DECOMPOSITION OF A COMPACT RING INTO THE SUM OF THE RADICAL AND AN INERTIAL SUBRING

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The classical Wedderburn—Mal'tsev theorem on the splitting off of the radical in finite-dimensional algebras over a field has been generalized in several directions. In a number of papers [1-3] conditions under which the analog of the theorem indicated is valid for infinite-dimensional algebras over a field were found. Some of these conditions always had topological character. In particular, D. Zelinsky [1] proved an analog of the Wedderburn—Mal'tsev theorem for compact associative rings of characteristic $p > 0$. A Wedderburn decomposition for hereditarily linearly compact rings was studied in [4] and necessary and sufficient conditions for its existence were found.

Another cycle of papers [5-8] is devoted to establishing analogs of the Wedderburn—Mal'tsev theorem for algebras of finite type over a ring K. As it turned out, such an analog can be established only when the ring K is Hensel. In addition a somewhat modified notion of the concept of semisimplicity is also necessary, since if the ring K has nonzero radical, there will not be any semisimple K-subalgebras of algebras over K of finite type at all. As analog of semisimplicity here one uses the notion of being unramified in the sense of Azumaya: an algebra A over a ring K is said to be unramified if $J(A) = J(K)\cdot A$, i.e., if A has smallest possible radical.

If R is a compact primary alternative or Jordan ring containing a unit 1 and having simple quotient ring $\bar{R} = R/J(R)$ by the quasiregular radical, then $p\cdot 1 \in J(R)$, where p is the characteristic of the ring $\bar{R}$. There is the smallest possible radical for R if $J(R) = p\cdot R$. In this case we call the compact primary ring R unramified. We call an arbitrary compact ring unramified if it is the complete direct sum with the Tikhonov topology of compact primary unramified rings.

The basic result of the paper asserts that in any compact alternative or Jordan ring R there is a compact unramified subring R_1 (Azumaya factor) such that $R_1 + J(R) = R$. It is proved that in a compact associative ring R all Azumaya factors are conjugate with respect to automorphisms of the form

$$x \longmapsto x^{a} = x - ax - xa' + axa', \tag{1}$$

where $\alpha \in J(R)$. For associative rings with identity Z. S. Lipkina [9] obtains analogous results in somewhat different terms.

Let $\mathfrak{M}$ be a manifold of rings, $\mathscr{X} = \Phi_{\mathfrak{M}}[X]$ be a free ring of countable rank, and A be a ring from $\mathfrak{M}$. We consider the ring $A * \mathscr{X}$ which is the free product in $\mathfrak{M}$ of the rings

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A and $\mathcal{X}$. We shall call the elements of the ring $A * \mathcal{X}$ nonassociative polynomials with coefficients in the ring A. If the sequence $x = (x_1, x_2, \ldots)$ specializes to the sequence $a = (a_1, a_2, \ldots)$, where $a_i \in A$ then each element $f(x) \in A * \mathcal{X}$ specializes to the element $f(a) \in A$. We shall call the sequence a a solution of the equation $f(x) = 0$ if $f(a) = 0$. We shall call the system of equations

$$f_\alpha(x) = 0, \quad \alpha \in I, \tag{2}$$

compatible on the subset $B \subseteq A$, if there exists a sequence $b = (b_1, b_2, \ldots)$ such that $b_i \in B$ and $f_\alpha(b) = 0$ for all $\alpha \in I$. The system of equations (2) is called locally compatible on B, if each of its finite subsystems is compatible on B.

Proposition 1. Let A be a compact ring and B be a closed subset of it. Then any system of equations which is locally compatible on B is compatible on this set.

