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SEMILATTICES OF COMPUTABLE INDEXATIONS OF CLASSES OF CONSTRUCTIVE MODELS

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The article establishes a connection between the structure of some subsemilattices of computable numerations of a suitable family of recursively enumerable sets. In particular, we prove that the semilattice of computable indexations of a class of finite models is isomorphic to the semilattice of computable numerations of some effectively definable family of recursively enumerable sets. We provide one sufficient condition for the existence of countably many incomparable elements in the semilattice of computable indexations of a class of constructive models.

We adopt definitions and notation from [1]-[4]. Let us recall some of these notions. We denote a constructive model by $(\mathfrak{M}_\nu, \nu)$, a class of constructive models by $\mathcal{K}^*$, and the corresponding class of abstract models (without constructivizations) of signature

$$\sigma = \{P_i^{n_i}, F_j^{m_j} \mid i \in I,\ j \in J\}.$$

by $\mathcal{K}$. Here, I and J either are finite or coincide with N, $P_i^{n_i}$ is a predicate of arity n_i, P_0^2 is the equality predicate, $F_j^{m_j}$ is a function of arity m_j, and n_i, m_j are general recursive functions of their indices.

A class $\mathcal{K}^*$ is said to be computable if there exist a map $\gamma : N^2 \to \bigcup_{\mathfrak{M} \in \mathcal{K}} |\mathfrak{M}|$ and computable families of recursive predicates $\mathbb{P} = \{p_i(n, x_0, x_1, \ldots, x_{n_i - 1}) \mid i \in I\}$ and general recursive functions $\mathcal{F} = \{f_j(n, x_0, x_1, \ldots, x_{m_j - 1}) \mid j \in J\}$ such that

1) for each fixed value $n \in N$ the numeration $\gamma_n(x) = \gamma(n, x)$ is a constructive numeration of some model $\mathfrak{M} \in \mathcal{K}$ with the families $\mathbb{P}$ and $\mathbb{F}$ as the corresponding predicates and functions on the numbers of elements of the model $\mathfrak{M}$;

2) for each constructive model $(\mathfrak{M}_\nu, \nu) \in \mathcal{K}^*$ there exists a value n for which the numerations γ_n and ν are autoequivalent ($\gamma_n \underset{a}{\equiv} \nu$);

3) for each n there is a constructive model $(\mathfrak{M}_\nu, \nu) \in \mathcal{K}^*$ such that $\gamma_n \underset{a}{\equiv} \nu$.

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Let $\mathcal{K}_0^* \subseteq \mathcal{K}^*$. The set

$$I^{\gamma}(\mathcal{K}_0^*) \leftrightharpoons \{n \mid (\mathfrak{M}_{\gamma_n}, \gamma_n) \in \mathcal{K}_0^*\}$$

is called the index set of the subclass $\mathcal{K}_0^*$ in the computable indexation γ of the class $\mathcal{K}^*$.

A model $\mathfrak{M}$ is said to be locally embedded in a model $\mathfrak{N}$ if each finite submodel $\mathfrak{M}_p$ of the model $\mathfrak{M}$ can be isomorphically embedded in the model $\mathfrak{N}$ $(\mathfrak{M} \xrightarrow{l} \mathfrak{N})$. If neither model can be locally embedded in the other model, these models are said to be finitely disginguishable.

The local cone defined by a model $(\mathfrak{M}_\nu, \nu)$ is the subclass $\mathcal{K}_\nu^* \leftrightharpoons \{(\mathfrak{M}_\alpha, \alpha) \mid (\mathfrak{M}_\alpha, \alpha) \in \mathcal{K}^*, \mathfrak{M}_\alpha \xrightarrow{l} \mathfrak{M}_\nu\}$ of the class $\mathcal{K}^*$. The local subclass defined by a model ($\mathfrak{M}_\nu, \nu$) is the subclass $L_\nu^* \leftrightharpoons \{(\mathfrak{M}_\alpha, \alpha) \mid (\mathfrak{M}_\alpha, \alpha) \in \mathcal{K}^*, \mathfrak{M}_\alpha \equiv_l \mathfrak{M}_\nu\}$.

Let η and γ be computable indexations of a class $\mathcal{K}^*$. We say that the indexation η is reduced to the indexation γ $(\eta \leq \gamma)$ if there exists a general recursive function $f(x)$ such that for all i the condition $\eta_i \underset{a}{\equiv} \gamma_{f(i)}$ holds. We say that the indexation η is reduced to γ in local classes ($\eta \underset{\text{l.c.}}{\leq} \gamma$) if there exists a general recursive function φ such that for each ν the inclusion $\varphi(I^{\eta}(L_\nu^*)) \subseteq I^{\gamma}(L_\nu^*)$ holds. If $\eta \underset{\text{l.c.}}{\leq} \gamma$ and $\gamma \underset{\text{l.c.}}{\leq} \eta$, then $\eta \underset{\text{l.c.}}{\equiv} \gamma$, i.e., the computable indexations η and γ are equivalent relative to reducibility in local classes. Equivalence of computable indexations relative to the usual reducibility is defined similarly.

We denote by $H(\mathcal{K}^*)$ the set of all computable indexations of the class $\mathcal{K}^*$. The relations $\equiv$ and $\underset{\text{l.c.}}{\equiv}$ are equivalence relations on the set $H(\mathcal{K}^*)$. For $\alpha \in H(\mathcal{K}^*)$ we denote the corresponding classes of equivalent elements by $\bar{\alpha}$ and $\tilde{\alpha}$, i.e.

$$\bar{\alpha} \leftrightharpoons \{\gamma \mid \gamma \in H(\mathcal{K}^*),\ \gamma \equiv \alpha\},$$

$$\tilde{\alpha} \leftrightharpoons \{\gamma \mid \gamma \in H(\mathcal{K}^*),\ \gamma \underset{\text{l.c.}}{\equiv} \alpha\}.$$

