the concept of representability of manifolds due to Jónsson [1].

Let $\mathfrak{U}$ be a manifold of algebras of type $\nu = \langle n_i \mid i \in I \rangle$. A representation of $\mathfrak{U}$ in a manifold $\mathfrak{V}$ of Ω-algebras is a collection $\rho = (\rho_i \mid i \in I)$ of terms of type ν of the functional symbols of Ω such that for every algebra $\langle A, \Omega \rangle$ of $\mathfrak{V}$ the algebra $\langle A, \{\rho_i^A \mid i \in I\} \rangle$ with a collection of polynomial operations ρ_i^A $(i \in I)$ defined by the terms ρ_i $(i \in I)$, belongs to $\mathfrak{U}$. The manifold $\mathfrak{U}$ is said to be representable in $\mathfrak{V}$ (symbolically $\mathfrak{U} \xrightarrow{\exists\rho} \mathfrak{V}$ or simply $\mathfrak{U} \to \mathfrak{V}$) if there is a representation ρ for $\mathfrak{U}$ in $\mathfrak{V}$.

From the definition of the property $S_{\delta\Sigma}$ and Taylor's theorem for strong Mal'tsev classes it follows directly that any $S_{\delta\Sigma}$-class K satisfies the condition

$$\mathfrak{U} \xrightarrow{\exists\rho} \mathfrak{V}, \quad \mathfrak{U} \in K \Rightarrow \mathfrak{V} \in K. \tag{4}$$

It is now easy to check that there is a manifold $\mathfrak{U}$ for which the Mal'tsev fiber $[\mathfrak{U}] = \{\mathfrak{V} \mid \mathfrak{V} \equiv \mathfrak{U}\}$ does not satisfy (4) and is therefore not an $S_{\delta\Sigma}$-class.*

In fact, suppose that any Mal'tsev fiber $[\mathfrak{U}]$ satisfies (4). Consider an arbitrary $\mathcal{S}$-class K. It is known [1] that there is a finitely defined (that is, of finite signature and with finitely many defining identities) manifold $\mathfrak{V}_0$ such that

$$K = \{\mathfrak{V} \mid \mathfrak{V}_0 \xrightarrow{\exists\rho} \mathfrak{V}\}.$$

In view of our assumption, the fiber $[\mathfrak{V}_0]$ must contain K. On the other hand, since K is closed with respect to $\mathcal{S}$-equivalence, $K \supseteq [\mathfrak{V}_0]$. Consequently, $K = [\mathfrak{V}_0]$. Thus, any $\mathcal{M}$-class, being the union of an increasing sequence of Mal'tsev fibers $[\mathfrak{V}_0] \subseteq [\mathfrak{V}_0'] \subseteq [\mathfrak{V}_0''] \subseteq \ldots$, must be an $\mathcal{S}$-class, since the inclusions of the fibers imply that $[\mathfrak{V}_0] = [\mathfrak{V}_0'] = [\mathfrak{V}_0''] = \ldots$. However, there are $\mathcal{M}$-classes (for example, the class of all congruence-distributive manifolds [1]) that are not $\mathcal{S}$-classes.

The resulting contradiction shows that the condition of closure with respect to $\mathcal{S}$-equivalence is not sufficient to characterize $S_{\delta\Sigma}$-classes. Theorem 3.8 of [4] can be revised in the following way.

THEOREM 1. A class K of manifolds of algebras is an $S_{\delta\Sigma}$-class if and only if it satisfies the condition

$$\mathcal{T}_S(\mathfrak{U}) \subseteq \mathcal{T}_S(\mathfrak{V}), \quad \mathfrak{U} \in K \Rightarrow \mathfrak{V} \in K. \tag{5}$$

*In Sec. 2 we show that for any nontrivial manifold $\mathfrak{U}$ of algebras the Mal'tsev fiber $[\mathfrak{U}]$ does not satisfy (4).

Proof. Suppose that K is an $S_{\delta\Sigma}$-class. Then it is representable in the form (3). Let us denote by $\theta_i^{(\lambda)}$ the strict Mal'tsev condition that defines the S-class $L_i^{(\lambda)}$. If $\mathcal{U} \in K$, then $\mathcal{U} \in K_\lambda$ for some λ, and so the conditions $\theta_1^{(\lambda)}, \theta_2^{(\lambda)}, \ldots$ are satisfiable in $\mathcal{U}$. If $\mathcal{T}_s(\mathcal{U}) \subseteq \mathcal{T}_s(\mathcal{V})$, then these conditions are also satisfiable in $\mathcal{V}$, so $\mathcal{V} \in K_\lambda \subseteq K$.