Proof. Let us assume that the system (2) is locally compatible on b. We consider the countable Cartesian power A^∞ endowed with the Tikhonov topology. Each polynomial $f_\alpha(x)$ generates a continuous map $f_\alpha : a \longmapsto f_\alpha(a)$ of A^∞ into A. The set $S_\alpha = \{a \in A^\infty \mid f_\alpha(a) = 0\}$ is the preimage of zero and hence is closed in A. The sets $S'_\alpha = S_\alpha \cap B$ are also closed. By hypothesis for any $\alpha_1, \ldots, \alpha_n \in I$ we have $S'_{\alpha_1} \cap \ldots \cap S'_{\alpha_n} \neq \emptyset$. Since A is compact, we have $\bigcap_{\alpha \in I} S'_\alpha \neq \emptyset$ which is what had to be proved.

Let $H = \{h_\alpha \mid \alpha \in I\}$ be a family of elements of the commutative topological group G, $\mathfrak{F}(I)$ be the set of all finite subsets of I, and

$$s_J = \sum_{\alpha \in J} h_\alpha, \quad J \in \mathfrak{F}(I).$$

The family H is said to be summable if the map $J \longrightarrow s_J$ has limit s with respect to the filter of sections of the set $\mathfrak{F}(I)$ ordered by the inclusion relation. In this case one writes $s = \sum_{\alpha \in I} h_\alpha$ and calls s the sum of the elements of the given family [10, p. 80].

LEMMA 1. In a compact alternative or Jordan ring A any orthogonal family of idempotents $E = \{e_\alpha \mid \alpha \in I\}$ is summable and its sum $e = \sum_{\alpha \in I} e_\alpha$ is also idempotent.

Proof. For $J \in \mathfrak{F}(I)$ let $e_J = \sum_{\alpha \in J} e_\alpha$. Obviously e_J is an idempotent such that $e_J e_\alpha = e_\alpha e_J = e_\alpha$. Since A is compact, the summability of the family E will be proved if we show that the map $J \longmapsto s_J$ has a unique limit point [11, p. 126].

We consider the sets of idempotents

$$E^\perp = \{u \in A \mid u^2 = u,\ ue_\alpha = e_\alpha u = 0,\ \alpha \in I\},$$

$$E^{\perp\perp} = \{v \in A \mid v^2 = v,\ uv = vu = 0,\ u \in E^\perp\}.$$

They are both closed. Moreover, it is obvious that $E^{\perp\perp} \supseteq E$. We write the system of equations:

$$\begin{cases} x^2 = x, \\ xe_\alpha = e_\alpha x = e_\alpha, \quad \alpha \in I. \end{cases}$$

Any finite subsystem of it contains in its notation only idempotents with indices from a finite subset $J \in \mathfrak{F}(I)$ and hence is compatible on $E^{\perp\perp}$. Let $e \in E^{\perp\perp}$ be a solution of it.

We show that e is the unique limit point of the map $J \longmapsto e_J$. Let $A_1(e)$ be a Pierce 1-component for the idempotent e. The set $A_1(e) = AU_e$ is compact, as the continuous image of a compact set, and hence is closed. Since $e_\alpha \in A_1(e)$ all the limit points of the map considered also belong to $A_1(e)$.

We note that we do not yet assert that e is a limit point. We show however that if f is a limit point of the map indicated above, then f necessarily coincides with e. Indeed it is obvious that $f^2 = f$ and $fe_\alpha = e_\alpha f = e_\alpha$ for all $\alpha \in I$. If $f \neq e$ then $e - f$ is an idempotent orthogonal to all e_α and consequently orthogonal to e. However, $e - f \in A_1(e)$ and this is impossible.

However by compactness there is a limit point. We have proved its uniqueness. Consequently, $e = \sum_{\alpha \in I} e_\alpha$. The lemma is proved.

For a completely disconnected compact associative ring the assertion of the lemma is contained in L. A. Skornyakov [12].

LEMMA 2. Let A be a compact alternative or Jordan ring, $E = \{e_\alpha \mid \alpha \in I\}$ be an orthogonal family of idempotents and e be its sum. Let $A_\alpha = AU_{e_\alpha}$ be a Pierce 1-component of the idempotent e_α. Then for any $q_\alpha \in A_\alpha$ the family $Q = \{q_\alpha \mid \alpha \in I\}$ is summable. If $q_\alpha \in A_\alpha$ is a closed subring of A_α where P is generated by the union of the sets $A_1(e)$ and is topologically isomorphic to the Tikhonov product $\prod_{\alpha \in I} P_\alpha$.

