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ERSHOV HIERARCHY AND THE T-JUMP

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Let $(O;<_O)$ be a Kleene system of ordinal notation [1], Σ_a^{-1}, Δ_a^{-1} classes of the Ershov hierarchy [2, 3], and ' the operation of the T-jump. The basic purpose of this paper is the proof of the following theorem.

THEOREM 1. 1) For any recursively enumerable (r.e.) set A and any nonleast $a \in O$ there exists an $R \in \Sigma_a^{-1}$ not T-equivalent to any set from Δ_a^{-1}, such that $R' \equiv_T A'$. 2) For any r.e. set A and any limit $a \in O$ there exists an $R \in \Delta_a^{-1}$, not T-equivalent to any set from $\bigcup_{b<_O a} \Sigma_b^{-1}$, such that $R' \equiv_T A'$.

This theorem completely describes jumps of sets from the Ershov hierarchy, since we noted [4] that if $a \in O$ is a successor for $b \in O$, then any Δ_a^{-1}-set is T-equivalent to some Σ_b^{-1}-set. For the case of differences of r.e. sets forming the second level of Ershov hierarchy, the theorem was proved by S. T. Ishmukhametov [5]. In his proof he made much use of the specifics of differences, which is not suitable, in our view, for other levels. Our proof is based on completely different ideas and is about three times shorter.

Theorem 1 implies the following assertion, which informally denotes the "independence" of the hierarchy of higher and lower degrees and the hierarchy of degrees induced by the Ershov hierarchy. Let L_n be the set of all T-degrees $d \leq 0'$ such that $d^{(n)} = 0^{(n)}$, $H_n = \{d \leq 0' \mid d^{(n)} = 0^{(n+1)}\}$, $M = \{d \leq 0' \mid \forall n (d \notin L_n \wedge d \notin H_n)\}$. Let S_a and D_a be aggregates of T-degrees containing sets from Σ_a^{-1} and Δ_a^{-1} respectively,

$$S_a^* = S_a \setminus D_a,\ D_a^* = D_a \setminus \bigcup_{b<_O a} S_a,\ H_n^* = H_n \setminus \bigcup_{m<n} H_m,\ L_n^* = L_n \setminus \bigcup_{m<n} L_m.$$

COROLLARY. 1) For any $n \in N$ and nonleast $a \in O$ the classes $S_a^* \cap L_{n+1}^*$, $S_a^* \cap H_n^*$, and $S_a^* \cap M$ are nonempty. 2) For any $n \in N$ and limit $a \in O$ the classes $D_a^* \cap L_{n+1}^*$, $D_a^* \cap H_n^*$, and $D_a^* \cap M$ are nonempty.

Let us pass to the proof of the theorem. We will introduce some notation. We identify sets with characteristics functions, i.e., $n \in X$ and $X(n) = 1$, as well as $n \notin N$ and $X(n) = 0$, are equivalent. Let $X^{(e)} = \{\langle e,x\rangle \mid \langle e,x\rangle \in X\}$, $X_a^{(e)} = \{\langle e,x\rangle \in X \mid \langle e,x\rangle \geq a\}$. $\psi(n)\downarrow$ $(\psi(n)\uparrow)$ denotes the partial function ψ is determinate (indeterminate) at the point n, and $\psi[n]$ denotes the restrictions of ψ to the set $\{x \mid x < n\}$. If X is a r.e. set $(X \in \Delta_2^0)$, then X^s is a finite set calculated in s steps in some effective enumeration of X (or in some limit calculation for

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X). φ_n, φ_n^s, φ_n^X, and $\varphi_n^{X,s}$ are standard [1]. In place of φ_n^X, $\varphi_n^{X,s}(C)$ and $\varphi_n^{X,s}[C]$ we will write $\Phi_n(X), \Phi_n^s(X,c)$, and $\Phi_n^s(X,[C])$. We define a recursive functional u as follows: $u(X,n,c,s)=0$ if $\Phi_n^s(X,c)\uparrow$; otherwise $u(X,n,c,s)$ is equal to the least number greater than all questions to the oracle in the calculation $\Phi_n^s(X,c)$. We set $u(X,n,[a],s)=\sup\{u(X,n,c,s)\mid c<a\}$.

<u>LEMMA 1.</u> [5]. For any r.e. set A there is a r.e. set $C\equiv_T A$ such that $e\in A'\to C^{(e)}$ is finite and $e\notin A'\to C^{(e)}=N^{(e)}$.

<u>Proof.</u> Set $\hat{\Phi}_e^0(A^0,e)=\Phi_e^0(A^0,e)$,

$$\hat{\Phi}_e^{s+1}(A^{s+1},e)=\begin{cases}\Phi_e^s(A^s,e) & \text{if } \Phi_e^s(A^s,e)\downarrow \wedge A^{s+1}[u(A^s,e,e,s)]=A^s[u(A^s,e,e,s)];\\ \uparrow & \text{otherwise.}\end{cases}$$

Since $A'=\{e\mid \Phi_e(A,e)\downarrow\}$, then e ∈ A' is equivalent to $\hat{\Phi}_e^s(A^s,e)\downarrow$ for almost all S. Therefore we can take as C the set $\{\langle e,x\rangle \mid x<\operatorname{card}\{s\mid \hat{\Phi}_e^s(A^s,e)\uparrow\}\}$. The lemma is proved.