The quotient set

$$L(\mathcal{K}^*) = H(\mathcal{K}^*)/_{\equiv} = \{\bar{\alpha} \mid \alpha \in H(\mathcal{K}^*)\}$$

with respect to the equivalence induced by reducibility of indexations forms an upper semilattice which is denoted by $\mathcal{L}(\mathcal{K}^*)$. The subsemilattice consisting of classes $\bar{\gamma} \leq \bar{\alpha}$ of the lattice $\mathcal{L}(\mathcal{K}^*)$ is denoted by $\mathcal{L}(\mathcal{K}^*, \alpha)$. If $\alpha \leq \beta$, then the subsemilattice consisting of classes $\bar{\gamma}$ of the lattice $\mathcal{L}(\mathcal{K}^*)$ satisfying the conditions $\bar{\alpha} \leq \bar{\gamma} \leq \bar{\beta}$ is denoted by $\mathcal{L}(\mathcal{K}^*, \alpha, \beta)$. If usual reducibility $\leq$ of computable indexations is replaced in the above definitions by the reducibility $\underset{\text{l.c.}}{\leq}$ in local classes, then we obtain the definitions of the semilattices $\mathcal{L}_{\text{l.c.}}(\mathcal{K}^*)$, $\mathcal{L}_{\text{l.c.}}(\mathcal{K}^*, \alpha)$ and $\mathcal{L}_{\text{l.c.}}(\mathcal{K}^*, \alpha, \beta)$.

THEOREM 1. For each computable class $\mathcal{K}^*$ of finite models there exists a computable family S of recursively enumerable sets such that $\mathcal{L}(\mathcal{K}^*) \cong \mathcal{L}(S)$. The family S is found effectively from the class $\mathcal{K}^*$.

Without loss of generality, we may assume that the signature σ of the studied models consists only of predicate symbols. We denote by $\sigma_n \subseteq \sigma$ a part of the signature containing no more than n predicate symbols of σ. We assume that $\sigma_1 \subseteq \sigma_2 \subseteq \sigma_3 \subseteq \ldots \subseteq \sigma_n \subseteq \sigma_{n+1} \subseteq \ldots \subseteq \sigma$ and $\sigma = \bigcup_{n=1}^{\infty} \sigma_n$. There are finitely many models of a finite cardinality K and a finite signature σ_n. Such models are isomorphic if and only if their diagrams coincide up to labeling their elements. Clearly, the verification whether such models are isomorphic or, equivalently, their diagrams coincide is completely effective.

One can choose a one-valued Gödel numeration of finite models of cardinality K of finite signatures σ_n such that one first enumerates all models of the signature σ_1, then those of σ_2, etc., in the order of increase of the number n of the signature σ_n. If, beginning with some n, we have $\sigma_n = \sigma_{n+\ell}$, then $\sigma_n = \sigma$ and there are finitely many models of this signature.

Suppose that $\bigcup_{i=1}^{\infty} R_i = N$ is an effective partition of the set N of natural numbers into an infinite sequence of infinite recursive sets R_i. We fix a Gödel numeration of finite models of finite signatures σ_n such that the numeration method of models of cardinality K described above uses only numbers in the set R_K. For the sake of brevity, we denote by $\mathfrak{M}_z$ the finite model with the Gödel number z in this fixed numeration.

We assign to an arbitrary model $\mathfrak{M}$ of the signature σ the set $S_{\mathfrak{M}} \leftrightharpoons \{z \mid \mathfrak{M}_z \hookrightarrow \mathfrak{M}\}$.

<u>LEMMA 1</u>. If $(\mathfrak{M}, \nu)$ is a constructive model, then the set $S_{\mathfrak{M}}$ is recursively enumerable and the enumerating function is effectively determined by the constructivization ν.

We denote by $\mathfrak{M}^t$ the submodel of a model $\mathfrak{M}$ of a bounded signature σ_t with the underlying set $\{\nu(0), \ldots, \nu(t)\}$. The set $S_{\mathfrak{M}^t} = \{z \mid \mathfrak{M}_z \hookrightarrow \mathfrak{M}^t\}$ is finite and found effectively by $\mathfrak{M}^t$. Since $S_{\mathfrak{M}} = \bigcup_{t=0}^{\infty} S_{\mathfrak{M}^t}$, the algorithm of enumeration of this set is obvious.

<u>LEMMA 2</u>. If $S_{\mathfrak{M}}$ is a recursively enumerable set corresponding to a finite model $\mathfrak{M}$, then the model $\mathfrak{M}$ is constructivizable and a constructivization ν of this model is effectively determined by an enumeration of the set $S_{\mathfrak{M}}$.

Suppose that a general recursive function $z(i) = z_i$ is given enumerating the set $S_{\mathfrak{M}} = \{z_0, z_1, \ldots\}$. We will construct the constructivization ν.

<u>Step 0</u>. We enumerate the elements of the model $\mathfrak{M}_{z_0}$, and define on them all predicates in the signature σ_{z_0}. Put $\varphi(0) = z_0$. Turn to the next step of the construction.

<u>Step $t+1$</u>. a) If $\mathfrak{M}_{\varphi(t)} \hookrightarrow \mathfrak{M}_{z_{t+1}}$, then put $\varphi(t+1) = z_{t+1}$. Enumerate the elements of $\mathfrak{M}_{z_{t+1}}$. Restore the diagram of this model in the signature $\sigma_{z_{t+1}}$ taking into account the embedding $\mathfrak{M}_{\varphi(t)} \hookrightarrow \mathfrak{M}_{z_{t+1}}$. Turn to the next step.

b) If $\mathfrak{M}_{\varphi(t)} \not\hookrightarrow \mathfrak{M}_{z_{t+1}}$, then we find the least K such that $\mathfrak{M}_{z_{t+1}} \hookrightarrow \mathfrak{M}_{z_K}$, $\mathfrak{M}_{\varphi(t)} \hookrightarrow \mathfrak{M}_{z_K}$. Put $\varphi(t+1) = z_K$. Enumerate the elements of the model $\mathfrak{M}_{z_K}$ and restore its diagram in the finite signature σ_{z_K} using the embedding $\mathfrak{M}_{\varphi(t)} \hookrightarrow \mathfrak{M}_{z_K}$. Turn to the next step of the construction.