Proof. As above, we consider the map

$$J \to q_J = \sum_{\alpha \in J} q_\alpha, \quad J \in \mathfrak{F}(I),$$

and let q be a limit point of it. Then it is obvious that $qe_\alpha = q_\alpha$ for all $\alpha \in I$ and if q' is another limit point of this map, then $(q - q')e_\alpha = 0$ for all $\alpha \in I$ so $(q - q')e = 0$ and consequently $q = q'$, since $q - q' \in A_1(e)$. Hence $q = \sum_{\alpha \in I} q_\alpha$.

We consider the map

$$\varphi : \prod_{\alpha \in I} P_\alpha \longrightarrow A_1(e),$$

defined as follows: if $u \in \prod_{\alpha \in I} P_\alpha$ and the projection of u on A_α is u_α then $\varphi(u) = \sum_{\alpha \in I} u_\alpha$. This map is injective, since if $v \in \prod_{\alpha \in I} P_\alpha$ and $v_\alpha \neq u_\alpha$ for at least one index $\alpha \in I$ then $\varphi(v)e_\alpha \neq \varphi(u)e_\alpha$ and consequently, $\varphi(v) \neq \varphi(u)$.

Now we note that $A_1(e)$ is a compact ring with unit e. By Theorem 1 of [13], it is completely disconnected and hence by Theorem 2 of [14], has a basis of neighborhoods of zero of open compact ideals. Let V be an open-closed neighborhood of zero in $A_1(e)$. Then $A_1(e) \setminus V$ is also compact. It is easy to see that the set E has zero as its only point of adherence, since any such point is idempotent of square equal to zero. Consequently, $E \cap (A_1(e) \setminus V)$ is discrete and hence finite. Let it consist of idempotents $e_{\alpha_1}, \ldots, e_{\alpha_n}$. Then the set

$$W=\{u\in\prod_{\alpha\in I}P_\alpha \mid u_{\alpha_i}\in V\ ,\ i=1,\dots,n\}$$

is open in $\prod_{\alpha\in I}P_\alpha$ and $\varphi(W)\subseteq V$. This proves the continuity of the map φ. Since $\prod_{\alpha\in I}P_\alpha$ is compact and $A_1(e)$ is Hausdorff, the map φ^{-1} is also continuous.

It is easy to see that the map φ acts isomorphically on the discrete direct sum $\sum_{\alpha\in I}P_\alpha\subseteq\prod_{\alpha\in I}P_\alpha$. Since it is an everywhere dense subset of the complete direct sum, the map φ is an isomorphism and the lemma is proved.

Let A be a ring, N be an ideal in A, and $F=\{f_\alpha \mid \alpha\in I\}$ be a set of pairwise orthogonal idempotents in A/N. One says that F can be lifted to A if there exists a set of pairwise orthogonal idempotents $E=\{e_\alpha\mid\alpha\in I\}$ in A such that $f_\alpha=e_\alpha+N$ for all $\alpha\in I$.

THEOREM 1. Let A be an alternative or Jordan compact ring, J(A) be its quasiregular radical. Then an arbitrary set $F=\{f_\alpha\mid\alpha\in I\}$ of pairwise orthogonal idempotents in A/J(A) with sum f can be lifted to A to a set of pairwise orthogonal idempotents $E=\{e_\alpha\mid\alpha\in I\}$ with sum e such that f = e + J(A).