Now let us consider the notation connected with the Ershov hierarchy. Let $\psi_n(x)$ be the remainder of the division of $\varphi_n(x)$ by 2 if $\varphi_n(x)\downarrow$ and $\psi_n(x)\uparrow$ if $\varphi_n(x)\uparrow$. Then $\{\psi_n\}$ is the principle computable numeration of all partial recursive functions (p.r.f.) that take only the values 0 and 1. The numeration $\{\psi_n\}$ generates for any a ∈ 0 the numeration $\{V_{a,n}\}$ of class Σ_a^{-1} in the following manner: $V_{a,n}(x)=0$ if $\forall b<_0 a\,(\psi_n\langle x,b\rangle\uparrow)$, otherwise $V_{a,n}(x)=\psi_n\langle x,b\rangle$, where b is the $<_0$-least element from $\{z<_0 a\mid \psi_n\langle x,z\rangle\downarrow\}$. Let $\{K_a^s\}$ be an effective enumeration by steps of the r.e. set $\{z\mid z<_0 a\}$. Exercise 11-55 from [1] shows this sequence can be considered uniform with respect to a. We define the canonically computable sequence of finite sets $\{V_{a,n}^s\}$: $V_{a,n}^s(x)=0$, if $\forall b\in K_a^s\,(\psi_n^s\langle x,b\rangle\uparrow)$; otherwise $V_{a,n}^s(x)=\psi_{n_s}\langle x,b\rangle$ where b is the $<_0$-least element from $\{z\in K_a^s\mid \psi_n^s\langle x,z\rangle\downarrow\}$. It is clear that $\lim_{s\to\infty} V_{a,n}^s=V_{a,n}$ for all $a\in O$, $n\in N$. The class of Δ_a^{-1}- sets coincides with the class of all $X\subseteq N$ for which there is an n such that $X=V_{a,n}$ and $\forall x\exists b<_0 a\,(\psi_n\langle x,b\rangle\downarrow)$. The description given is found in [4, 6] and is based on the ideas of [7]. It is simpler than the original definition of [2] in that it does not refer to parity. The possibility of defining Σ_a^{-1} and Δ_a^{-1} without referring to parity was already clear from [3].

We will construct the unknown set R by steps, determining its limit calculation $\{R^s\}$. For this we construct the p.r.f. θ, taking only the values 0 and 1. At step s we determine its part proof and set θ^s if $R^s(x)=0$; otherwise $\forall b\in K_a^s\,(\theta^s\langle x,b\rangle\uparrow)$, $R^s(x)=\theta^s\langle x,b\rangle$, where b is the $<_0$-least element from $\{z\in K_a^s\mid \theta^s\langle x,z\rangle\downarrow\}$. For the proof of assertion 1) from the theorem we will construct θ with a calculation such that R satisfies for all e, $e=\langle x,m,n\rangle$, the following requirements:

$$T_e:\ \text{if } V_x\in\Delta_a^{-1} \text{ then } R\neq\Phi_m(V_x)\vee V_x\neq\Phi_n(R);$$
$$P_e:\ \langle e,x\rangle\in C\leftrightarrow\langle e+1,x\rangle\in R \quad \text{for almost all } x,$$

where $\{V_x\}$ is a numeration of Σ_a^{-1} sets, $(V_x=V_{a,x})$, and C is the set from Lemma 1. The requirements P_e ensure that $C'\leq_T R'$ and T_e that R will not be T-equivalent to any Δ_a^{-1}-set.

The condition $R' \leq_T C'$ will be satisfied due to the fact that at each step s at which $\varphi^s_e(R^{s-1}_e, e)\downarrow$, we forbid changing R by numbers less than $u(R^{s-1}, e, e, s)$ with a priority of e, thus preserving the calculation of $\varphi_e(R, e)$.

In the construction we use the apparatus of "Chinese boxes" [8]. Let σ be the set of all finite sequences of zeroes and ones, including the empty sequence ϕ. We denote the sequences by σ, τ, ... The length of σ we denote by $lh(\sigma)$; σ is identified with the mapping from $\{x \mid x < lh(\sigma)\}$ into {0, 1}. By σ*τ we denote the sequence obtained from σ by assigning τ to the right of it. Set $\sigma \subseteq \tau$ if $\tau = \sigma * \sigma_1$ for some $\sigma_1 \in \mathcal{O}$; $\sigma | \tau$ if $\sigma = \sigma_1 * 0 * \sigma_2$ and $\tau = \sigma_1 * 1 * \sigma_3$ for some $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{O}$; $\sigma < \tau$ if $\sigma \subset \tau$ or $\sigma | \tau$. We will note the obvious properties of the entities introduced:

LEMMA 2. The relation < is a linear order on $\mathcal{O}$. For any $\sigma, \tau \in \mathcal{O}$ we have: $\tau | \sigma * 0 \leftrightarrow \tau | \sigma$; $\tau < \sigma * 0 \leftrightarrow \tau \leq \sigma$; $\tau | \sigma * 1 \leftrightarrow \tau | \sigma \vee \sigma * 0 \subseteq \tau$; $\tau < \sigma * 1 \leftrightarrow \tau \leq \sigma \vee \sigma * 0 \subseteq \tau$.

We will describe subsidiary entities that arise in the construction. Let us imagine that with sequence σ a "box" B_σ is associated, consisting of three "divisions" B^{-1}_σ, B^0_σ and B^1_σ in which at step s certain numbers from $N^{(0)}$ can appear. Numbers from B^i_σ, $i \in \{0, 1\}$, can in the course of the construction be moved to the box $B_{\sigma * i}$ at which time they leave the box B_σ. Numbers falling into B^{-1}_σ cannot be moved. Numbers enter the box $B_{\sigma * i}$ in increasing order $y_0 < y_1 < y_2 < y_3 < \dots$ and are distributed among divisions as follows: y_0 in $B^{-1}_{\sigma * i}$, y_1 in $B^0_{\sigma * i}$, y_2 in $B^1_{\sigma * i}$, y_3 in $B^{-1}_{\sigma * i}$ etc.

At step s, along with the p.r.f. θ described above the p.r.f. c(σ, s) and the t.r.f. p(σ, s) and q(σ, s) are constructed, where σ ∈ σ. Further, set $r(\sigma, s) = \sup\{p(\tau, s), q(\tau, s) \mid \tau < \sigma\}$. If at step s the values of certain entities are not described, then their values remain the same as at set s − 1. At step 0 we set $B^{-1}_\phi = \{<0,0>\}$, $B^0_\phi = \{<0, 2x> \mid x > 0\}$, $B^1_\phi = \{<0, 2x+1> \mid x \geq 0\}$, $B_\sigma = \phi$ for $\sigma \neq \phi$; $c(\phi, 0) = <0,0>$, $c(\sigma, 0)\uparrow$ for proof; $\sigma \neq \phi$; $p(\sigma, 0) = q(\sigma, 0) = 0$ for all $\sigma \in \mathcal{O}$; $\theta^0 = \phi$.

We will describe a step s > 0. Every such step will be of one of types (σ, i), where $\sigma \in \mathcal{O}$, $i \leq 5$, and for each such pair (σ, i) there must be infinitely many types of type (σ, i). Let $e = lh(\sigma)$, $e = <x, m, n>$.

