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The Global Existence of Yang–Mills–Higgs Fields in 4-Dimensional Minkowski Space

II. Completion of Proof*

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Abstract. In this paper we complete the proof of global existence of Yang–Mills–Higgs fields in 4-dimensional Minkowski space by showing that an appropriate norm of the solutions cannot blow up in a finite time. A key step in the proof is the demonstration that the L^∞ norm of the curvature is bounded *a priori*. Our results apply to any compact guage group and to any invariant Higgs self-coupling which is positive and of no higher than quartic degree.

I. Introduction

In this paper we shall complete the proof of global existence of Yang–Mills–Higgs (YMH) fields which we began in Ref. 1 (referred to hereinafter as paper 1). In paper 1 we established local existence, uniqueness and smoothness properties of YMH fields in the temporal gauge, improving earlier results [2, 3] for this system by essentially one order of differentiability. To extend the argument to a global existence proof we must show that the $(H_2 \times H_1 \times H_2 \times H_1)$ norm of $(A_i, \dot{A}_i, \phi, \dot{\phi})$ does not blow up in a finite time. To accomplish this we shall first derive an *a priori* bound on the norms $\|{}^{(4)}F(t)\|_{L^\infty}$ and $\|D\phi(t)\|_{L^\infty}$ where ${}^{(4)}F$ is the curvature of the Yang–Mills potential ${}^{(4)}A$ and $D\phi$ is the covariant gradient of the Higgs field ϕ. Given this estimate we can easily complete the proof by showing that a suitably defined higher order "energy" does not blow up.

To derive an estimate on the "curvatures" $({}^{(4)}F, D\phi)$ we adopt a method inspired by Jörgens' treatment of the non-linear wave equation [4]. We write an integral

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equation for the values $({}^{(4)}F(p), D\phi(p))$ of the curvatures at an arbitrary point p within the domain of local existence by using the retarded (or advanced) fundamental solution of the linear wave equation, treating the non-linear terms as "sources" for the solution. This expresses the curvatures at p in terms of integrals over the past (or future) light cone from p to the initial surface and in terms of data on the initial surface itself. We then show that the most troublesome terms in the light cone integrals can be bounded by expressions of the form

$$\left(E_0 \int_{t_0}^{t} ds(\| {}^{(4)}F(s) \|_{L^\infty} + \| D\phi(s) \|_{L^\infty}) \right),$$

where E_0 is the energy of the solution. This result yields an integral inequality from which the bound on $\| {}^{(4)}F(t) \|_{L^\infty}$ and $\| D\phi(t) \|_{L^\infty}$ readily follows.

One can define the norms so that $\| {}^{(4)}F \|_{L^\infty}$ and $\| D\phi \|_{L^\infty}$ are gauge invariant. In deriving the integral inequality described above, we make use of this invariance to transform the potentials $({}^{(4)}A, \phi)$ to a convenient gauge for making the estimates. A gauge which is especially suited to this purpose is Cronström's gauge [5] which is defined so that $(x^\mu - x_p^\mu)A_\mu(x) = 0$. A remarkable feature of this gauge condition is that it allows one to express the potential ${}^{(4)}A$ explicitly in terms of the curvature ${}^{(4)}F$. By introducing Cronström's gauge (relative to the light cone vertex p) we can eliminate the potentials from the light cone integrals and thereby derive the inequality described above. In the appendix we show that Cronström's gauge condition can always be imposed throughout the domain of local existence. This method works because the fundamental solution to the ordinary wave equation is, in Cronström's gauge, a parametrix for the covariant wave equations satisfied by the curvatures ${}^{(4)}F$ and $D\phi$ [6].

Other recent work on the global existence of Yang–Mills fields has been carried out by Christodoulou and Choquet-Bruhat [7]. They make special use of the conformal invariance of the Yang–Mills equations to prove global existence for solutions with sufficiently small initial data. Though their method seems limited to this class of solutions it has the advantage of being able to treat Dirac fields coupled to YMH fields. Our approach on the other hand leans heavily on the positivity of energy for YMH fields and does not seem readily extendible to the Dirac case. In addition Glassey and Strauss [8], using a particular ansatz for the form of the potentials considered, have proven the global existence of a special class of solutions of the Yang–Mills equations.