Conversely, suppose that K satisfies (5). Then K is closed with respect to S-equivalence and splits into pairwise disjoint Mal'tsev fibers. Suppose that $\mathcal{U} \in K$ and that $[\mathcal{U}]$ is the fiber containing $\mathcal{U}$. If $\mathcal{T}_s(\mathcal{U}) = \{\theta_1, \theta_2, \ldots\}$ and L_i is the S-class defined by the condition θ_i, then

$$[\mathcal{U}] \subseteq L_1 \cap L_2 \cap \ldots .$$

The intersection $L_{\mathcal{U}} = L_1 \cap L_2 \cap \ldots$ is an S_δ-class contained in K, by (5). Consequently, $K = \bigcup L_{\mathcal{U}} \, (\mathcal{U} \in K)$ is an $S_{\delta\Sigma}$-class.

COROLLARY 1. The following properties are equivalent:

$$S_{\delta\Sigma} \leftrightarrow \mathcal{M}_{\delta\Sigma} \leftrightarrow \mathcal{M}_{\sigma\delta\Sigma} \leftrightarrow S_{\sigma\delta\Sigma}.$$

In fact, $S_{\delta\Sigma} \to \mathcal{M}_{\delta\Sigma} \to \mathcal{M}_{\sigma\delta\Sigma}$, since $S_\delta \to \mathcal{M}_\delta \to \mathcal{M}_{\sigma\delta}$. We show that $\mathcal{M}_{\sigma\delta\Sigma} \to S_{\delta\Sigma}$. By Theorem 1 it is sufficient to show that any $\mathcal{M}_{\sigma\delta\Sigma}$-class K satisfies (5).

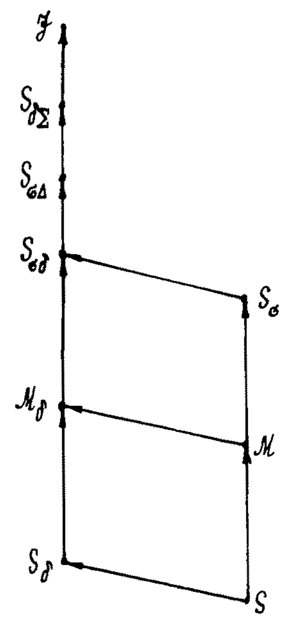
Suppose that K admits a representation of the form (3) in which all the $L_i^{(\lambda)}$ are $\mathcal{M}_\sigma$-classes. Since $\mathcal{M}_\sigma$-classes are unions of S-classes, they satisfy (5). Hence K also satisfies (5).

The equivalence $\mathcal{M}_{\sigma\delta\Sigma} \leftrightarrow S_{\sigma\delta\Sigma}$ follows from the equivalence $\mathcal{M}_\sigma \leftrightarrow S_\sigma$.

Let us agree to call any class of manifolds of algebras that satisfies (4) a $\mathcal{J}$-class.

Taking account of the fact that $\mathcal{M}_{\sigma\Delta} \leftrightarrow S_{\sigma\Delta}$ and any $S_{\sigma\Delta}$-class satisfies (5), we obtain the next corollary.

COROLLARY 2. We have the following diagram of properties:



In Sec. 4 we shall prove that the property $\mathcal{J}$ is distinct from the property $S_{\mathcal{J}\Sigma}$. The fact that the properties $S_{6\Delta}$ and $S_{6\mathcal{J}}$ are distinct was established by Baldwin and Berman [4, Theorem 3.4]. Thus, all the properties in the diagram are pairwise distinct, except possibly the properties $S_{6\Delta}$ and $S_{\mathcal{J}\Sigma}$; it has still not been confirmed that these are distinct.

We draw the attention of the reader to a paper by Czedli [5]. From it, it follows that a class K of manifolds whose lattices of congruences of terms satisfy Jónsson's quasiidentity

$$\theta+\varphi=\theta+\psi \longrightarrow \theta+\varphi=\theta+(\varphi\cap\psi)$$

is an $S_{6\Delta}$-class. Jónsson's question [1] of whether this is an $\mathcal{M}_{\mathcal{J}}$-class is still open.