Type $(\sigma, 0)$. Let s' be the largest of the steps <s of type (σ, s), if such exists, otherwise s' = 0. If the set $(C^s \setminus C^{s'})^{(e)}$ is nonempty and in B^0_σ there is a number at step s − 1, then we move the least such number to $B_{\sigma * 0}$. Otherwise we do nothing.

Type (σ, 1). If in B^1_σ there is a number at step s − 1, then we move the least such number to $B_{\sigma * 1}$. Otherwise we do nothing.

Type (σ, 2). If $c(\sigma, s-1)\uparrow$ and in B^{-1}_σ there is at step s − 1 a number x such that $x \geq r(\sigma, s-1)$ and $\forall t < s\,(x \neq c(\sigma, t))$, then we set c(σ, s) equal to the least such x. Otherwise we do nothing.

Type σ, 3). For every x satisfying the condition

$$<e, x> \in C^s \wedge <e+1, x> \notin R^{s-1} \wedge r(\sigma * 0, s-1) \leq <e+1, x> \leq z,$$

where τ is the supremum of all numbers that by step s have been in $B_{\sigma * c}$, we do the following. We put $\langle e+1, x\rangle$ in R (i.e., we set $\theta^s \langle\langle e+1, x\rangle, 1\rangle = 1$, 1 is the least element in $(O; <_0)$). If $\sigma * 0 \mid \tau$, $c(\tau, s-1)\downarrow$, then we set $c(\tau, s)\uparrow$.

Type (σ, 4). We do nothing if the condition

$$c(\sigma, s-1)\downarrow \wedge \phi_e^s(R^{s-1}, e)\downarrow \wedge R^{s-1}[w_s] = R_\sigma^{s-1}[w_s]$$

is false, where $w_s = \sup\{q(\sigma, s-1), u(R^{s-1}, e, e, s)\}$, $R_\sigma^{s-1} = R^{s-1} \cup (\cup\{N^{(e+1)}_{r(\sigma[e+1], s-1)} \mid \sigma(e) = 0\})$.

Assume the condition is true. We set $q(\sigma, S) = w_s$; if $w_s > q(\sigma, s-1)$, $\sigma < \tau$ and $c(\tau, s-1)\downarrow$ then we set set $c(\tau, s)\uparrow$.

Type (σ, 5). If $c(\sigma, s-1)\uparrow$ we do nothing. Otherwise let c = c(σ, s − 1) and we will consider one of six cases.

Case 1. At all steps t ≤ s of type (σ, 5), c(σ, t − 1) = c, the following condition is false

$$R^{t-1}(c) = \Phi_m^t(V_x^t, c) \wedge V_x^t[u_t] = \Phi_n^t(R^{t-1}, [u_t]) \wedge$$
$$\wedge \forall y \le v_t \exists b \in K_a^t (\psi_x^t \langle y, b\rangle\downarrow) \wedge R^{t-1}[v_t] = R_\sigma^{t-1}[v_t],$$

where $u_t = u(V_x^t, m, c, t)$, $v_t = \sup\{p(\sigma, t-1), u(R^{t-1}, n, [u_t], t)\}$, R_σ^{t-1} is the same as at a step of type (σ, 4). In this case we do nothing.

Case 2. Case 1 does not occur and at all steps t < s of type (σ, 5), c(σ, t − 1) = c case 1 occurs.

Then (I_s) is true. We take for each y ≤ u_s a $b_y \in K_a^s$ such that $\psi_x^s\langle y, b_y\rangle\downarrow$. Let b be the $<_0$-largest element from $\{b_y \mid y \le u_s\}$. We set $\theta^s\langle c, b\rangle = 1$; $p(\sigma, s) = v_s$; if $\tau > \sigma$ and $c(\tau, s-1)\downarrow$, then we set $c(\tau, s)\uparrow$.

We further assume that cases 1 and 2 do not occur; therefore there exists a step $s_0 < s$ of type $(\sigma, 5)$, $c(\sigma, s_0 - 1) = c$ at which case 2 occurs.

Case 3. Case 1 and 2 do not occur and at all steps t of type $(\sigma, 5)$, $s_0 < t < s$, $c(\sigma, t-1) = c$, the condition

$$R^{t-1}(c) = \Phi_m^t(V_x^t, c) \wedge V_x^t[u_{s_0}] = \Phi_n^t(R^{t-1}, [u_{s_0}]) \wedge R^{t-1}[v_t] = R_\sigma^{t-1}[v_t]$$

is false, where $v_t = \sup\{p(\sigma, t-1), u(R^{t-1}, n, [u_{s_0}], t)\}$, R_σ^{t-1} is as in (I_t).

In this case we do nothing.

Case 4. Cases 1-3 do not occur and at all t of type (σ, 5), s_0 < t < s, c(σ, t − 1) = c, case 3 occurs.

Then the conditions (I_{s_0}) and (2_s) are true. We find y < u_{s_0}, for which $V_x^s(y) \ne V_x^{s_0}(y)$. Let b be the $<_0$-least element from $\{z \in K_a^s \mid \psi_x^s\langle y, z\rangle\downarrow\}$. We set $\theta^s\langle c, b\rangle = 0$, $p(\sigma, s) = v_s$.

If $\tau > \sigma$ and $c(\tau, s-1)\downarrow$, we set $C(\tau, s)\uparrow$.

We further assume that cases 1-4 do not occur; therefore at each step s_1 of type $(\sigma, 5)$, $s_0 < s_1 < s$, $C(\sigma, s_1) = c$, case 4 occurs, and in particular the number y is determinate.

Case 5. Cases 1-4 do not occur and at all steps t of type $(\sigma, 5)$, $s_1 < t \leq s$, $C(\sigma, t-1) = c$, the condition (2_y) is false.

Then we do nothing.

Case 6. Cases 1-5 do not occur.

Let s' be the largest step of type $(\sigma, 6)$, $s_1 \leq s' < s$, at which case 4 or 6 has occurred, and b' the $<_0$-least element from $\{z \in K_a^{s'} \mid \theta^{s'}\langle c, z\rangle\downarrow\}$. Let b be the $<_0$-least element from $\{z \in K_a^s \mid \psi_x^s\langle y, z\rangle\downarrow\}$. We set $\theta^s\langle c, b\rangle = 1$ if $\theta^{s'}\langle c, b'\rangle = 0$ and $\theta^s\langle c, b\rangle = 0$ if $\theta^{s'}\langle c, b'\rangle = 1$. If $\tau > \sigma$, $C(\tau, s-1)\downarrow$, then we set $C(\tau, s)\uparrow$. Let us pass to the next step.