Proof. The case when F consists of one idempotent is analyzed in Lemma 1 of [15]. Let $F=\{f_1,\dots,f_n\}$ and $n\geq 2$. Arguing by induction, let us assume that there exist pairwise orthogonal idempotents $e_1,\dots,e_i$ such that $f_j=e_j+J(A)$ for j = 1, ..., i. We consider the idempotent $e=e_1+\dots+e_i$ and with respect to it we consider the Pierce decomposition of the ring A:

$$A=A_1\oplus A_{1/2}\oplus A_0 \tag{3}$$

(in order to be definite we consider the Jordan case). Let g be a preimage of f_{i+1} and $g=g_1+g_{1/2}+g_0$ be the decomposition of the element g in correspondence with the Pierce decomposition (3). The element $g_0=gU_{1-e}$ lies in A_0 and $f_{i+1}=g_0+J(A)$. Hence, since $J(A_0)=A_0\cap J(A)$ (cf. [16, p. 393]), by Lemma 1 of [15] we can lift f_{i+1} to an idempotent $e_{i+1}\in A_0$ such that $f_{i+1}=e_{i+1}+J(A)$. It will be the one sought since its location in A_0 guarantees its orthogonality to $e_1,\dots,e_i$.

Let F be arbitrary. We consider its maximal subset $F_0=\{f_\alpha\mid\alpha\in I'\}$ which can be lifted to A to $E_0=\{e_\alpha\mid\alpha\in I'\}$ and such a subset exists by Zorn's lemma. We want to prove that $F_0=F$. Let us assume the contrary: there exists an $h\in F\setminus F_0$. Let g be a preimage of h in A. The set $F_0\cup\{h\}$ can be lifted to A if and only if the system of equations

$$\begin{aligned}&(g-x)^2=g-x,\\&(g-x)e_\alpha=0,\qquad \alpha\in I',\\&e_\alpha(g-x)=0,\qquad \alpha\in I',\end{aligned}\tag{4}$$

is solvable for $x\in J(A)$. We have proved that it is locally compatible on J(A). By the compactness of A and the fact that J(A) is closed, which is proved in [17], the system (4) is compatible on J(A), which contradicts the inequality $F_0\neq F$.

By Lemma 1, E is summable and it is proper to speak of the sum $e = \sum e_\alpha$. Since J(A) is closed, the canonical homomorphism from A to A/J(A) is continuous. Hence e goes under this homomorphism into f. The theorem is proved.

The following proposition to a large extent clarifies the analogy between compact rings and algebras of finite type.

Proposition 2. Let A be a compact alternative or Jordan ring. Then

a) if $a \in J(A)$ then its quasiinverse a' belongs to the closure $(\bar{a})$ of the subring generated by a;

b) if B is a closed subring of A, then $J(B) \supseteq J(A) \cap B$

Proof. It is proved in [17] that the radical J(A) is topologically nilpotent. Hence for the sequence of elements $y_n = -(a+a^2+\dots+a^n)$ we have $a \circ y_n = -a^{n+1}$ and $a \circ y_n \to 0$. If y is a limit point of the sequence $\{y_n\}$ then $a \circ y = 0$ and $y \in (\bar{a})$. The second point follows from the first. The proposition is proved.

Let A be a compact alternative or Jordan ring with unit 1. By K = K(A) we shall denote the closed subring of A generated by the unity. If A is primary, then its quotient-ring $\bar{A} = A/J(A)$ by the radical is simple and by Theorem 3 of [14] it is finite. If p is the characteristic of the ring $\bar{A}$ then the ring A is called unramified if $J(A) = p \cdot A$.

THEOREM 2. Let A be a compact primary alternative or Jordan ring. Then

a) The subring K is isomorphic either to the ring of integral p-adic numbers ZP or the ring of residues Z_{p^m} (where p is the characteristic of the ring $\bar{A}$) and is Hensel;

b) If A is unramified and $\{\bar{a}_1, \dots, \bar{a}_n\}$ is a basis of $\bar{A}$ over the field $\bar{K} = K/J(K)$ then $\{a_1, \dots, a_n\}$ where a_i is an arbitrary preimage of $\bar{a}_i$ in A, is a free basis for A as a K-module;

c) If A is unramified, then the topology i A is J(A)-adic.