In this connection we observe that any class of manifolds definable by a universal $\mathcal{P}$-formula [3] satisfies (4) and is thus a $\mathcal{J}$-class. It is not known whether this class is actually an $S_{\mathcal{J}\Sigma}$-class.

2. S-Theories and Representability of Manifolds

From Taylor's theorem for strong Mal'tsev classes [1] it follows that if a manifold $\mathcal{U}$ can be represented in a manifold $\mathcal{V}$, then $\mathcal{T}_S(\mathcal{U})\subseteq\mathcal{T}_S(\mathcal{V})$.

In fact, suppose that $\mathcal{U}\xrightarrow{\exists p}\mathcal{V}$ and that the strict Mal'tsev condition (1) is satisfiable in $\mathcal{U}$. Then $\mathcal{U}$ belongs to the strict Mal'tsev class K determined by (1). Since $\mathcal{U}$ is representable in $\mathcal{V}$, by Taylor's theorem $\mathcal{V}$ also belongs to K. Consequently, (1) is satisfiable in $\mathcal{V}$, and thus $\mathcal{T}_S(\mathcal{U})\subseteq\mathcal{T}_S(\mathcal{V})$.

From this we see that if the manifolds $\mathcal{U}$ and $\mathcal{V}$ are equivalent (that is, $\mathcal{U}\xrightarrow{\exists p}\mathcal{V}$ and $\mathcal{V}\xrightarrow{\exists p}\mathcal{U}$), then they are also S-equivalent ($\mathcal{U}\equiv\mathcal{V}$).

For a finitely based manifold $\mathcal{U}$ the first of these assertions has a converse.

THEOREM 2. If a manifold $\mathcal{U}$ is finitely based, then the inclusion $\mathcal{T}_S(\mathcal{U})\subseteq\mathcal{T}_S(\mathcal{V})$ implies that $\mathcal{U}$ is representable in $\mathcal{V}$.

Proof. Suppose that $\mathcal{T}_S(\mathcal{U})\subseteq\mathcal{T}_S(\mathcal{V})$ and that $\mathcal{U}$ is defined by a finite set Σ of identities of functional symbols f_i $(i\in I)$. It is known [1] that the manifold $\mathcal{U}_0$ obtained from $\mathcal{U}$ by omitting those symbols f_i that do not occur in any identity of Σ is finitely defined and equivalent to $\mathcal{U}$ (that is, $\mathcal{U}_0\xrightarrow{\exists p}\mathcal{U}$ and $\mathcal{U}\xrightarrow{\exists p}\mathcal{U}_0$). Consequently, $\mathcal{U}_0\equiv\mathcal{U}$, and so $\mathcal{T}_S(\mathcal{U}_0)\subseteq\mathcal{T}_S(\mathcal{V})$. Since the relation of representability is transitive, it is sufficient to prove that $\mathcal{U}_0$ is representable in $\mathcal{V}$.

Suppose that the signature of $\mathcal{U}_0$ consists of 0-ary functional symbols (constants) $c_1,\ldots,c_K$ and functional symbols $f_1,\ldots,f_m$ that are at least 1-ary. Following [4], in all the equalities of Σ we replace each constant c_i by $q_i(x)$ with unary functional symbol q_i. We denote by θ the conjunction of all the resulting equalities and the equalities $q_1(x)=q_1(y),\ldots,$ $q_K(x)=q_K(y)$. The strict Mal'tsev condition

$$(\exists f_1)\ldots(\exists f_m)(\exists q_1)\ldots(\exists q_K)(\forall x)(\forall y)\ldots\theta \tag{6}$$

is satisfiable in $\mathcal{U}_0$ and, by virtue of the inclusion $\mathcal{J}_S(\mathcal{U}_0) \subseteq \mathcal{J}_S(\mathcal{V})$, is also satisfiable in $\mathcal{V}$. The fact that (6) is satisfiable in $\mathcal{V}$ implies that $\mathcal{U}_0$ is representable in $\mathcal{V}$.

COROLLARY 1. If $\mathcal{J}_S(\mathcal{U}) \subseteq \mathcal{J}_S(\mathcal{V})$, then all finitely based manifolds that contain $\mathcal{U}$ are representable in $\mathcal{V}$.