The description of the construction is complete. The set of all numbers that have in the course of the construction been in box B_σ, we also denote by B_σ. We define $\mu: N \to \{0, 1\}$ as follows: $\mu(e) = 0$ if $B_{\mu[e]*0}$ is infinite, otherwise $\mu(e) = 1$.

LEMMA 3. For any e the set $B_{\mu[e]}$ is infinite, the set $\bigcup_{\sigma|\mu[e]} B_\sigma$ is finite and $\mu(e) = 0$ is equivalent to $C^{(e)}$ being infinite.

Proof. The proof is carried out by induction with respect to e. For e = 0 we have $\mu[e] = \phi$, so that $B_{\mu[e]} = N^{(0)}$, $\bigcup_{\sigma|\mu[e]} B_\sigma = \phi$. If $C^{(0)}$ is infinite, then for infinitely many steps of type $(\phi, 0)$ an element from B_ϕ^0 will be moved to B_0. Therefore B_0 is infinite and $\mu(0) = 0$. If $C^{(0)}$ is finite, then only on a finite set of steps of type $(\phi, 0)$ will elements B_ϕ^0 move to B_0. Therefore B_0 is finite and $\mu(e) = 1$. Assume that for e the assertion is proved and let us prove it for e + 1. First let $C^{(e)}$ be infinite. Then $\mu(e) = 0$ and for infinitely many steps of type $(\mu[e], 0)$ numbers from $B_{\mu[e]}^0$ move to $B_{\mu[e]*0} = B_{\mu[e+1]}$. Therefore $B_{\mu[e+1]}$ is infinite. Further $\sigma|\mu[e+1] \longleftrightarrow \sigma|\mu[e]$ by Lemma 2; therefore $\bigcup_{\sigma|\mu[e+1]} B_\sigma = \bigcup_{\sigma|\mu[e]} B_\sigma$ is finite. Now let $C^{(e)}$ be finite, $\mu(e) = 1$. Then only at a finite number of steps of type $(\mu[e], 0)$ does an element from $B_{\mu[e]}^0$ move to $B_{\mu[e]*0}$; therefore $\bigcup_{\sigma \supseteq \mu[e]*0} B_\sigma = B_{\mu[e]*0}$ is finite. At steps of type $(\mu[e], 1)$ all elements of the infinite set $B_{\mu[e]}$ move to $B_{\mu[e]*1} = B_{\mu[e+1]}$, i.e.. $B_{\mu[e+1]}$ is infinite. By Lemma 2, $\sigma|\mu[e+1] \longleftrightarrow \sigma|\mu[e] \vee \mu[e]*0 \subseteq \sigma$. Therefore $\bigcup_{\sigma|\mu[e+1]} B_\sigma = \left(\bigcup_{\sigma|\mu[e]} B_\sigma\right) \cup \left(\bigcup_{\sigma \supseteq \mu[e]*0} B_\sigma\right)$. The set $\bigcup_{\sigma|\mu[e]} B_\sigma$ is finite by induction, and $\bigcup_{\sigma \supseteq \mu[e]*0} B_\sigma$ by what was proven above, that is $\bigcup_{\sigma|\mu[e+1]} B_\sigma$ is finite as well. It remains to show that $\mu(e+1) = 0 \leftrightarrow C^{(e+1)}$ is finite. This is proved as in the basis of induction.

LEMMA 4. For any x there is a sequence σ such that starting from some step, the number $\langle 0, x\rangle$ is continually in B_σ^{-1} or B_σ^0.

Proof. Assume the contrary: the number $\langle 0, x\rangle$ in the course of the construction moves infinitely many times, in particular, it never falls into B_τ^{-1}, $\tau \in \sigma$. Let $y_0(\tau, i) < y_1(\tau, i) < \ldots$ be a direct enumeration of the set B_τ^i $(\tau \in \sigma, i \in \{0, 1\})$. Let r and $i \in \{0, 1\}$ be such that $\langle 0, x\rangle = y_n(\phi, i)$. Applying to n Euclid's alogirthm, we find q_0, q_1, q_2, r_0, r_1 such that $n = 3q_0 + r_0$, $q_0 = 3q_1 + r_1$, $q_1 = 3q_2$, where $r_0, r_1 \in \{1, 2\}$ (we assume that the algorithm terminates at the third step). Taking into account the rule of distribution of numbers from B_τ among divisions B_τ^{-1}, B_τ^0, B_τ^1 we obtain

$$<0,x> = y_n(\phi,i) = y_{q_0}(i,r_0-1) = y_{q_1}(i*(r_0-1),r_1-1) = y_{q_2}(i*(r_0-1)*(r_1-1),-1),$$

i.e., $<0,x> \in B_\sigma^{-1}$, $\sigma = i*(r_0-1)*(r_1-1)$. Contradiction.

LEMMA 5. For any $\sigma \in \mathcal{O}$ and $s \in N$ we have: $p(\sigma,s) \leq p(\sigma,s+1)$; $q(\sigma,s) \leq q(\sigma,s+1)$; $r(\sigma,s) \leq r(\sigma,s+1)$; if $c(\sigma,s)\downarrow$ then $c(\sigma,s) \geq r(\sigma,s)$.

Proof. The first three assertions are obvious. Let $c = c(\sigma,s)\downarrow$ and assume the contrary: $c < r(\sigma,s)$. Let $t \leq s$ be a step at which $c(\sigma,t) = c$, $c(\sigma,t-1)\uparrow$. Then t is a step of type $(\sigma, 2)$, hence $c \geq r(\sigma,t-1)$. Since $c < r(\sigma,s)$ then there is a $\tau < \sigma$ and a least step t_1, $t < t_1 \leq s$, for which $p(\tau,t_1) > p(\tau,t_1-1)$ or $q(\tau,t_1) > q(\tau,t_1-1)$. Let us consider the first case (the second analogously). Then t_1 is a step of type $(\tau,5)$ at which case 2 or 4 occurs. Since $\sigma > \tau$ then $c(\sigma,t_1)\uparrow$ by construction at step t_1. By construction at steps $t_2 \geq t_1$ of type $(\sigma, 2)$ we have $c(\sigma,t_2) \neq c(\sigma,t) = c$. Hence $c(\sigma,s) \neq c$. Contradiction

LEMMA 6. Let s be a step of type $(\sigma, 4)$, at which the condition of this step is true. Then for any $t > s$, $c(\sigma,t) = c(\sigma,s)$, and any $z < q(\sigma,s)$, $z \neq c(\sigma,s)$, we have $R^t(z) = R^{t-1}(z)$.