Proof. a) If p is the characteristic of the ring $\bar{A}$ then $p \cdot 1 \in J(A)$ and by Proposition 2, $p \cdot 1 \in J(K)$ so $J(K) = p \cdot K$ and $K/J(K) = Z_p$. Since in a compact ring the radical is topologically nilpotent, $p^n \cdot 1 \to 0$ so either $p^m \cdot 1 = 0$ for some m and $K = Z_{p^m}$ or $K = Z^p$. Both these rings are Hensel as is any complete local ring [18, p. 323].

b) We consider the K-submodule $A_1 = Ka_1 + \dots + Ka_n$. Since $J(A) = p \cdot A$, one has A = A_1 + pA. Iterating this relation, we get $A = A_1 + p^k \cdot A$ for any k. However, $p^k \cdot A \subseteq J(A)^k$ and J(A) is topologically nilpotent. Consequently, for any neighborhood of zero U we have A = A_1 + U, i.e., A_1 is everywhere dense in A. But A_1 is compact, since it is a sum of compact subsets and consequently closed. Hence A = A_1 and A = $Ka_1 + \dots + Ka_n$. But now by Theorem 6 of [5] we can deduce from the linear independence of $\{\bar{a}_1, \dots, \bar{a}_n\}$ over $\bar{K}$ the linear independence of $\{a_1, \dots, a_n\}$ over K.

c) Now we note that $J(A)^k = p^k \cdot A$ and hence the powers of the radical have finite index in A and are consequently open. Since the radical is furthermore topologically nilpotent, the topology in A coincides with the J(A)-adic one.

The theorem is proved.

We note that the fact that the primary compact ring A is unramified in the sense of the definition above is equivalent with its being unramified in the sense of Azumaya as an algebra over K, which motivates our terminology.

In the following theorem we formulate results of [5-8] in a form which is convenient for use later.

<u>THEOREM 3</u> (Azumaya-Zhelyabin). Let A be an alternative or Jordan algebra of finite type over the Hensel ring K such that $\bar{A} = A/J(A)$ is a simple finite-dimensional separable algebra over the field $\bar{K} = K/J(K)$ with basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$ and set of structural constants $\{\bar{\gamma}_{ij}^{(k)}\}$. Let $\{\gamma_{ij}^{(k)}\}$ be a set of preimages of the structural constants in K. Then in A one can find preimages $e_1, \ldots, e_n$ of the basis elements $\bar{e}_1, \ldots, \bar{e}_n$ such that $e_i \cdot e_j = \sum_{k=1}^{n} \gamma_{ij}^{(k)} e_k$.

Now we can begin the proof of the basic results of the section. Following Azumaya, the subring A_1 of the ring A will be called inertial if $A_1 + J(A) = A$.

<u>LEMMA 3.</u> Let A be a compact primary alternative or Jordan ring. Then if $\{\bar{e}_1, \ldots, \bar{e}_m\}$ is a basis of the $\bar{K}$-algebra $\bar{A}$ such that $\bar{e}_1 = 1$ and

$$\bar{e}_i \cdot \bar{e}_j = \sum_{k=1}^{m} \gamma_{ij}^{(k)} \cdot \bar{e}_k, \qquad \gamma_{ij}^{(k)} \in Z,$$

then in A one can find an inertial unramified primary closed subring A_1 with the same unit which is an algebra of finite type over K with free basis $\{e_1, \ldots, e_m\}$ and structural constants $\{\gamma_{ij}^{(k)}\}$ where e_i is a preimage of $\bar{e}_i$.

<u>Proof.</u> As noted in the proof of Lemma 2, a compact ring with unit is completely disconnected and has a basis of neighborhoods of zero of open compact ideals. Let $\{I_\alpha\}$ be such a basis in the ring A. We consider the chain of ideals $\{J_i\}$ defined as follows: $J_1 = J(A)$, $J_{i+1} = (J_i)^2 + (J_i)^2 \cdot A$ (in the alternative case the second summand is superfluous). In the finite ring A/I_α the image of the radical J(A) is nilpotent and consequently solvable in the sense of Penico (cf., e.g., [19]). Hence one can find an $n = n(\alpha)$ such that $J_n \subseteq I_\alpha$. Consequently, $\bigcap J_n = (0)$ and $J_n^2 \subseteq J_{n+1}$.