In fact, if $\mathcal{U}_0$ is a finitely based manifold and $\mathcal{U} \subseteq \mathcal{U}_0$, then $\mathcal{U}_0 \xrightarrow{\exists p} \mathcal{U}$, so $\mathcal{J}_S(\mathcal{U}_0) \subseteq \mathcal{J}_S(\mathcal{U}) \subseteq \mathcal{J}_S(\mathcal{V})$. By Theorem 2, $\mathcal{U}_0$ is representable in $\mathcal{V}$.

In Sec. 4 we shall show that the condition that $\mathcal{U}$ is finitely based is essential in Theorem 2 and that S-equivalent manifolds need not be equivalent. We shall also show that the fact that all finitely based hypermanifolds of a given manifold $\mathcal{V}$ are representable in $\mathcal{U}$ does not, generally speaking, imply that $\mathcal{U}$ is representable in $\mathcal{V}$.

COROLLARY 2. For any nontrivial manifold $\mathcal{U}$ the Mal'tsev fiber $[\mathcal{U}]$ does not contain trivial manifolds and is therefore not a $\mathcal{J}$-class.

In fact, suppose that $\mathcal{E}$ is a trivial manifold and that $\mathcal{J}_S(\mathcal{E}) = \mathcal{J}_S(\mathcal{U})$. Since $\mathcal{E}$ is finitely based, by Theorem 2 we have $\mathcal{E} \xrightarrow{\exists p} \mathcal{U}$. Consequently, all the algebras of $\mathcal{U}$ consists of one element, which contradicts the fact that $\mathcal{U}$ is nontrivial. Thus, $[\mathcal{U}]$ does not contain trivial manifolds. Since $\mathcal{U}$ has a submanifold $\mathcal{E}_0$ of one-element algebras, $\mathcal{U} \xrightarrow{\exists p} \mathcal{E}_0$ and $\mathcal{E}_0 \notin [\mathcal{U}]$. Consequently, $[\mathcal{U}]$ does not satisfy (4), that is, it is not a $\mathcal{J}$-class.

With each manifold $\mathcal{U}$ of algebras we also associate the classes of manifolds

$$\vec{\mathcal{U}} = \{\mathcal{V} \mid \mathcal{U} \xrightarrow{\exists p} \mathcal{V}\} \quad \text{and} \quad \bar{\mathcal{U}} = \{\mathcal{V} \mid \mathcal{J}_S(\mathcal{U}) \subseteq \mathcal{J}_S(\mathcal{V})\}.$$

Clearly, $\vec{\mathcal{U}} \subseteq \bar{\mathcal{U}}$ and for any finitely based manifold $\mathcal{U}$ we have $\vec{\mathcal{U}} = \bar{\mathcal{U}}$.

Since the relation of representability is transitive, the class $\vec{\mathcal{U}}$ is the smallest $\mathcal{J}$-class to which $\mathcal{U}$ belongs.

The class $\bar{\mathcal{U}}$ is the smallest S_δ-class of which $\mathcal{U}$ is an element. In fact, suppose that $\mathcal{J}_S(\mathcal{U}) = \{\theta_1, \theta_2, \ldots\}$ and that K_i is the S-class defined by the strict Mal'tsev condition θ_i. Then the intersection $K_1 \cap K_2 \cap \ldots$ is an S_δ-class containing $\mathcal{U}$ and $\bar{\mathcal{U}} = K_1 \cap K_2 \cap \ldots$, since if $\mathcal{V} \in K_1 \cap K_2 \cap \ldots$, then $\mathcal{J}_S(\mathcal{U}) \subseteq \mathcal{J}_S(\mathcal{V})$ and so $\mathcal{V} \in \bar{\mathcal{U}}$. By Theorem 1, any S_δ-class to which $\mathcal{U}$ belongs contains $\bar{\mathcal{U}}$ as a subclass.

We also observe that $\bar{\mathcal{U}}$ is an S-class if and only if $\mathcal{U} \equiv \mathcal{U}_0$ for some finitely based manifold $\mathcal{U}_0$. In fact, if $\bar{\mathcal{U}}$ is an S-class, then $\bar{\mathcal{U}} = \vec{\mathcal{U}}_0$ for some finitely based manifold $\mathcal{U}_0$. Since $\vec{\mathcal{U}}_0 = \bar{\mathcal{U}}_0$, we have $\bar{\mathcal{U}} = \bar{\mathcal{U}}_0$, so $\mathcal{U} \equiv \mathcal{U}_0$. Conversely, if $\mathcal{U} \equiv \mathcal{U}_0$ for some finitely based manifold $\mathcal{U}_0$, then, as we mentioned in the proof of Theorem 2, $\mathcal{U} \equiv \mathcal{U}_0'$ for some finitely defined manifold $\mathcal{U}_0'$. Consequently, $\bar{\mathcal{U}} = \bar{\mathcal{U}}_0' = \vec{\mathcal{U}}_0'$ is an S-class.