Clearly, $\mathfrak{M}_{\varphi(0)} \subseteq \mathfrak{M}_{\varphi(1)} \subseteq \ldots$ and $\mathfrak{M} = \bigcup_{t=0}^{\infty} \mathfrak{M}_{\varphi(t)}$. The construction process provides a constructive numeration ν of the model $\mathfrak{M}$. The lemma is proved.

LEMMA 3. Finite models $(\mathfrak{M}, \nu)$ and $(\mathfrak{N}, \mu)$ are constructively isomorphic if and only if $S_{\mathfrak{M}} = S_{\mathfrak{N}}$ is a recursively enumerable set.

Indeed, the constructive isomorphism of $(\mathfrak{M}, \nu)$ and $(\mathfrak{N}, \mu)$ implies that $\mathfrak{M} \cong \mathfrak{N}$ but then we have $\mathfrak{M}_z \hookrightarrow \mathfrak{M} \Longleftrightarrow \mathfrak{M}_z \hookrightarrow \mathfrak{N}$, i.e., $z \in S_{\mathfrak{M}} \Longleftrightarrow z \in S_{\mathfrak{N}}$. By Lemma 1, we conclude that the set $S_{\mathfrak{M}}$ is recursively enumerable.

Suppose that recursively enumerable sets $S_{\mathfrak{M}} = S_{\mathfrak{N}}$ are given. According to Lemma 2, from the enumerable set $S_{\mathfrak{M}}$ we can effectively find a constructivization ν of the model $\mathfrak{M}$. Similarly, from the enumeration of the set $S_{\mathfrak{N}}$ we find a constructivization μ of the model $\mathfrak{N}$. The models $\mathfrak{M}$ and $\mathfrak{N}$ are finite; therefore, finitely many extensions were made in the number of elements while the remaining extensions were made in the signature. If $m = \|\mathfrak{M}\|$, then $m = max\{\ell \mid S_{\mathfrak{M}} \cap R_\ell \neq \emptyset\}$. If $n = max\{\ell \mid S_{\mathfrak{N}} \cap R_\ell \neq \emptyset\}$, then the equality $S_{\mathfrak{M}} = S_{\mathfrak{N}}$ implies that $m = n$. For a model of cardinality m and a given signature σ_K the only element in the intersection $S_{\mathfrak{M}} \cap R_m$ is the number of the model $\mathfrak{M}\restriction_{\sigma_K}$. But $S_{\mathfrak{M}} \cap R_m = S_{\mathfrak{N}} \cap R_m$ and for the model $\mathfrak{N}\restriction_{\sigma_K}$ there also is a unique number in the set $S_{\mathfrak{N}} \cap R_m$. Thus, the numbers of the models $\mathfrak{M}\restriction_{\sigma_K}$ and $\mathfrak{N}\restriction_{\sigma_K}$ coincide, i.e., $\mathfrak{M}\restriction_{\sigma_K} \cong \mathfrak{N}\restriction_{\sigma_K}$. Considering these isomorphisms from the moment when all elements of the sets $|\mathfrak{M}|$ and $|\mathfrak{N}|$ have been enumerated, we obtain a constructive isomorphism between $(\mathfrak{M}, \nu)$ and $(\mathfrak{N}, \mu)$. The lemma is proved.

We introduce a family of recursively enumerable sets:

$$S \leftrightharpoons \{ S_{\mathfrak{M}_\nu} \mid (\mathfrak{M}_\nu, \nu) \in \mathcal{K}^* \} .$$

According to Lemma 1, each computable indexation γ determines a computable numeration γ' of the family S. Lemmas 2 and 3 show that this correspondence is one-to-one.

Let α and β be computable indexations of a class $\mathcal{K}^*$ with $\alpha \leq \beta$ and let $f(x)$ be the corresponding general recursive function. Then for each n we have $\alpha_n \underset{a}{\equiv} \beta_{f(n)}$. By virtue of Lemma 3, we obtain $\alpha'_n = \beta'_{f(n)}$, i.e., $\alpha' \leq \beta'$. The converse is also true. So $\alpha \leq \beta \Longleftrightarrow \alpha' \leq \beta'$ and $\alpha \equiv \beta \Longleftrightarrow \alpha' \equiv \beta'$. Thus, the map $\varphi \colon \mathcal{L}(\mathcal{K}^*) \longrightarrow \mathcal{L}(S)$ defined by the rule $\varphi(\overline{\gamma}) = \overline{\gamma'}$ is a semilattice isomorphism. The theorem is proved.

For a class $\mathcal{K}^*$ of constructive models we introduce the following notation for the family of recursively enumerable sets:

$$S(\mathcal{K}^*) \leftrightharpoons \{S_{\mathfrak{M}_\nu} \mid (\mathfrak{M}_\nu, \nu) \in \mathcal{K}^*\}.$$

COROLLARY. If $\mathcal{K}^*$ is a class of constructive finite models, then the class $\mathcal{K}^*$ is computable if and only if $S(\mathcal{K}^*)$ is a computable family.

THEOREM 2. If $\mathcal{K}^*$ is a computable class of finitely distinguishable constructive models and α is its computable indexation, then

$$\mathcal{L}(\mathcal{K}^*, \alpha) \cong \mathcal{L}(S(\mathcal{K}^*), \alpha')$$

for a suitable computable numeration α' of the family $S(\mathcal{K}^*)$.

Note that Lemma 1 in Theorem 1 has been proved for an arbitrary constructive model. Furthermore, if $(\mathfrak{M}_\nu, \nu)$ is constructively isomorphic to a model $(\mathfrak{M}_\mu, \mu)$, then $S_{\mathfrak{M}_\nu} = S_{\mathfrak{M}_\mu}$. Thus, from each computable indexation α of the class $\mathcal{K}^*$ one can effectively determine a computable numeration α' of the family $S(\mathcal{K}^*)$.

We define a map $\varphi: \mathcal{L}(\mathcal{K}^*, \alpha) \to \mathcal{L}(S(\mathcal{K}^*), \alpha')$ as follows. If $\beta \leq \alpha$, then $\varphi(\bar{\beta}) = \overline{\beta'}$. Let us verify that it is a semilattice isomorphism.