Let $A_n = A/J_n$ and $K_n = K(A_n)$. The ring A_n is primary and by Theorem 2 the ring $K_n = K(A_n)$ is Hensel. Moreover, A_n has solvable radical $J(A_n) = J(A)/J_n$. Since in addition the quotient-ring $A_n/J(A_n)$ is finite, by the Zhevlakov—Shestakov theorem (Theorem 6 of [20]) the ring A_n is locally finite over K_n.

Let $\{f_1, \ldots, f_m\}$ be a set of preimages in A_n of the elements $\{\bar{e}_1, \ldots, \bar{e}_m\}$. Then the K_n-subalgebra A_n' generated by the elements $f_1, \ldots, f_m$, is finite over K_n and by Proposition 2, $J(A_n') = A_n' \cap J(A_n)$. Using the Azumaya-Zhelyabin theorem we can choose from A_n' elements $g_i \in f_i + J(A_n)$ such that

$$g_i \cdot g_j = \sum_{k=1}^{m} \gamma_{ij}^{(k)} \cdot g_k.$$

Let $e_i^{(n)}$ be a preimage in A of the element g_i of A_n. Then

$$e_i^{(n)} \cdot e_j^{(n)} - \sum_{k=1}^{m} \gamma_{ij}^{(k)} \cdot e_k \in J_n \tag{5}$$

and $e_i^{(n)}$ is a preimage in A of the element $\bar{e}_i$. Let e_i be a limit point of the sequence $e_i^{(1)}, e_i^{(2)}, \ldots, e_i^{(n)}, \ldots$. Then e_i is also a preimage in A of the element $\bar{e}_i$ and passing to the limit in (5), we get

$$e_i \cdot e_j = \sum_{k=1}^{m} \gamma_{ij}^{\prime(k)} \cdot e_k . \tag{6}$$

Let $A_0 = Ke_1 + \ldots + Ke_m$. This compact subring of A by (6) satisfies the condition $A_0/pA_0 = \bar{A}$ and hence is unramified. The units in A and A_0 coincide since otherwise their difference is an idempotent which lies in the radical which is impossible. By Theorem 2, $\{e_1, \ldots, e_n\}$ is a linearly independent basis in A_0. That A_0 is inertial is obvious. The theorem is proved.

THEOREM 4. Let A be a compact primary alternative or Jordan ring. Then in A one can find an inertial closed unramified primary subring A_1 with the same unit whose center L is a primary associative-commutative ring which is a free module of finite type over $K = K(A) = K(A_1)$ which is isomorphic to Z^p or Z_{p^m} and A_1 is isomorphic to one of the following rings:

1) a ring of matrices L_n, $n \geq 1$;

2) a split Cayley—Dickson ring C(L) over L;

3) the ring $B(f) = L \cdot 1 + V$ of a bilinear form f on the free L-module V which is an inner product in the sense of [21];

4) a Jordan matrix ring $H(M_n, J_a)$, $n \geq 3$, where the ring (M, j) with involution j is one of the following rings:

a) $L \oplus L^0$, where L^0 is a ring isomorphic to L, and j is the involution of interchanging components;

b) M is a primary commutative extension of degree 2 of the ring L; j is an automorphism of period 2 and $L = M^j$ is the fixed subring of this automorphism;

c) Q(L) is a split quaternion ring over L with standard involution j;

d) C(L) is a split Cayley—Dickson ring with standard involution j and n = 3.

Proof. Lemma 3 is the basic part of it. It is necessary to use the fact that finite simple alternative and Jordan rings split and to use the corresponding structure theory which is found, for example, in [19]. We leave the routine work on the choice of suitable bases to the reader.

THEOREM 5. In any compact alternative or Jordan ring A there is an unramified inertial closed subring.