We have not attempted here to characterize the general solution of the initial value constraint equations. In Ref. (9) however, one of us solved this problem within the context of certain weighted Sobolev spaces. A more extensive treatment would be needed to solve the corresponding problem in ordinary Sobolev spaces. Nevertheless it's clear that the constraints possess infinite dimensional families of non-trivial solutions in these spaces. As an example one can take (in the notation used herein) $E_i = \pi = 0$ and choose A_i and ϕ arbitrarily. This corresponds to "time symmetric" initial data. The constraints are of course preserved by the evolution equations.

II. Global Bounds on Growth of Norms

A. Preliminaries

We shall adopt here the same notation used in paper 1, writing for example

$$
\begin{aligned}
{}^{(4)}A &= A^{(a)}_{\mu}\theta_a dx^{\mu} = A_{\mu}dx^{\mu} \\
{}^{(4)}F &= F^{(a)}_{\mu\nu}\theta_a dx^{\mu} \wedge dx^{\nu} = F_{\mu\nu}dx^{\mu} \wedge dx^{\nu}
\end{aligned}
\tag{2.1}
$$

for the Yang–Mills potential and its curvature. Here $\{\theta_a\}$ is a basis for a real matrix representation of the Lie algebra $\mathscr{g}$ of an arbitrary compact Lie group G. Thus

$$[\theta_a, \theta_b] = f^{abc}\theta_c \tag{2.2}$$

for some constants f^{abc}. We choose the basis so that the θ_a are real $d \times d$ antisymmetric matrices obeying[1]

$$\mathrm{Tr}\{\theta_a, \theta_b\} = \delta_{ab} \tag{2.3}$$

and so that the f^{abc} are completely antisymmetric.

The Higgs field $\phi = \{\phi_\kappa\}, \kappa = 1, \ldots, d$, takes values in the real d-dimensional vector space associated to the given representation of $\mathscr{g}$. The covariant derivative of ϕ is defined by

$$D_\mu\phi = \partial_\mu\phi + A_\mu\phi, \tag{2.4}$$

and we define a gauge invariant contraction "." by

$$\phi\cdot\phi = \phi_\kappa\phi_\kappa, \quad (D_\mu\phi)\cdot(D_\nu\phi) = (D_\mu\phi)_\kappa(D_\nu\phi)_\kappa, \tag{2.5}$$

etc.

The Lagrangian for the Yang–Mills–Higgs (YMH) equations, with spacetime metric $\eta_{\mu\nu}$ of signature $(- + + +)$, is

$$\mathscr{L} = \mathrm{Tr}\{-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\} - \tfrac{1}{2}(D_\mu\phi)\cdot(D^\mu\phi) - P(\phi), \tag{2.6}$$

where $P(\phi)$ is a gauge invariant, positive polynomial in ϕ of no higher than quartic degree. For any sufficiently smooth G-valued function $\mathscr{U}$ over spacetime the gauge transformations are defined by

$$
\begin{aligned}
A'_\mu &= \mathscr{U}A_\mu\mathscr{U}^{-1} + \mathscr{U}\partial_\mu\mathscr{U}^{-1}, \\
\phi' &= \mathscr{U}\phi, \\
F'_{\mu\nu} &= \mathscr{U}F_{\mu\nu}\mathscr{U}^{-1}, \quad (D_\mu\phi)' = \mathscr{U}(D_\mu\phi),
\end{aligned}
\tag{2.7}
$$

and the invariance condition on P means $P(\mathscr{U}\phi) = P(\phi)\forall\phi$ and $\forall\mathscr{U}\in G$.