Baldwin and Berman [4] constructed an example of an $S_{\delta\sigma}$-class that is not an $S_{\sigma\delta}$-class, but the question of the truth of the implication $S_{\sigma\delta} \to S_{\delta\sigma}$ remains open. In this connection we have the following corollary.

COROLLARY 3. Any $S_{\delta\Sigma}$ -class K that contains only a countable set of Mal'tsev fibers is an $S_{\delta\sigma}$ -class. In particular, any countably fibered $S_{\sigma\delta}$ -class is an $S_{\delta\sigma}$ -class.

In fact, if in each fiber of the class K we choose a manifold $\mathfrak{U}_i$, we find that $K=\bigcup\overline{\mathfrak{U}}_i$ is the union of a countable set of S_δ -classes $\overline{\mathfrak{U}}_i$.

We observe that in any $S_{\delta\Sigma}$ -class K the set $\{\overline{\mathfrak{U}}|\mathfrak{U}\in K\}$ of S_δ-subclasses $\overline{\mathfrak{U}}$ always has the same cardinality as the set $\{[\mathfrak{U}]|\ \mathfrak{U}\in K\}$ of fibers $[\mathfrak{U}]$. In fact, the equality $\overline{\mathfrak{U}}=\overline{\mathfrak{V}}$ implies that $\mathfrak{U}\equiv\mathfrak{V}$, and so the correspondence $[\mathfrak{U}]\to\overline{\mathfrak{U}}$ is one-to-one. Since the set of strong Mal'tsev classes is countable, we see that the cardinality of the set of fibers in any $S_{\delta\Sigma}$ -class does not exceed 2^ω. We do not know whether this cardinality is actually obtainable. We shall give an example of the nontriviality of a finitely fibered S_δ -class $\overline{\mathfrak{U}}$ in Sec. 3.

3. Representability in Post Manifolds and the Problem of the Completeness of S -Theories

Let A be an infinite set of cardinality α and Ω_A the set of functions on A consisting of all one-place functions $g: A\to A$ and any two-place Cantor function $n: A^2\to A$. Thus, together with the function $n(x,y)$ the set Ω_A also contains the functions ℓx and $r x$, which satisfy the following identity relations on A:

$$\ell n(x,y)=x,\quad r n(x,y)=y,\quad n(\ell x,\ r x)=x.$$

It is known [6, Sec. 4] that in the Post iterative algebra $\mathcal{P}_A$ over A the set Ω_A is complete, that is, it generates the whole algebra $\mathcal{P}_A$. In fact, suppose that

$$n^2(x,y)=n(x,y),\ n^m(x_1,\dots,x_m)=n^{m-1}(n(x_1,x_2),x_3,\dots,x_m)\ (m\geqslant 3),$$

$$\begin{aligned}
\ell_{m1}x &= \ell\ell\dots\ell\ell x && (\ell \text{ occurs } m-1 \text{ times}),\\
\ell_{m2}x &= r\ell\dots\ell\ell x && (\ell \text{ occurs } m-2 \text{ times}),\\
&\dots\dots\dots\dots\\
\ell_{m,m-1}x &= r\ell x,\\
\ell_{mm}x &= r x.
\end{aligned}$$

If $f: A^m\to A$ is an arbitrary function and

$$g_f x = f(\ell_{m1}x,\dots,\ell_{mm}x),$$

then the function $g_f x$ belongs to the set Ω_A and for all $x_1,\dots,x_m$ of A we have

$$f(x_1,\dots,x_m)=g_f n^m(x_1,\dots,x_m). \tag{7}$$

We shall consider the algebra $\langle A,\Omega_A\rangle$ with support A and set of basic operations Ω_A in some signature Ω of cardinality 2^α.

The manifold $\mathcal{P}_A=\mathrm{var}(\langle A,\Omega\rangle)$, generated by the algebra $\langle A,\Omega\rangle$, is called a Post manifold.