Proof. Assume the contrary: $z < q(\sigma,s)$, $z \neq c(\sigma,s)$, $R^t(z) \neq R^{t-1}(z)$. First let $z \in N^{(0)}$. Then $z = c(\tau,t-1)$ and t is a step of type $(\tau, 5)$, at which one of the cases 2, 4, 6 occurs. Since $z \neq c(\sigma,s)$, then $\sigma \neq \tau$. If $\sigma < \tau$, then, by Lemma 5, $z = c(\tau,t-1) \geq r(\tau,t-1) \geq q(\sigma,t-1) \geq q(\sigma,s)$. This is inconsistent with the condition $y < q(\sigma,s)$. If $\tau < \sigma$, then $c(\sigma,t)\uparrow$ by construction at step t, which is inconsistent with the condition of the lemma. Now let $z = <e+1,x>$ for some e, x. Then t is a step of type $(\tau,3)$, $e = lh(\tau)$ and the condition of step t is true, in particular $z \notin R^{t-1}$, $r(\tau*0,t-1) \leq z$. The cases $\sigma < \tau*0$, $\tau*0 \mid \sigma$, $\tau*0 \subseteq \sigma$ are possible. In the first case $z \geq r(\tau*0,t-1) \geq q(\sigma,t-1) \geq q(\sigma,s)$, which is inconsistent with $z < q(\sigma,s)$. In the second case $c(\sigma,t)\uparrow$, which is inconsistent with the condition of the lemma. Finally, let $\tau*0 \subseteq \sigma$. Since $e = lh(\tau)$, then $\sigma(e) = 0$. We have $z \geq r(\tau*0,t-1) = r(\sigma[e+1],t-1) \geq r(\sigma[e+1], s-1)$; therefore $z \in N^{(e+1)}_{r(\sigma[e+1],s-1)} \subseteq R_\sigma^{s-1}$. By the condition of step s we have $R_\sigma^{s-1}[w_s] = R^{s-1}[w_s]$, $w_s = q(\sigma,s)$. Therefore $z \in R^{s-1}$ and $z \in R^{t-1}$, which is inconsistent with the condition $z \notin R^{t-1}$.

The following lemma is proved in the same way as Lemma 6.

LEMMA 7. Let s be a step of type $(\sigma, 5)$, at which one of the cases 2, 4, 6 occurs. For any $t > s$, $c(\sigma,t) = c(\sigma,s)$, and any $z < p(\sigma,s)$, $z \neq c(\sigma,s)$, we have $R^t(z) = R^{t-1}(z)$.

LEMMA 8. Let $c \in N^{(0)}$, $s_0 < s_1 < s_2 < s_3 < \ldots$ be a sequence of steps of type $(\sigma,5)$, $c(\sigma,s_i) = c$, at which cases 2, 4, 6, 6, ... occur respectively. Then $b_0 >_0 b_1 >_0 \ldots$, where b_i is the value of the variable b at step s_i.

Proof. We have (1_{s_0}) and (2_{s_1}), therefore

$$\phi_m^{s_1}(V_x^{s_1},c) = R^{s_1-1}(c) = R^{s_0}(c) = 1 \neq R^{s_0-1}(c) = \phi_m^{s_0}(V_x^{s_0},c).$$

Hence $V_x^{s_1}[u_{s_0}] \neq V_x^{s_0}[u_{s_0}]$, that is, at step s_1 we find $y < u_{s_0}$, $V_x^{s_1}(y) \neq V_x^{s_0}(y)$. By construction at steps s_0 and s_1, $\psi_x^{s_0}<y,b_y>\downarrow$, $b_y \in K_a^{s_0}$, $b_y \leq_0 b_0$, b_1 is the least $<_0$-element from $\{z \in K_a^{s_1} \mid \psi_x^{s_1}<y,z>\downarrow\}$. Therefore $b_1 <_0 b_0$. By Lemma 7, $R^{s_2-1}(z) = R^{s_0-1}(z)$ for all

$z<\rho(\sigma,s_0)=U_{s_0}$, $z\neq c$. But $R^{s_2-1}(c)=R^{s_1}(c)=0=R^{s_0-1}(c)$, therefore in fact $R^{s_2-1}[U_{s_0}]=R^{s_0-1}[U_{s_0}]$. Hence and from (1_{s_0}) and (2_{s_2}) we obtain

$$V_x^{s_2}[u_{s_0}]=\phi_n^{s_2}(R^{s_2-1}, [u_{s_0}])=\phi_n^{s_0}(R^{s_0-1}, [u_{s_0}])=V_x^{s_0}[u_{s_0}].$$

Therefore $V_x^{s_2}(y)=V_x^{s_0}(y)\neq V_x^{s_1}(y)$. Hence and from the fact that $b_1\in K_a^{s_1}$, $\psi_x^{s_1}<y,b_1>\downarrow$, b_2 is the $<_0$-least element from $\{z\in K_a^{s_2}\mid \psi_x^{s_2}<y,z>\downarrow\}$, we obtain $b_2<_0 b_1$. In the presence of steps $s_3,\ldots$ the reasoning is analogous.

LEMMA 9. The set R lies in Σ_a^{-1}.

Proof. From Lemma 8 it follows that the p.r.f. θ is defined correctly. Since R(c) = 0 in the case $\forall b<_0 a\,(\theta<c,b>\uparrow)$ and $R(c)=\theta<c,b>$ otherwise, where b is the $<_0$-least element from $\{z<_0 a\mid \theta<c,z>\downarrow\}$, then $R\in\Sigma_a^{-1}$.