From $\mathscr{L}$ one derives, in the usual way, a gauge invariant energy-momentum tensor $T^{\mu\nu}$ given by

$$
\begin{aligned}
T^{\mu\nu} = {} & \mathrm{Tr}\{F^{\mu\alpha}F^{\nu}{}_{\alpha} - \tfrac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\} + (D^\mu\phi)\cdot(D^\nu\phi) \\
& - \tfrac{1}{2}\eta^{\mu\nu}(D_\alpha\phi)\cdot(D^\alpha\phi) - \eta^{\mu\nu}P(\phi),
\end{aligned}
\tag{2.8}
$$

1 For convenience we have defined the trace operation Tr to be the negative of the usual matrix trace

which satisfies

$$\partial_\nu T^{\mu\nu} = 0, \tag{2.9}$$

as a consequence of the equations of motion

$$\nabla_\nu F^{\mu\nu} = -((D^\mu \phi)\cdot\theta_a\phi)\theta_a \tag{2.10}$$

and

$$(D_\mu D^\mu \phi)_\kappa = \frac{\partial P}{\partial \phi_\kappa}, \tag{2.11}$$

where

$$\nabla_\gamma F_{\alpha\beta} = \partial_\gamma F_{\alpha\beta} + [A_\gamma, F_{\alpha\beta}]. \tag{2.12}$$

The curvature also obeys the Bianchi identity

$$\nabla_\gamma F_{\alpha\beta} + \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} = 0. \tag{2.13}$$

Taking the covariant divergence of (2.13) and making use of the field equation (2.10), one derives

$$\nabla^\gamma \nabla_\gamma F_{\alpha\beta} = ((F_{\alpha\beta}\phi)\cdot\theta_a\phi)\theta_a + [(D_\beta\phi)\cdot\theta_a(D_\alpha\phi) - (D_\alpha\phi)\cdot\theta_a(D_\beta\phi)]\theta_a + 2[F^\gamma{}_\alpha, F_{\gamma\beta}], \tag{2.14}$$

where, written out explicitly,

$$\nabla^\gamma \nabla_\gamma F_{\alpha\beta} = \eta^{\mu\nu}\partial_\mu\partial_\nu F_{\alpha\beta} + 2\partial_\gamma([A^\gamma, F_{\alpha\beta}]) - [\partial_\gamma A^\gamma, F_{\alpha\beta}] + [A^\gamma, [A_\gamma, F_{\alpha\beta}]]. \tag{2.15}$$

In a similar way one derives

$$D_\mu D^\mu (D_\alpha \phi) = ((D_\alpha\phi)\cdot\theta_a\phi)\theta_a\phi - 2F_\alpha{}^\mu(D_\mu\phi) + D_\alpha\left(\frac{\partial P}{\partial\phi}\right), \tag{2.16}$$

where $\left(\frac{\partial P}{\partial \phi}\right)_\kappa = \frac{\partial P}{\partial \phi_\kappa}$. Equations (2.14) and (2.16) will play a key role in the analysis below.

If we contract $T^{\mu\nu}$ with the timelike killing field $\hat{X} = \frac{\partial}{\partial t}$ we get a vector field

$$J^\alpha = X^\beta T^\alpha_\beta = T^\alpha_0, \tag{2.17}$$

which satisfies the continuity equation

$$\partial_\alpha J^\alpha = 0. \tag{2.18}$$

If we integrate this equation over the interior of the past light cone K_p from a point p to the ($t = t_0 =$ constant) initial data surface and use Gauss' theorem, we may express the result as the vanishing of a surface integral over the boundary of this region. To write this explicitly, let us translate the coordinate system until p lies at the origin and introduce spherical spatial coordinates (r, θ, ϕ) centered at p and a system of basis vector fields

$$\hat{\ell} = -\frac{\partial}{\partial t} + \frac{\partial}{\partial r}, \quad \hat{m} = +\frac{\partial}{\partial t} + \frac{\partial}{\partial r},$$
$$\hat{e}_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \hat{e}_\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{2.19}$$

Relative to the spacetime metric these vector fields have the inner products

$$\hat{\ell}\cdot\hat{\ell}=\hat{m}\cdot\hat{m}=0,\quad \hat{\ell}\cdot\hat{m}=2,$$
$$\hat{\ell}\cdot\hat{e}_A=\hat{m}\cdot e_A=0,\quad \hat{e}_A\cdot\hat{e}_B=\delta_{AB},\tag{2.20}$$

where $\{\hat{e}_A\}=\{\hat{e}_\theta,\hat{e}_\phi\}$ and $\hat{\ell}\cdot\hat{m}=\ell^\mu m_\mu$, etc. The null field $\hat{\ell}$ is tangent to the light cone K_p.