THEOREM 3. Any nontrivial manifold $\mathcal{U}$ of algebras is representable in the Post manifold $\mathcal{P}_A$ constructed for any infinite algebra A of $\mathcal{U}$.

Proof. Let Φ be the signature of $\mathcal{U}$, and $\langle A,\Phi\rangle$ an arbitrary infinite algebra of $\mathcal{U}$. For each functional symbol $f\in\Phi$ in the algebra $\langle A,\Phi\rangle$ there is defined a basic operation $f(x_1,\ldots,x_m)$, representable in the form (7). We show that the collection of Ω-terms $g_f n^m(x_1,\ldots,x_m)$ $(f\in\Phi)$, which we write briefly in the form $\{g_f n^m \mid f\in\Phi\}$, serves as a representation for $\mathcal{U}$ in $\mathcal{P}_A$. For this we need to prove that for each algebra $B=\langle B,\Omega\rangle$ of the Post manifold

$$\mathcal{P}_A = HSP(\langle A,\Omega\rangle)$$

the algebra $\langle B,\{g_f n^m \mid f\in\Phi\}\rangle$ with the collection of polynomial operations $g_f n^m(x_1,\ldots,x_m)$ $(f\in\Phi)$, defined by the Ω-terms $g_f n^m$ $(f\in\Phi)$, belongs to $\mathcal{U}$.

If $B=A$, then, by (7) $\langle B,\{g_f n^m \mid f\in\Phi\}\rangle$ is an algebra $\langle A,\{f \mid f\in\Phi\}\rangle$ of $\mathcal{U}$.

Let $B=\Pi\langle A,\Omega\rangle$ be a Cartesian power of the algebra $\langle A,\Omega\rangle$. Then for any elements $\bar{x}_k=(x_1^k,x_2^k,\ldots)$ $(k=1,\ldots,m)$ of B we have

$$g_f n^m(\bar{x}_1,\ldots,\bar{x}_m)=(g_f n^m(x_1^1,\ldots,x_m^1),g_f n^m(x_1^2,\ldots,x_m^2),\ldots)=(f(x_1^1,\ldots,x_m^1),f(x_1^2,\ldots,x_m^2),\ldots).$$

Consequently, $\langle B,\{g_f n^m \mid f\in\Phi\}\rangle$ is an algebra $\Pi\langle A,\{f \mid f\in\Phi\}\rangle$ of $\mathcal{U}$.

Suppose that for an algebra $B=\langle B,\Omega\rangle$ of $\mathcal{P}_A$ the algebra $\langle B,\{g_f n^m \mid f\in\Phi\}\rangle$ belongs to $\mathcal{U}$. We first consider a subalgebra $\langle C,\Omega\rangle$ of the algebra $\langle B,\Omega\rangle$. The subset $C\subseteq B$, being closed with respect to operations of Ω, is also closed with respect to all polynomial operations defined by the Ω-terms $g_f n^m$ $(f\in\Omega)$. Consequently, $\langle C,\{g_f n^m \mid f\in\Phi\}\rangle$ is a subalgebra of the algebra $\langle B,\{g_f n^m \mid f\in\Phi\}\rangle$ of $\mathcal{U}$ and so it also belongs to $\mathcal{U}$.

We now consider a congruence θ of the algebra $\langle B,\Omega\rangle$. Since congruences are stable with respect to all polynomial operations, θ is a congruence of the algebra $\langle B,\{g_f n^m \mid f\in\Phi\}\rangle$. Since the latter algebra belongs to $\mathcal{U}$, we see that for the factor algebra $\langle B/\theta,\Omega\rangle$ the derived algebra $\langle B/\theta,\{g_f n^m \mid f\in\Phi\}\rangle$ in the representation under investigation also belongs to $\mathcal{U}$. This proves Theorem 3.

COROLLARY. If the sets A and B are infinite and have the same cardinality, then the Post manifolds $\mathcal{P}_A$ and $\mathcal{P}_B$ are equivalent (that is, $\mathcal{P}_A \xrightarrow{\exists\rho} \mathcal{P}_B$ and $\mathcal{P}_B \xrightarrow{\exists\rho} \mathcal{P}_A$).

In fact, by virtue of the equality $|A|=|B|$, $\mathcal{P}_B$ has an algebra with support A and, by Theorem 3, it is representable in $\mathcal{P}_A$. Similarly, $\mathcal{P}_A$ is representable in $\mathcal{P}_B$.