Proof. The quotient ring $\bar{A}=A/J(A)$ by Corollary 1 of Theorem 4 of [17] is the topological direct sum of finite simple rings $\bar{A}_\beta$ with units $\bar{e}_\beta$, $\beta\in I$. The set of orthogonal idempotents $\{\bar{e}_\beta \mid \beta\in I\}$ by Theorem 1 lifts to A to a set of orthogonal idempotents $\{\bar{e}_\beta \mid \beta\in I\}$ with sum e. Let $A_{1\beta}$ be a Pierce 1-component for the idempotent e_β (again to be definite we consider the Jordan case). By McCrimmon's theorem [16, p. 393], $J(A_{1\beta})=A_{1\beta}\cap J(A)$ and hence $A_{1\beta}/J(A_{1\beta})=\bar{A}_\beta$, i.e., $A_{1\beta}$ is a primary ring. By Theorem 4, in $A_{1\beta}$ there is a compact primary unramified inertial subring Q_β. It remains to use Lemma 2. The theorem is proved.

We cannot yet prove a theorem of uniqueness (conjugacy) for closed inertial unramified subrings of commutative alternative and Jordan rings due to the fact that this question has not been solved in the discrete case. For associative rings one can do this.

THEOREM 6. Let Q and Q' be two closed inertial unramified subrings of an associative compact ring A. Then there exists an element $a\in J(A)$ with quasiinverse a' such that the inner automorphism (1) induces a topological isomorphism of Q and Q'.

Proof. Let $\bar{A}=A/J(A)=\prod_{\beta\in I}\bar{A}_\beta$ and $\bar{A}_\beta$ be the unit of the ring $\bar{A}_\beta$. Let Q_β and Q_β' be the preimages of $\bar{A}_\beta$ in Q and Q' under the canonical homomorphism of A to $\bar{A}$. By Lemma 2, $Q=\prod_{\beta\in I}Q_\beta$ and $Q'=\prod_{\beta\in I}Q'_\beta$. Obviously Q_β and Q_β' are primary unramified subrings in Q and Q' respectively. Let e_β be the unit in Q_β and f_β be the unit in Q_β'. Obviously $e_\beta\equiv f_\beta \bmod J(A)$. We consider the system of equations $e_\beta^x=f_\beta$, $\beta\in I$. By virtue of Theorem 3 of [5] this system is locally compatible of J(A). By Proposition 1 this system is compatible on J(A) and there exists an element $c\in J(A)$ such that $e_\beta^c=f_\beta$ for all $\beta\in I$.

Now we consider the subrings Q^c and Q'. They have common units f and common system of pairwise orthogonal idempotents $\{f_\beta \mid \beta\in I\}$. We consider the primary ring $A_\beta=f_\beta A f_\beta$. It remains primary unramified rings A_β and f_β which have common identity Q_β^c with Q'_β and are matrix rings over commutative rings L_β and L_β' respectively. These rings are free modules over the closed subring K_β generated by f_β and the ranks of these modules are identical. Let $\{e'_{ij}\}$ be the matrix units in $Q_\beta{}^c$. By Theorem 25 of [5] we can find in Q_β^c matrix units $\{e_{ij}\}$ such that $e_{ij}\equiv e'_{ij} \bmod J(A_\beta)$. But then by Theorem 4 of the same paper there exists an element $b_\beta\in J(A_\beta)$ such that $e_{ij}^{b_\beta}=e'_{ij}$. Here of course $L_\beta^{b_\beta}=L'_\beta$ since the ring of coefficients of the matrix subring is the centralizer of the system of matrix units.

Since $J(A_\beta)=A_\beta\cap J(A)$ one has $b_\beta\in J(A)$. Be Lemma 2 the family $\{b_\beta \mid \beta\in I\}$ is summable and has sum b. Then $(Q^c)^b=Q'$ and for a = c + b − cb we have $Q^a=(Q^c)^b=Q'$. Since in both Q and Q' the topology is defined by powers of the radical as a basis of neighborhoods of zero, the isomorphism indicated is topological. The theorem is proved.

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