LEMMA 10. For any e there exists a step t_e such that for all $s\geq t_e$, $\sigma\leq\mu[e]$, we have $\rho(\sigma,s)=\rho(\sigma,t_e)$, $q(\sigma,s)=q(\sigma,t_e)$, $c(\sigma,s)=c(\sigma,t_e)$ and $c(\mu[e],t_e)\downarrow$.

Proof. The proof is carried out by induction with respect to e. Let $e=0$, $\mu[e]=\phi$. Then $c(\phi,s)=<0,0>$ for all s. Therefore the function $\lambda s.\rho(\phi,s)$ changes value no more than twice (in cases 2 and 4). Let s_1 be such that $\rho(\phi,s)=\rho(\phi,s_1)$ for $s\geq s_1$. By Lemma 9 we find $s_2\geq s_1$ for which $R^s(<0,0>)=R^{s_2}(<0,0>)$ for $s\geq s_2$. After step s_2 the function $\lambda s.\,q(\phi,s)$ changes value no more than once. In fact, let $s_3\geq s_2$, $q(\phi,s_3)>q(\phi,s_3-1)$. Then s_3 is a step of type (ϕ, 4), at which the condition of this step is true. By the choice of s_2 and by Lemma 6 $R^t[q(\phi,s_3)]=R^{s_3-1}[q(\phi,s_3)]$ for $t\geq s_3$. Since $q(\phi,s_3)\geq u(R^{s_3-1},e,e,s_3)$ then $u(R^{t-1},e,e,s)=u(R^{s_3-1},e,e,s)$ for $t\geq s_3$ and therefore $q(\phi,t)=q(\phi,s_3)$.

Assume the assertion to be proven for e and let us prove it for e + 1. By Lemmas 3 and 9 we find an $s_1\geq t_e$ such that $R^s(z)=R^{s_1}(z)$ for all $s\geq s_1$ and all z satisfying the condition

$$z<r(\mu[e]*0,t_e)\vee z\leq \sup\Big(\bigcup_{\tau|\mu[e]} B_\tau\Big)\vee \exists\sigma\leq\mu[e]\,(z=c(\sigma,t_e)).$$

The cases μ(e) = 0 and μ(e) = 1 are possible. Let $\mu(e)=0$, $\mu[e+1]=\mu[e]*0$. Since $B_{\mu[e+1]}$ is infinite, then $c(\mu[e+1],s_2)\downarrow$ for some $s_2\geq s_1$. We will assert that $c(\mu[e+1],s)=c(\mu[e+1],s_2)$ for $s\geq s_2$. Assume the contrary. Then there is a least $s>s_2$, $c(\mu[e+1],s)\uparrow$. Then s is a step of one of the types $(\sigma,3)$, $(\sigma,4)$, $(\sigma,5)$. In the first case $<i+1,x>\in R^s\setminus R^{s-1}$, $\sigma*0|\mu[e+1]$ and $<i+1,x>\leq \sup B_{\sigma*0}$ for $i=lh(\sigma)$ and some x. Since $\sigma*0\,|\,\mu[e]$ by Lemma 2, this is inconsistent with choice of s_1. Let s be a step of type $(\sigma,4)$. Then $\sigma<\mu[e+1]$ and $q(\sigma,s)>q(\sigma,s-1)$. Since $\sigma\leq\mu[e]$ by Lemma 2, this is inconsistent with the choice of t_e. If s is a step of type $(\sigma,5)$, then $\sigma<\mu[e+1]$ and at step s one of the cases 2, 4, 6 occurs. But then $\sigma\leq\mu[e]$ and $R^s(c(\sigma,t))\neq R^{s-1}(c(\sigma,t))$. This is inconsistent with choice of t_e. Then $\forall s\geq s_2(c(\mu[e+1],s)=c(\mu[e+1],s_2))$ is proven. The further proof is the same as in the basis of induction.

Let $\mu(e)=1$, $\mu[e+1]=\mu[e]*1$. From Lemmas 2-4 it follows that there is an $s_2\geq s_1$ such that when $s\geq s_2$ we have $c(\sigma,s)=c(\sigma,s_2)$ for $\sigma|\mu[e+1]$ and $R^s(z)=R^{s_2}(z)$ for all z satisfying the condition $z<r(\mu[e]*0,t_e)\vee z\leq\sup\Big(\bigcup_{\tau|\mu[e+1]}B_\tau\Big)\vee\exists\sigma<\mu[e+1]\,(z=c(\sigma,s_2))$. From the description of steps of type (σ, 4) and (σ, 5) it follows that if $\sigma\supseteq\mu[e]*0$ and $c(\sigma,s_2)\uparrow$ then $\rho(\sigma,s)=\rho(\sigma,s_2)$ and $q(\sigma,s)=q(\sigma,s_2)$ for all $s\geq s_2$. Let $\sigma_1<\ldots<\sigma_k$ be a lexicographic ordering of the set $\{\sigma \ni \mu[e]*0 \mid c(\sigma,S_2)\downarrow\}$. Carrying out for $\sigma_1,\dots,\sigma_K$ in turn the reasoning from the basis of induction, we find at the end a step $S_3 \geq S_2$ such that for $S \geq S_3$ we have $\forall\sigma < \mu[e+1]$ $(p(\sigma,S)=p(\sigma,S_3) \wedge q(\sigma,S)=q(\sigma,S_3))$ and $R^S(z)=R^{S_3}(z)$ for all $z<r(\mu[e+1],S_3)$. The further reasoning is the same as in the case $\mu(e)=0$ (replacing s_1 with s_3).

LEMMA 11. If $\mu(e) = 0$, then $N^{(e+1)}_{r(\mu[e+1],t_e)} \subseteq R$. $A' \leq_T R'$.

Proof. By Lemma 10, $r(\mu[e+1],s)=r(\mu[e+1],t_e)$ for $s \geq t_e$. By Lemma 3 the set $B_{\mu[e]*0} = B_{\mu[e+1]}$ is infinite. Hence and from the description of steps of type $(\mu[e], 3)$ we obtain $N^{(e+1)}_{r(\mu[e+1],t_e)} \subseteq R$. Since $<e+1,x> \in R \rightarrow <e,x> \in C$, we have: $C^{(e)}$ is finite $\rightarrow R^{(e+1)}$ is finite and $C^{(e)} = N^{(e)} \rightarrow R^{(e+1)} = {}^*N^{(e+1)}$. Hence and from Lemma 1 we obtain

$$e \notin A' \leftrightarrow C^{(e)} = N^{(e)} \leftrightarrow R^{(e+1)} = {}^*N^{(e+1)} \leftrightarrow \exists y\, \forall x \geq y\, (<e+1,x> \in R),$$

i.e., $A' \in \Pi^{0,R}_2$. It is clear that $A' \in \Sigma^{0,R}_2$. Therefore $A' \in \Delta^{0,R}_2$, $A' \leq_T R'$.