The cardinality $\alpha=|A|$ is called the order of $\mathcal{P}_A$.

Henceforth a Post manifold of infinite order α will be considered up to equivalence and denoted by $\mathcal{P}_\alpha$.

The aggregate $\mathcal{T}$ of strict Mal'tsev conditions will be called a Mal'tsev theory (or briefly an S-theory) if there is a manifold $\mathcal{V}$ of algebras such that $\mathcal{T}=\mathcal{T}_S(\mathcal{V})$.

An $\mathcal{S}$-theory $\mathcal{T}$ will be called trivial if $\mathcal{T}=\mathcal{T}_S(\mathcal{E})$, where $\mathcal{E}$ is a manifold of one-element algebras.

By analogy with equational theories, an $\mathcal{S}$-theory $\mathcal{T}$ is naturally said to be complete if it is nontrivial and for any two nontrivial manifolds $\mathcal{U}$ and $\mathcal{V}$ the inclusions $\mathcal{T}\subseteq\mathcal{T}_S(\mathcal{U})$ and $\mathcal{T}\subseteq\mathcal{T}_S(\mathcal{V})$ imply that $\mathcal{T}_S(\mathcal{U})=\mathcal{T}_S(\mathcal{V})$.

It is clear that an $\mathcal{S}$-theory $\mathcal{T}_S(\mathcal{U})$ of a manifold $\mathcal{U}$ is complete if and only if $\mathcal{U}$ is nontrivial and the $\mathcal{S}_\mathcal{J}$-class $\overline{\mathcal{U}}$ consists of exactly two fibers, the fiber $[\mathcal{E}]$ of trivial manifolds and the fiber $[\mathcal{U}]=\{\mathcal{V}\mid\mathcal{V}\equiv\mathcal{U}\}$.

THEOREM 4. All Post manifolds of infinite order are $\mathcal{S}$-equivalent and have a complete $\mathcal{S}$-theory $\mathcal{T}_S(\mathcal{P}_\omega)$, which contains any other nontrivial $\mathcal{S}$-theory.

Proof. Consider a Post manifold $\mathcal{P}_\alpha$ of arbitrary infinite order α, and suppose that a strict Mal'tsev condition θ is satisfiable in it. The class K_θ of manifolds defined by this condition is strong, and so $K_\theta=\vec{\mathcal{U}}_0$ for some finitely defined manifold $\mathcal{U}_0$. Since $\mathcal{P}_\alpha\in K_\theta$, we have $\mathcal{U}_0\xrightarrow{\exists\mathcal{P}}\mathcal{P}_\alpha$. Consequently, $\mathcal{U}_0$ is nontrivial and so it has an infinite countable algebra (for example, a $\mathcal{U}_0$-free algebra of rank ω). By Theorem 3, $\mathcal{U}_0$ is representable in the Post manifold $\mathcal{P}_\omega$. Consequently, $\mathcal{P}_\omega\in K_\theta$, that is, the condition $\mathcal{P}_\omega$ is satisfiable in θ. Since the choice of θ in $\mathcal{T}_S(\mathcal{P}_\alpha)$ is arbitrary, we obtain $\mathcal{T}_S(\mathcal{P}_\alpha)\subseteq\mathcal{T}_S(\mathcal{P}_\omega)$. On the other hand, $\mathcal{P}_\omega$, like any other manifold that has an infinite countable algebra, has algebras of any infinite cardinality (see [7, Corollary 6]). By Theorem 3, $\mathcal{P}_\omega$ is representable in $\mathcal{P}_\alpha$, so $\mathcal{T}_S(\mathcal{P}_\omega)\subseteq\mathcal{T}_S(\mathcal{P}_\alpha)$. Thus, $\mathcal{T}_S(\mathcal{P}_\alpha)=\mathcal{T}_S(\mathcal{P}_\omega)$ for any infinite cardinal α.

Suppose that $\mathcal{T}_S(\mathcal{P}_\omega)\subseteq\mathcal{T}_S(\mathcal{U})$ for some nontrivial manifold $\mathcal{U}$. By Theorem 3, $\mathcal{U}$ is representable in $\mathcal{P}_\alpha$ for some cardinal α. Consequently, $\mathcal{T}_S(\mathcal{U})\subseteq\mathcal{T}_S(\mathcal{P}_\alpha)=\mathcal{T}_S(\mathcal{P}_\omega)$. Thus, $\mathcal{T}_S(\mathcal{U})=\mathcal{T}_S(\mathcal{P}_\omega)$, that is, the $\mathcal{S}$-theory $\mathcal{T}_S(\mathcal{P}_\omega)$ is complete.