Suppose that $\beta \leq \alpha$, $\gamma \leq \alpha$ are computable indexations of the class $\mathcal{K}^*$ and $\varphi(\bar{\beta}) = \varphi(\bar{\gamma})$. But then $\beta' \leq \gamma'$ and $\gamma' \leq \beta'$. The reducibility $\beta' \leq \gamma'$ has a general recursive function $f(x)$ such that for each n we have $\beta'_n = \gamma'_{f(n)}$. So for all n we have $S_{\mathfrak{M}_{\beta_n}} = S_{\mathfrak{M}_{\gamma_{f(n)}}}$. Models $(\mathfrak{M}_{\beta_n}, \beta_n), (\mathfrak{M}_{\gamma_{f(n)}}, \gamma_{f(n)})$ lie in class $\mathcal{K}^*$ of finitely distinguishable models and are not distinguished by finite models. Therefore, they are constructively isomorphic, i.e., $\beta_n \underset{a}{\equiv} \gamma_{f(n)}$. Since this autoequivalence holds for all values of n, we deduce that β is reduced to γ by means of the function f.

Similarly, $\gamma' \leq \beta'$ implies the reducibility $\gamma \leq \beta$. Thus, we have shown that the map φ is one-to-one and preserves order on semilattices, i.e.

$$\bar{\beta} \leq \bar{\gamma} \iff \varphi(\bar{\beta}) \leq \varphi(\bar{\gamma}).$$

It remains to verify that φ is a map onto the semilattice $\mathcal{L}(S(\mathcal{K}^*), \alpha')$. Let β' be a computable numeration of the family $S(\mathcal{K}^*)$ reduced to α' by means of a general recursive function $f(x)$. Then $\beta'_n = \alpha'_{f(n)} = S_{\mathfrak{M}_{\alpha_{f(n)}}}$. We define a computable indexation $\gamma_n = \alpha_{f(n)}$. Since $\mathcal{K}^*$ is a class of finitely distinguishable models, the correspondence $\mathfrak{M}_\nu \longleftrightarrow S_{\mathfrak{M}_\nu}$ is one to one. The numeration β' effectively enumerates all sets in the class $S(\mathcal{K}^*)$, so γ would index all constructive models in the class $\mathcal{K}^*$. The reducibility $\gamma \leq \alpha$ is obvious from the definition. It is easily seen that $\gamma' \equiv \beta'$. So $\varphi(\bar{\gamma}) = \overline{\beta'}$. The theorem is proved.

THEOREM 3. Let $\mathcal{K}^*$ be a class of constructive models, let $\alpha \underset{\ell.c.}{\leq} \beta$ be computable indexations of the class $\mathcal{K}^*$. Then there exist a family $S(\mathcal{K}^*)$ of recursively enumerable sets and its computable indexations α' and β' such that $\mathcal{L}_{\ell.c.}(\mathcal{K}^*, \alpha, \beta) \cong \mathcal{L}(S(\mathcal{K}^*), \alpha', \beta')$.

Note that $\alpha \underset{\ell.c.}{\leq} \gamma \underset{\ell.c.}{\leq} \beta$ implies $\alpha' \leq \gamma' \leq \beta'$ for α', γ', β' and $S = S(\mathcal{K}^*)$ which are defined like in Theorem 2. We define the map

$$\varphi : \mathcal{L}_{\text{l.c.}}(\mathcal{K}^*, \alpha, \beta) \to \mathcal{L}(S(\mathcal{K}^*), \alpha', \beta'),$$

like in Theorem 2: $\varphi(\overline{\gamma}) = \overline{\gamma'}$. The verification that φ preserves the semilattice partial order and φ is one-to-one is analogous to verification of these properties in Theorem 2.

We will verify the surjectivity of the map φ. Let γ' be a computable numeration of the family $S(\mathcal{K}^*)$ such that $\gamma' \leq \beta'$ by means of some general recursive function $f(x)$. Put $\eta = \alpha + (\beta f)$, where $(\beta f)_n = \beta_{f(n)}$. Then we have obvious equalities $\eta' = \alpha' + (\beta f)' = \alpha' + \gamma'$. Since $\alpha' \leq \gamma'$ we have $\eta' \leq \gamma'$ and $\gamma' \leq \eta'$. Thus, $\overline{\eta'} = \overline{\gamma'}$, i.e., $\varphi(\overline{\eta}) = \overline{\gamma'}$. The theorem is proved.

Let $\Theta \in \{\Delta_n^0, \Sigma_n^0, \Pi_n^0 \mid n \in \omega\}$.

A set B is said to be Θ-simple in a set A if $B \subseteq A$, $B \in \Theta$ and for each $x \in \omega$ the condition $W_x \subseteq A \setminus B$ implies the finiteness of the r.e. set W_x. If B is a Σ_1^0-simple subset of a set A, then it is said to be a simple subset of A.

Clearly, for an arithmetic set A lying in a class Θ a Θ-simple subset always exists; for instance, the set A itself is an example. However, a Θ-set A does not always have a Θ_1-simple subset, where $\Theta_1 \subsetneqq \Theta$. For instance, in a productive set there is no simple Σ_1^0-subset.

<u>THEOREM 4</u>. Suppose that $\mathcal{K}^*$ is a computable class of constructive models, γ is its computable indexation, $(\mathfrak{M}_\nu, \nu) \in \mathcal{K}^*$, $|L_\nu^*| \geq 2$, and $I^\gamma(\mathcal{K}_\nu^*)$ has a Σ_2^0-simple subset A. Then the class $\mathcal{K}^*$ has infinitely many computable indexations incomparable relative to reducibility.

We denote by $P(n,x)$ the recursive predicate defining the set A. Namely,

$$n \in A \iff (\exists x)(\forall y > x)\, P(n,y);$$
$$n \notin A \iff (\forall x)(\exists y > x)\, \neg P(n,y).$$

The domain of a function φ is denoted by $\delta\varphi$; the graph of a function φ computed in t steps, by φ^t; the function computing the Cantor numbers of an ordered triple $\langle m,i,j\rangle$ by $C_3(m,i,j)$, and the function computing the Cantor number of an ordered pair $\langle i,p\rangle$ by $c(i,p)$.