LEMMA 12. For all $X \in \Delta^{-1}_a$ we have $R \not\equiv_T X$.

Proof. Assume the contrary. Then there is an x such that $R \equiv_T V_x$ and $\forall y \exists b <_0 a\, (\psi_x <y,b> \downarrow)$. Let m and n be such that $R = \Phi_m(V_x)$ and $V_x = \Phi_n(R)$. Let $e=<x,m,n>$. By Lemma 10 $c(\mu[e],s)=c(\mu[e],t_e)\downarrow$ for $s \geq t_e$. Hence and from Lemma 11 it follows that there exists an infinite sequence of steps $s_0 < s_1 < s_2 < s_3 < \dots$ of type $(\mu[e],5)$, $s_0 \geq t_e$ at which the cases 2, 4, 6, 6... respectively occur. By Lemma 8 $b_0 \, {}_0{>}_1 \, {}_0{>} \cdots$ which is inconsistent with the foundedness of the relation $<_0$.

LEMMA 13. $R' \leq_T A'$.

Proof. We must prove that with an oracle A' it is possible to know whether $\Phi_e(R,e)$ is determinate. By Lemmas 1 and 3 $\mu \equiv_T A'$; therefore we can find $\mu[e]$. From the proof of Lemma 10 it is clear that t_e is found effectively with respect to A'. Let $c=c(\mu[e],t_e)\downarrow$. We find with the oracle ϕ' the least $t \geq t_e$ such that $R^s(c)=R^t(c)$ for $s \geq t$. It is sufficient to verify that $\Phi_e(R,e)\downarrow$ is equivalent to there being a step $s \geq t$ of type $(\mu[e], 4)$ at which the condition of this step is true. Let $\Phi_e(R,e)\downarrow$, $s_1 \geq t$ be such that $\Phi^s_e(R^{s-1}_t,e)\downarrow$ for $s \geq s_1$. By Lemmas 6 and 11 and the choice of t we find a $S_2 \geq S_1$ such that $R^t[w]=R^s[w]=R^s_{\mu[e]}[w]$ for any $s \geq s_2$ where $w=q(\mu[e],t_e)$. Therefore at the least step $s \geq s_2$ of type $(\mu[e],4)$ the condition of this step is satisfied. Conversely, let $s \geq t$ be a step of type $(\mu[e],4)$, at which $\Phi^s_e(R^{s-1},e)\downarrow \wedge R^{s-1}[w_s] = R^{s-1}_{\mu[e]}[w_s]$ Since $w_s \geq u(R^{s-1},e,e,s)$ and by Lemma 6 $R^{s_1}[w_s]=R^{s-1}[w_s]$ for all $s_1 \geq s$, then $\Phi_e(R,e) = \Phi^s_e(R^{s-1},e)\downarrow$.

With Lemma 13 the proof of the first assertion of the theorem is complete. Let us prove the second. Let a be a limit, $a=3 \cdot 5^{\alpha}$. Then $\bigcup_{b <_0 a} \Sigma^{-1}_b = \bigcup_{K \in N} \Sigma^{-1}_{\varphi_d(K)}$; therefore it is sufficient to construct $R \in \Delta^{-1}_a$, $R' \equiv_T A'$, R is not T-equivalent to any set from $\bigcup_K \Sigma^{-1}_{\varphi_d(K)}$. We denote $V_{\varphi_d(K),x}$ more briefly by $V_{K,x}$. To construct R we change the following in the construction. 1) At step 0 we define θ^0 as follows. If $y \in N^{(0)}$, let $\sigma \in \mathfrak{S}$ be such that y falls into B^{-1}_σ (σ is uniquely determined and is found effectively, see the proof of Lemma 4). Set $\theta^0 <y, 2^{\varphi_d(K)}>$, where $<K,x,m,n> = \ell h(\sigma)$, If $y \notin N^{(0)}$, set $\theta^0<y,2>=0$ (2 is the second element in magnitude in $(\mathcal{O}; <_{\mathcal{O}}))$. 2) At steps of type (σ, 5) we consider $lh(\sigma)=\langle k, x, m, n\rangle$; in the condition (2_t) instead of V_x we take $V_{k,x}$, and the condition (I_t) is replaced with

$$R^{t-1}(c)=\Phi_m^t(V_{k,x}^t,c)\wedge V_{k,x}^t[u_t]=\Phi_n^t(R^{t-1}[u_t])\wedge R^{t-1}[v_t]=R_\sigma^{t-1}[v_t].$$

The proof of the lemmas is hardly changed. Change 1) ensures that $\forall y\exists b<_{\mathcal{O}} a(\theta\langle y,b\rangle\downarrow)$, therefore $R\in\Delta_a^{-1}$. Lemma 12 now appears as follows: $R\not\equiv_T X$ for $X\in\bigcup_k\Sigma^{-1}_{\varphi_a(k)}$. Otherwise $R\equiv_T V_{k,x}$ for some k, x. Let m, n be such that $R=\Phi_m(V_{k,x}), V_{k,x}=\Phi_n(R)$ and set $e=\langle k,x,m,n\rangle$. Reasoning further as in Lemma 12, we obtain an inconsistency. For the rest the proof given above is suitable for the second assertion of the theorem.

In conclusion we prove that the degrees of Σ_a^{-1}-sets are rarely arranged in the ordering of degrees ≤ 0'. For the case of differences this was proven in [5].

THEOREM 2. For any not least a ∈ 0 there is an R and $\tilde{R}$ from Σ_a^{-1} such that $\tilde{R}<_T R$, $\tilde{R}'\equiv_T\emptyset'$, $R'\equiv_T\emptyset''$ and there does not exist $X\in\Delta_a^{-1}$, $\tilde{R}\leq_T X\leq_T R$. 2) For any limit a ∈ 0 there is an R, $\tilde{R}\in\Delta_a^{-1}$, such that $\tilde{R}<_T R$, $\tilde{R}'\equiv_T\emptyset'$, $R'\equiv_T\emptyset''$ and there does not exist $X\in\bigcup_{b<_{\mathcal{O}}a}\Sigma_b$, $\tilde{R}\leq_T X\leq_T R$.