Finally, if $\mathcal{T}_S(\mathcal{V})$ is an arbitrary nontrivial $\mathcal{S}$-theory, then $\mathcal{V}$ is representable in $\mathcal{P}_\alpha$ for some α, by Theorem 3. Hence, $\mathcal{T}_S(\mathcal{V})\subseteq\mathcal{T}_S(\mathcal{P}_\alpha)=\mathcal{T}_S(\mathcal{P}_\omega)$. This proves Theorem 4.

Thus, there is a unique complete Mal'tsev theory, the $\mathcal{S}$-theory of Post manifolds of infinite order.

It is known [8] that all Post manifolds of finite order are arithmetic (that is, congruence-commutative and congruence-distributive). From Theorem 4 and the definition of Post algebras of infinite order we also obtain the following corollary.

COROLLARY. All Post manifolds of infinite order are arithmetic, and in each of them all free algebras $F_\tau(\mathcal{P}_\alpha)$ of finite rank $\tau=1,2,\dots$ are isomorphic.

4. Counterexamples and Some Corollaries

First of all we show that $\mathcal{S}$-equivalent manifolds $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ need not be equivalent.

THEOREM 5. If the infinite cardinals α and β satisfy the inequality $2^\alpha\leqslant\beta$, then the Post manifolds $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ are not equivalent (namely, $\mathcal{P}_\beta$ cannot be represented in $\mathcal{P}_\alpha$).

Proof. Suppose that $\mathcal{P}_\beta$ is representable in $\mathcal{P}_\alpha$. Consider the manifold $\mathcal{M}_F$ of left F-modules over a field F-modules over a field $\mathbf{F}$ of cardinality β. By Theorem 3, $\mathcal{M}_F$ is representable in $\mathcal{P}_\beta$. Since the relation of representability is transitive, $\mathcal{M}_F$ is representable in $\mathcal{P}_\alpha$. However, this is impossible, since any nonzero F-module M has cardinality $|M|>\alpha$ by virtue of the relations

$$|M| \geqslant |F| = \beta \geqslant 2^\alpha > \alpha.$$

COROLLARY 1. The $\mathcal{J}$-class $\vec{\mathcal{P}}_\alpha$ is not an $S_{\delta\Sigma}$-class for any $\alpha \geqslant 2^\omega$.

In fact, the $\mathcal{J}$-class $\vec{\mathcal{P}}_\alpha$ does not contain the manifold $\mathcal{P}_\omega$ and so it is not closed with respect to S-equivalence. By Theorem 1, $\vec{\mathcal{P}}_\alpha$ is not an $S_{\delta\Sigma}$-class.

In particular, for any $\alpha \geqslant 2^\omega$ we have the strict inclusion

$$\vec{\mathcal{P}}_\alpha \subset \bar{\mathcal{P}}_\alpha .$$

Since for any finitely based manifold $\mathcal{U}$ we have $\vec{\mathcal{U}} = \bar{\mathcal{U}}$ by Theorem 2, we obtain the next corollary.

COROLLARY 2. For any cardinal $\alpha \geqslant 2^\omega$ the equational theory of the Post manifold $\mathcal{P}_\alpha$ does not have a finite basis of identities.

We do not know whether the Post manifold $\mathcal{P}_\omega$ has a finite basis of identities or whether the S_δ-class $\vec{\mathcal{P}}_\omega$ is an S-class.

In conclusion, we show that "local representability" of a manifold does not imply "representability in the large."

We have already mentioned in Sec. 2 that any finitely based manifold $\mathcal{U}$ is equivalent to some finitely defined manifold $\mathcal{U}_0$. If $\mathcal{U}_0$ is nontrivial, then it has an infinite countable algebra and, by Theorem 3, it is representable in $\mathcal{P}_\omega$. Consequently, if $\alpha \geqslant 2^\omega$, then all finitely based manifolds that contain the Post manifold $\mathcal{P}_\alpha$ are representable in $\mathcal{P}_\omega$ whereas, by Theorem 5, $\mathcal{P}_\alpha$ is not representable in $\mathcal{P}_\omega$.

Finally, we observe that Mal'tsev theories form by inclusion a lattice, which the author proposes to consider in a later article.

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