We will construct countably many computable indexations γ^i, $i \in \omega$, of the class $\mathcal{K}^*$ which will be pairwise imcomparable relative to reducibility. Here, labels of two kinds will be used, $\langle m,i,j\rangle$ and $[m,i,j]$, where $i \neq j$. The two labels, $\langle m,i,j\rangle$ and $[m,i,j]$, are assumed to be mutually incomparable and lying in one equivalence class. We fix some effective ordering of these classes in the type ω. In the construction, smaller labels will be of greater priority.

For each constructivization γ_k^i being constructed, its successor γ_y will be appointed, according to which its construction will be performed. The successor of a numeration under construction may be changed but only finitely many times. The successors themselves may undergo a "transfer" from some constructivizations to others, but, again, only finitely many times.

Attaching of several copies of a label $\langle m,i,j\rangle$ will mean that we intend to violate the reducibility of the indexation γ^i to the indexation γ^j by means of the function φ_m. The labels $[m,i,j]$ play an auxiliary role and show that the above reducibility may not be violated immediately after the introduction of the labels $\langle m,i,j\rangle$.

The part of the model $(\mathfrak{M}_{iK},\gamma^i_K)$ constructed at the step t is denoted by $\mathfrak{M}^t_{iK}$. In the case of introduction of the labels $[m,i,j]$, we will define values of an auxiliary partial function $\ell(m,i,j,t)$ and finite models $D(m,i,j,t,\ell)$ for all $\ell \le \ell(m,i,j,t)$.

The construction will be performed for a progressively expanding collection of indexations and an increasing number of constructive numerations for each of these indexations. At the same time, we will study a list of labels $\langle m,i,j\rangle$ requiring consideration, i.e., a list for considering labels for which the reducibility of the indexation γ^i to the indexation γ^j by means of the function φ_m has not yet been violated.

<u>Construction of the Models</u> $(\mathfrak{M}_{iK},\gamma^i_K)$

<u>Step 0</u>. For all $i,j\in\omega$ put $\mathfrak{M}^0_{ij}=\varnothing$. Include all labels $\langle m,i,j\rangle$, where $i\neq j$, in the list for consideration. Turn to the next step of the construction.

We denote by $d^t_{\langle m,i,j\rangle}$ the number of numerations at the step t which carry labels no greater than $\langle m,i,j\rangle$. We introduce the quantity

$$\delta^t_{\langle m,i,j\rangle} \leftrightharpoons \max\{2d^t_{\langle m,i,j\rangle},\, 2c_3(m,i,j)\}+1.$$

<u>Step $t+1$.</u> It consists of four stages.

1. Consider the numerations $\gamma^j_{\varphi_m(2K)}$ which carry labels of the form $[m,i,j]$, where the corresponding labels $\langle m,i,j\rangle$ are excluded from the list for consideration. For all such numerations we verify the embedding $D(m,i,j,t,\ell)\hookrightarrow \mathfrak{M}^{t+1}_\nu$ under the corresponding values of $\ell\le\ell(m,i,j,t)$.

a) If there are no $[m,i,j]$ such that for all $\ell\le\ell(m,i,j,t,\ell)$ the embedding $D(m,i,j,t,\ell)\hookrightarrow \mathfrak{M}^{t+1}_\nu$ is established, then put

$$\ell(m,i,j,t+1)=\ell(m,i,j,t),$$
$$D(m,i,j,t+1,\ell)=D(m,i,j,t,\ell)$$

for all $\ell\le\ell(m,i,j,t)$. Turn to the next stage.

b) If among the labels considered there are $[m,i,j]$ such that for all $\ell\le\ell(m,i,j,t)$ we have the embedding $D(m,i,j,t,\ell)\hookrightarrow \mathfrak{M}^{t+1}_\nu$, then choose the least such label $[m,i,j]$. For labels $[m',i',j']$ less than $[m,i,j]$ put $\ell(m',i',j',t+1)=\ell(m',i',j',t)$ and $D(m',i',j',t+1,\ell)=D(m',i',j',t,\ell)$ for all $\ell\le\ell(m',i',j',t)$. For all successors γ_s of the numerations $\gamma^j_{\varphi_m(2K)}$ carrying the chosen label $[m,i,j]$ verify the truth of the predicate $P(s,t)$ If it is true, verify the embedding $\mathfrak{M}^t_{j\,\varphi_m(2k)} \hookrightarrow \mathfrak{M}^{t+1}_\nu$ for the corresponding numeration $\gamma^j_{\varphi_m(2k)}$.

If there is a numeration $\gamma^j_{\varphi_m(2k)}$ among them such that $\mathfrak{M}^t_{j\,\varphi_m(2k)} \hookrightarrow \mathfrak{M}^{t+1}_\nu$, then the successor of the numeration $\gamma^j_{\varphi_m(2k)}$ is changed from γ_s to ν. Then μ is appointed the successor of the numeration γ^i_{2k}. Perform the embeddings $\mathfrak{M}^t_{i\,2k} \hookrightarrow \mathfrak{M}_\mu$, $\mathfrak{M}^t_{j\,\varphi_m(2k)} \hookrightarrow \mathfrak{M}^{t+1}_\nu$. Turn to stage 2.

If, however, for all such numerations $\gamma^j_{\varphi_m(2k)}$ we have $\mathfrak{M}^t_{j\,\varphi_m(2k)} \not\hookrightarrow \mathfrak{M}_\nu$, then we define for them $D(m,i,j,t+1,\ell) = \mathfrak{M}^t_{j\,\varphi_m(2k)}$ for the appropriate values $\ell \leq \ell(m,i,j,t)$. Include the label $\langle m,i,j\rangle$ in the list for consideration and turn to stage 2.

2. In the list for consideration find the least label $\langle m,i,j\rangle$ such that $c_3(m,i,j) \leq t$ and for k ∈ ω we have

$$k \leq S^t_{\langle m,i,j\rangle} \Longrightarrow 2k \in \delta\varphi^t_m .$$

If there is no such label, turn to stage 3.