Proof. We will prove only 1) since 2) is proven by the same modification of the construction as in Theorem 1). The set R is constructed by the construction of Theorem 1 with the changes indicated below. Set $\tilde{R}=R^{(0)}$, $\tilde{R}^s=(R^s)^{(0)}$. As A in the construction we take ϕ' and replace steps of type (σ, 4) with steps of type $(e,4), e\in N$.

Step s > 0 of Type (e, 4). We do nothing if the condition $\Phi_e^s(\tilde{R}^{s-1},e)\downarrow\wedge\exists\sigma(lh(\sigma)=e\wedge c(\sigma,s-1)\downarrow)$ is false. Otherwise let σ be the <-least square for which $lh(\sigma)=e, c(\sigma,s-1)\downarrow$. Set $q(\sigma,s)=\sup\{q(\sigma,s-1),u(\tilde{R}^{s-1},e,e,s)\}$. If $q(\sigma,s)>q(\sigma,s-1)$, $\tau>\sigma$ $c(\tau,s-1)\downarrow$, set $c(\tau,s)\uparrow$.

The proof of all lemmas except Lemma 13 remains almost unchanged. By Lemma 11, $R'\equiv_T\emptyset''$. The condition $\tilde{R}\leq_T R$ is obvious. From the proof of Lemma 12 it follows that there does not exist $X\in\Delta_a^{-1}$, $\tilde{R}\leq_T X\leq_T R$. In fact, otherwise there is an x such that $\tilde{R}\leq_T V_x\leq_T R$, $\forall y\exists b<_{\mathcal{O}} a(\psi_x\langle y,b\rangle\downarrow)$. Let m and n be such that $\tilde{R}=\Phi_m(V_x), V_x=\Phi_n(R)$, and let $e=\langle x,m,n\rangle$. Since $\tilde{R}(c)=R(c)$ for $c\in N^{(0)}$, the proof of Lemma 12 leads to an inconsistency. It remains to verify that $\tilde{R}'\leq_T\emptyset'$, i.e., with the oracle $\emptyset'$ we can know whether $\phi_e(\tilde{R}, e)$ is determinate. Let us find a <-least sequence σ of length e and a step s_0 such that $c(\sigma,s_0)\downarrow$ and $\forall t\geq s_0(c(\sigma,t)=c(\sigma,s_0))$. Such σ and s_0 exist by Lemma 10 and can be found effectively with respect to ϕ' (taking into account Lemma 3). Let us further find with the oracle ϕ' a step $s_1\geq s_0$ such that $R^s(c(\sigma,s_0))=R^{s_1}(c(\sigma,s_0))$ for all s ≥ s_1. Just as in Lemma 13 we verify that $\Phi_e(R,e)\downarrow$ is equivalent to there existing a step $s\geq s_1$ of type (e, 4) for which $\Phi_e^s(\tilde{R}^{s-1},e)\downarrow$. The theorem is proved.

Remark. The set R is T-equivalent to the direct sum of the set $\tilde{R}$ and the high r.e. set $R\setminus\tilde{R}$.

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ALGORITHMIC DEGREE OF UNARS

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INTRODUCTION

In this paper, questions are studied of the existence of unars with distinct algorithmic degrees of different type. All the necessary information can be found in [5, 6, 8].

Let $\mathfrak{A}$ be a model. The relation P in $\mathfrak{A}$ is called stable if it is invariant with respect to the action of the group of automorphisms of $\mathfrak{A}$. By $St(\mathfrak{A})$ is denoted the set of all stable relations of $\mathfrak{A}$. Following [7], let us give the following definitions.

Definition 1. a) The constructivizations ν and μ of $\mathfrak{A}$ are called essentially equivalent ($\nu \underset{C}{\sim} \mu$) if for any $S \in St(\mathfrak{A})$, $\nu^{-1}S$ is a recursive set if and only if the set $\mu^{-1}S$ is recursive.

b) The constructivization ν automatically reduces to μ. $\nu \leq_{\Pi} \mu$ if there exists a recursive function ψ with the property: if φ_x is a p.r.f. with a Kleene number x is the characteristic function of the set $\mu^{-1}S$ for some $S \in St(\mathfrak{A})$, then $\varphi_{\psi(x)}$ is the characteristic function for the set $\nu^{-1}S$. If $\nu \leq_{\Pi} \mu$ and $\mu \leq_{\Pi} \nu$ then we will write that $\nu \underset{\Pi}{\sim} \mu$.

c) The constructivization ν uniformly reduces to μ. $\nu \leq_{P} \mu$, if for some computable operation $\mathcal{F}$ we have $\mathcal{F}(\mu^{-1}S) = \nu^{-1}S$ for all $S \in St(\mathfrak{A})$. Let us write that $\nu \underset{P}{\sim} \mu$ if $\nu \leq_{P} \mu$ and $\mu \leq_{P} \nu$.

Let us denote the fact of the autoequivalence of ν and μ as $\nu \underset{A}{\sim} \mu$. The sign "$\leftrightharpoons$" is read as "this is by definition."

Definition 2. Let $\theta \in \{C, \Pi, P, A\}$. Then $\theta\text{-}\dim\mathfrak{A} \leftrightharpoons$ is the maximal number of not θ-equivalent constructivizations of $\mathfrak{A}$. The number $\theta\text{-}\dim\mathfrak{A}$ is called the θ-degree, and the set $\{\theta\text{-}\dim\mathfrak{A} \mid \theta \in \{C, \Pi, P, A\}\}$ the algorithmic degree of $\mathfrak{A}$.

1. Influence of Constants on the A-Degree

Let $a_1, \ldots, a_n \in |\mathfrak{A}|$, $n \in N$. It is evident that $A\text{-}\dim\mathfrak{A} \leq A\text{-}\dim(\mathfrak{A}; a_1, \ldots, a_n)$. Examples of models for which the sign "$\leq$" in a given relation could be replaced by "$<$" were not known.

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