In this notation the conservation law described above yields the equation[2]

$$\begin{aligned}&\int_{K_p} r^2 dr d\Omega\{\tfrac{1}{2}\mathrm{Tr}[\tfrac{1}{4}(\hat{\ell}\cdot{}^{(4)}F\cdot\hat{m})^2+(\hat{\ell}\cdot{}^{(4)}F\cdot\hat{e}_A)^2+\tfrac{1}{2}(\hat{e}_A\cdot{}^{(4)}F\cdot\hat{e}_B)^2]\\ &\quad+\tfrac{1}{2}[(D_{\hat{\ell}}\phi)\cdot(D_{\hat{\ell}}\phi)+(D_{\hat{e}_A}\phi)\cdot(D_{\hat{e}_A}\phi)]+P(\phi)\}|_{t=-r}\\ &=\int_{B_p} r^2 dr d\Omega\{\tfrac{1}{2}\mathrm{Tr}[E^iE_i+B^iB_i]+\tfrac{1}{2}\pi\cdot\pi+\tfrac{1}{2}(D_i\phi)\cdot(D^i\phi)+P(\phi)\}|_{t=t_0},\end{aligned}\tag{2.21}$$

where $d\Omega=\sin\theta d\theta d\phi$ and we have defined

$$E_i=F_{0i},B^i=\frac{\varepsilon^{ijk}}{2}F_{jk},\pi=D_0\phi,\hat{\ell}\cdot{}^{(4)}F\cdot\hat{m}=\ell^\mu F_{\mu\nu}m^\nu,\text{etc.}\tag{2.22}$$

and

$$D_{\hat{\ell}}\phi=\ell^\mu D_\mu\phi,\text{etc.}\tag{2.23}$$

Here B_p represents the solid sphere in the initial surface $t=t_0$ which is bounded by the intersection of K_p with this surface. Thus B_p is a solid sphere of radius $r_0=|t_0|$.

The right hand side of Eq. (2.21) represents the energy contained within the region B_p at $t=t_0$. Since $P(\phi)$ is positive by assumption, this energy is bounded by the total (conserved) energy E_0,

$$E_0=\int_{R^3} d^3x\{\tfrac{1}{2}\mathrm{Tr}[E^iE_i+B^iB_i]+\tfrac{1}{2}\pi\cdot\pi+\tfrac{1}{2}(D_i\phi)\cdot(D^i\phi)+P(\phi)\},\tag{2.24}$$

of the solution considered.

The left hand side of Eq. (2.21) represents the flux of energy through the cone K_p. Note especially that the integrand in this flux integral consists (except for $P(\phi)$) of a sum of squares of projections of the curvature ${}^{(4)}F$ and the gradient $D_\mu\phi$ of ϕ. Not all of the projections occur in this integral. In particular, $\hat{m}\cdot{}^{(4)}F\cdot\hat{e}_A$ and $D_{\hat{m}}\phi$ are absent. Roughly speaking, this means that the flux integral measures energy flowing "across the cone" K_p but not energy flowing "along the cone." That such flux integrals are always bounded by the total energy of the solution will play a crucial role in the argument below.