If there is such a label, then the construction is done according to one of the following cases.

a') If there exist k_1 and k_2 such that $k_1 < k_2 \leq S^t_{\langle m,i,j\rangle}$, $\varphi_m(2k_1) = \varphi_m(2k_2)$ and the numerations $\gamma^i_{2k_1}$, $\gamma^i_{2k_2}$ carry no smaller labels, then the embeddings $\mathfrak{M}^t_{i\,2k_2} \hookrightarrow \mathfrak{M}_\mu$, $\mathfrak{M}^t_{i\,2k_1} \hookrightarrow \mathfrak{M}_\nu$ are performed. We appoint μ successor of numeration $\gamma^i_{2k_2}$ and ν successor of numeration $\gamma^i_{2k_1}$. To all numerations γ^i_k free of smaller labels and such that $k \leq 2S^t_{\langle m,i,j\rangle}$ we attach the labels $\langle m,i,j\rangle$. All greater labels, if they have been used in the construction, are included in the list for consideration and removed from the numerations being constructed while these numerations themselves are marked with the labels $\langle m,i,j\rangle$. The label $\langle m,i,j\rangle$ is excluded from the list for consideration. Turn to stage 3.

b') If the condition of case a') does not hold and there exists k such that $k \leq S^t_{\langle m,i,j\rangle}$, $\varphi_m(2k)$ is even, and the numerations γ^i_{2k}, $\gamma^j_{\varphi_m(2k)}$ carry no smaller labels, then we appoint ν the successor of the numeration $\gamma^j_{\varphi_m(2k)}$ and μ the successor of the numeration γ^i_{2k}. Perform the embeddings $\mathfrak{M}^t_{j\,\varphi_m(2k)} \hookrightarrow \mathfrak{M}_\nu$, $\mathfrak{M}^t_{i\,2k} \hookrightarrow \mathfrak{M}_\mu$. Remove all labels $\langle m,i,j\rangle$ from the numerations being constructed if they have been used in construction. Include these labels in the list for considerations. The numerations freed from the labels are marked with the labels $\langle m,i,j\rangle$. We also mark the label-free numerations γ^i_s, γ^j_ℓ such that $s,\ell \leq max\{2S^t_{\langle m,i,j\rangle}, \varphi_m(2k)\}$ with the labels $\langle m,i,j\rangle$. The label $\langle m,i,j\rangle$ is excluded from the list for consideration and we turn to stage 3.

c') If the conditions of cases a') and b') do not hold and there is k such that $k \leq S^t_{\langle m,i,j\rangle}$, $\varphi_m(2k)$ is an odd number, the numerations γ^i_{2k}, $\gamma^j_{\varphi_m(2k)}$ carry no smaller labels, then the successor γ_y of the numeration $\gamma^j_{\varphi_m(2k)}$, if it was there, is moved to the numeration γ^j_{2p+1}. Also, this constructivization has not yet been involved in the construction, $2p+1 > 2S^t_{\langle m,i,j\rangle}$, and p has the least possible value. We appoint μ to be the successor of the numeration γ^i_{2k} and perform the embedding $\mathfrak{M}^t_{i\,2k} \hookrightarrow \mathfrak{M}_\mu$. We attach the label $\gamma^j_{\varphi_m(2k)}$ to the numeration $[m,i,j]$. Remove all labels greater than $\langle m,i,j\rangle$ if they were involved in the construction. Include them in the list for consideration. The numerations freed from labels are marked with the labels $\langle m,i,j\rangle$. Attach the labels $\langle m,i,j\rangle$ to all numerations γ^i_r, γ^j_l which are free of labels and have indices r, $l \leq max\{2S^t_{\langle m,i,j\rangle}, 2p+1\}$. We define

$$\ell(m,i,j,t+1) = \begin{cases} \ell(m,i,j,t)+1 & \text{, if the value } \ell(m,i,j,t) \text{ was defined;} \\ 0 & \text{otherwise;} \end{cases}$$

$$D(m,i,j,t+1,\ell(m,i,j,t+1)) \leftrightharpoons \mathfrak{M}^t_{j\,\varphi_m(2k)}.$$

Exclude the label $\langle m,i,j\rangle$ from the list for consideration. Turn to stage 3.

3. The numeration γ^i_{2p} which has not received a successor, with $c(i,2p) \leq t$, adopts the constructivization ν as its successor. For $i \leq t$ we find a numeration γ_k which has not been appointed the successor of a constructive numeration of the indexation γ^i already in construction and had the smallest possible value of k. We appoint γ_k the successor of the numeration γ^i_{2p+1} with the smallest possible index $2p+1$ having no successor. Turn to stage 4.

4. For each numeration γ^i_k already having a successor γ_y but carrying no label of the form $[m,s,i]$ or carrying the label $[m,s,i]$ but with the false value of the predicate $P(y,t)$ we enumerate the elements of the finite model

$$\mathfrak{M}^{t+1}_{i\,k} \leftrightharpoons \mathfrak{M}^t_{i\,k} \cup \mathfrak{M}^t_{\gamma_y}.$$

If, however, the numeration γ^i_k with a successor γ_y carries the label $[m,s,i]$ and the predicate $P(y,t)$ is true or the numeration γ^i_k has no successor, then put $\mathfrak{M}^{t+1}_{i\,k} \leftrightharpoons \mathfrak{M}^t_{i\,k}$. Turn to the next step of the construction.

Define

$$\mathfrak{M}_{ik} \leftrightharpoons \bigcup_{t=0}^{\infty} \mathfrak{M}^t_{i\,k}.$$

Clearly,

$$\mathfrak{M}^0_{i\kappa} \subseteq \mathfrak{M}^1_{i\kappa} \subseteq \ldots \subseteq \mathfrak{M}^t_{i\kappa} \subseteq \mathfrak{M}^{t+1}_{i\kappa} \subseteq \ldots \subseteq \mathfrak{M}_{i\kappa}$$

and γ^i_κ is a constructive numeration of the model $\mathfrak{M}_{i\kappa}$. Since the construction just described is uniform relative to i and κ, γ^i is a computable indexation of the class

$$\mathcal{K}^*_i \rightleftharpoons \{(\mathfrak{M}_{i\kappa}, \gamma^i_\kappa) \mid \kappa \in \omega\}.$$

The construction is complete.

LEMMA 1. If the conditions of stage 2 are fulfilled at some step, then the conditions of one of the cases, a'), b'), or c'), are necessarily fulfilled at this step.

Suppose that the condition of stage 2 is fulfilled at step $t+1$ for a label $\langle m,i,j\rangle$, but the condition of case a') is not fulfilled. After the step t, labels not exceeding $\langle m,i,j\rangle$ will be attached to $d^t_{\langle m,i,j\rangle}$ numerations. So in the set

$$\{2\kappa \mid \kappa \leq S^t_{\langle m,i,j\rangle},\ 2\kappa \in S\varphi^t_m\}$$

the number of numbers $2k$ such that the numeration $\gamma^i_{2\kappa}$ carries a label not exceeding $\langle m,i,j\rangle$ is no greater than $d^t_{\langle m,i,j\rangle}$. The remaining numbers in this set constitute a collection of at least $d^t_{\langle m,i,j\rangle}+1$ numbers. Since the condition of a') does not hold, the values of the function φ_m on distinct numbers of this set are distinct. Thus, for at least one 2κ in the set

$$\{2\kappa \mid \kappa \leq S^t_{\langle m,i,j\rangle},\ 2\kappa \in S\varphi^t_m\}$$

the numeration $\gamma^j_{\varphi_m(2\kappa)}$ does not carry a label not exceeding $\langle m,i,j\rangle$. If $\varphi_m(2\kappa)$ is an even number, then, obviously, the condition of case b') holds. If $\varphi_m(2\kappa)$ is odd, then the condition of case c') is fulfilled. The lemma is proved.

A label $\langle m,i,j\rangle$ is said to be stabilized (at a given step) if its new copies no longer appear at subsequent steps of the construction while its existing copies are not removed from the numerations being constructed.

LEMMA 2. Every label is stabilized.

First, note that the removal of the labels $\langle z,\ell,s\rangle$ and $[z,\ell,s]$ occurs simultaneously during the implementation of stage 2. On the other hand, if the label $[z,\ell,s]$ is attached to the numeration $\gamma^s_{\varphi_z(2\kappa)}$, then the label $\langle z,\ell,s\rangle$ is attached to the numeration $\gamma^\ell_{2\kappa}$. Thus, the stabilization of labels can be established by induction on their ordering.

Suppose that all labels less than $\langle m,i,j\rangle$ have already stabilized towards step t_0 of the construction. Thus, the removal of the labels $\langle m,i,j\rangle$ cannot occur at the subsequent steps. New copies of this label may appear in the construction only if it is included in the list for consideration. When new copies of the label $\langle m,i,j\rangle$ are attached, it is excluded from the list for consideration at the same time. Thus, the label $\langle m,i,j\rangle$ is not stabilized only if it is included in the list for consideration and excluded from it infinitely many times. We will show that this is impossible.

Suppose that the label $\langle m,i,j\rangle$ is included in the list for consideration and excluded from it infinitely many times. It may be included in the list only if the second part of b) holds. At the same time, a) and the first part of b) cannot be fulfilled because, otherwise, it would never be included in the list for consideration. The label $\langle m,i,j\rangle$ may be excluded from the list for consideration only at stage 2. Cases a') and b') cannot be fulfilled because after their implementation the condition of stage 1 can no longer hold for this label $\langle m,i,j\rangle$, i.e., it can no longer be included in the list for consideration. Thus, c') and the second part of b) are fulfilled, alternating, infinitely many times for the label $\langle m,i,j\rangle$. But in this case we have an infinite sequence of numerations $\gamma^j_{k_1}, \gamma^j_{k_2}, \dots$ carrying the labels $[m,i,j]$. Each of these numerations $\gamma^j_{k_s}$ has a successor γ_{z_s} which is not changed any longer because otherwise the first part of b) would be fulfilled. The successors of the numerations of the indexation γ^j are always declared with distinct γ-indices. So we obtain a recursively enumerable sequence of distince γ-indices $z_1, z_2, \dots$.

Since b) is fulfilled infinitely many times for $\gamma^j_{k_s}$, we conclude that, on one hand, $\mathfrak{M}_{\gamma_{z_s}} \hookrightarrow \mathfrak{M}_\nu$, on the other hand, the formula $(\forall x)(\exists y > x)\neg P(z_s, y)$ holds, i.e., $z_s \in I^\gamma(\mathcal{K}^*_\nu)\setminus A$. But there cannot be such an infinite enumerable sequence of indices because the set A is Σ^0_2-simple in $I^\gamma(\mathcal{K}^*_\nu)$ by the hypotheses of the theorem. The obtained contradiction proves the stabilization of the label $\langle m,i,j\rangle$.

<u>LEMMA 3</u>. For each $i\in\omega$ the inclusion $\mathcal{K}^*_i \subseteq \mathcal{K}^*$ holds.

Indeed, each numeration γ^i_k acquires a successor. If this numeration carries a label of the form $\langle z, \ell, s\rangle$, then the successor of this numeration may be changed only if this label is removed and a smaller label is attached. If this numeration carries a label of the form $[z,\ell,s]$, then its successor may be changed only once provided that the same label remains on the numeration, and the new successor is ν. Since there are infinitely many general recursive functions, infinitely many labels are attached. In each indexation γ^i the set of γ^i-indices of the numerations carrying labels is an initial segment of the natural numbers. All labels are stabilized. So each numeration γ^i_k gets a label at some step t which is never removed from it. But then after some step $t_1 \geqslant t$ the successor γ_y of the numeration γ^i_k is never changed. Therefore, we have $\gamma^i_k \underset{a}{\equiv} \gamma_y$ and $(\mathfrak{M}_{i\,k}, \gamma^i_k) \in \mathcal{K}^*$. The lemma is proved.

<u>LEMMA 4</u>. For each $i\in\omega$ the inclusion $\mathcal{K}^* \subseteq \mathcal{K}^*_i$ holds.

Fix an arbitrary value $i\in\omega$. According to stage 3 of the construction, each numeration γ_y becomes, at some step, the successor of a constructivization γ^i_k for a suitable value of k. Note that if the successor γ_y is moved at some step from a numeration γ^i_k to a numeration γ^i_m and the constructivization γ^i_k carries the label $\langle n,\ell,s\rangle$ or $[n,\ell,s]$ at this step of the construction, then the numeration γ^i_m will carry a smaller label. If γ^i_k is label-free, then, at the step when the successor γ_y of the numeration γ^i_k is moved, γ^i_m will acquire some label. The removal of a label is accompanied, as has already been pointed out, by attaching a smaller label. So the successor γ_y may undergo movements only finitely many times. Let γ^i_z be a constructivization such that γ_y is declared to be its successor at some step of the construction and is never moved again. Then, as a result of the construction, we have $\gamma^i_z \underset{a}{\equiv} \gamma_y$, i.e.,

$$(\mathfrak{M}_{\gamma_y}, \gamma_y) \in \mathcal{K}^*_i$$

The lemma is proved.

LEMMA 5. Each γ^i is a computable indexation of the class $\mathcal{K}^*$.

As has already been pointed out, γ^i is a computable indexation of the class $\mathcal{K}^*_i$. By virtue of Lemmas 3 and 4, we have the equality $\mathcal{K}^*_i = \mathcal{K}^*$. The lemma is proved.

LEMMA 6. If $i, j \in \omega$ are distinct, then $\gamma^i \not\leq \gamma^j$.

Suppose, to the contrary, that for some distinct $i, j \in \omega$ we have the reducibility $\gamma^i \leq \gamma^j$. Suppose that the appropriate general recursive function is $\varphi_m(x)$ with the Kleene number m. Consider the label $\langle m, i, j \rangle$. At some step t this label stabilizes. Since the function φ_m is general recursive, stage 2 was implemented at the step t, i.e., one of the cases a'), b'), or c') is fulfilled at this step. If case a') holds, then as a result of the construction we would have $k_1 \neq k_2$ such that $\gamma^i_{k_1} \underset{a}{\not\equiv} \gamma^i_{k_2}$ but $\varphi_m(k_1) = \varphi_m(k_2)$. If case b') holds, then, as a result of the construction, we would have k such that $\gamma^i_k \underset{a}{\not\equiv} \gamma^j_{\varphi_m(k)}$.

If case c') holds, then we obtain, as a result of the construction, $k_1, \ldots, k_\ell$ such that the numerations $\gamma^j_{\varphi_m(k_s)}$, $1 \leq s \leq \ell$, carry the labels $[m, i, j]$ and $\ell = \ell(m, i, j, t)$. If some $s \leq \ell$ and all $t_1 \geq t$ satisfy

$$D(m, i, j, t_1, s) = D(m, i, j, t, s) \not\hookrightarrow \mathfrak{M}^{t_1}_\nu,$$

then case a) always holds for the label $[m, i, j]$ hereafter and

$$\mathfrak{M}_{j\,\varphi_m(k_s)} \not\hookrightarrow \mathfrak{M}_{i\,k_s} \cong \mathfrak{M}_\nu.$$

A fortiori,

$$\mathfrak{M}_{j\,\varphi_m(k_s)} \not\cong \mathfrak{M}_{i\,k_s} \text{ and } \gamma^i_{k_s} \underset{a}{\not\equiv} \gamma^j_{\varphi_m(k_s)}.$$

If at some step $t_2 \geq t$ we have

$$D(m, i, j, t_2, s) = D(m, i, j, t, s) \hookrightarrow \mathfrak{M}_\nu$$

for all $s \leq \ell$, then the condition of case b) holds. The second part of this case cannot be fulfilled since in that case the label $\langle m, i, j \rangle$ would be again included in the list for consideration, which contradicts its stabilization. Thus, the first part of b) is fulfilled at the step t_2. But then, as a result of the construction, we have $\gamma^i_{\kappa_s} \underset{a}{\equiv} \mu$, $\gamma^j_{\varphi_m(\kappa_s)} \underset{a}{\equiv} \nu$ for a suitable $s \leq \ell$. Therefore, in this case there is again κ_s such that $\gamma^i_{\kappa_s} \underset{a}{\not\equiv} \gamma^i_{\varphi_m(\kappa_s)}$.

There are no other possibilities. In all considered cases we have arrived at a contradiction to the definition of a reducing function. Thus, $\gamma^i \not\leq \gamma^j$. The lemma is proved.

An application of Lemmas 5 and 6 concludes the proof of the theorem.

COROLLARY 1. Suppose that $\mathcal{K}^*$ is a computable class of constructive models, γ its computable indexation, $(\mathfrak{M}_\nu, \nu) \in \mathcal{K}^*$, $|\mathcal{L}^*_\nu| \geq 2$, and $I^\gamma(\mathcal{K}^*_\nu)$ contains no infinite recursively enumerable subsets. Then the class $\mathcal{K}^*$ has infinitely many computable indexations pairwise incomparable relative to reducibility.

COROLLARY 2. Suppose that $\mathcal{K}^*$ is a computable class of constructive models, γ its computable indexation, $(\mathfrak{M}_\nu, \nu) \in \mathcal{K}^*$, $|\mathcal{L}^*_\nu| \geq 2$, and $I^\gamma(\mathcal{K}^*_\nu) \in \Delta^0_2$. Then the class $\mathcal{K}^*$ has countably many computable indexations pairwise incomparable relative to reducibility.

COROLLARY 3. Suppose that $\mathcal{K}^*$ is a computable class of constructive models, γ its computable indexation, $(\mathfrak{M}_\nu, \nu) \in \mathcal{K}^*$, $|\mathcal{L}^*_\nu| \geq 2$, and the index set $I^\gamma(\mathcal{K}^*_\nu)$ has a Σ^0_2-simple subset. Then the semilattice $\mathcal{L}(\mathcal{K}^*)$ of computable indexations of the class $\mathcal{K}^*$ has infinitely many incomparable elements.

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