If K_p is, as above, a light cone from p to the initial data surface which lies within the domain of local existence of some solution $({}^{(4)}A,\phi)$ then we can define, on an open set S_p containing the set bounded by $K_p\cup B_p$, the Cronström transform $({}^{(4)}\hat{A},\hat{\phi})$ of the given solution. The Cronström [5] transform is simply the gauge transform of $({}^{(4)}A,\phi)$ defined so that

$$(x^\mu-x^\mu_p)\hat{A}_\mu(x)=0,\quad \hat{A}_\mu(x_p)=0\tag{2.25}$$

2 This and other conservation laws for the Yang–Mills equations have been studied extensively by R. Glassey and W. Strauss in Commun. Math. Phys. **67**, 51–67 (1979)

on S_p. In the appendix we prove that a unique gauge transformation U (with $U(x_p) =$ id) always exists which transforms (A_μ, ϕ) to Cronström's gauge on suitably shaped regions of spacetime. We shall make use of Cronström's gauge in the next section to derive an *a priori* estimate on the (gauge invariant) L^∞ norms of ${}^{(4)}F$ and $D_\mu\phi$.

A remarkable feature of Cronström's gauge is that it allows one to express the potential $\hat{A}_\mu$ explicitly in terms of the curvature $\hat{F}_{\mu\nu}$. Translating the origin of coordinates to p as before, we can express this relationship as [5]

$$\hat{A}_\nu(x) = \int_0^1 d\lambda\, \lambda x^\mu \hat{F}_{\mu\nu}(\lambda x). \tag{2.26}$$

Differentiating this formula and using the field equations one can also derive

$$\partial_\mu \hat{A}^\mu(x) = \int_0^1 d\lambda \{\lambda^2 x^\alpha [\hat{F}_{\alpha\mu}(\lambda x), \hat{A}^\mu(\lambda x)] - \lambda^2 x^\alpha ((D_\alpha \hat{\phi}(\lambda x)) \cdot \theta_a \hat{\phi}(\lambda x))\theta_a\}. \tag{2.27}$$

These formulas were given in Cronström [5]. For completeness we sketch their derivation in the appendix.

In paper 1 we showed that if the temporal gauge $(A_0 = 0)$ initial data $u_0 \equiv (A_i, E_i, \phi, \pi)|_{t=0}$ lies in the Sobolev space $\mathscr{H} \equiv (H_{s+1} \times H_s \times H_{s+1} \times H_s) \equiv (H_{s+1} \times H_s)^2$ for $s \geqq 1$ then there is a unique solution $u(t)$ of the integral equation associated to the YMH system on some interval (t_a, t_b) containing the initial surface $t = 0$ and having $u(0) = u_0$. Furthermore, either $\|u(t)\|_{\mathscr{H}} \to \infty$ as $t \to t_a$ or $t \to t_b$ (or both) or else $(t_a, t_b) = (-\infty, \infty)$ and the "abstract" solution $u(t)$ is global. Within this context we also showed that if the initial data is restricted to lie $(H_{s+1+k} \times H_{s+k})^2$ for $s \geqq 1, k \geqq 2$ and to satisfy the initial value constraint equation (i.e., the $\mu = 0$ component of Eq. (2.10) above), then the abstract solution $u(t)$ defines C^k potentials (A_i, ϕ) and C^{k-1} momenta (E_i, π) on $(t_a, t_b) \times R^3$ which satisfy the temporal gauge YMH equations (including constraint) in the classical sense. If $k \geqq 3$ then $(F_{\mu\nu}$ and $D_\alpha\phi)$ will in turn satisfy Eqs. (2.14) and (2.16) above in the classical sense. Paper 1 actually treated a wider class of (distributional) solutions of the YMH equations but for simplicity we shall in this paper restrict our attention to the classical solutions.

As we shall show in the appendix, the gauge transformation to Cronström's gauge is a G-valued C^k function U which takes $(A_\mu, F_{\mu\nu}, \phi, D_\alpha\phi) \in (C^k \times C^{k-1} \times C^k \times C^{k-1})$ to $(\hat{A}_\mu, \hat{F}_{\mu\nu}, \hat{\phi}, D_\alpha\hat{\phi}) \in (C^{k-1}, C^{k-1}, C^k, C^{k-1})$ throughout the domain of U. Thus for $k \geqq 3$ (e.g., for $u_0 \in (H_{2+k} \times H_{1+k})^2, k \geqq 3$) the transformed fields will also satisfy their respective second order field equations in the classical sense. The global existence of such solutions may be established by showing that their $(H_2 \times H_1)^2$ norms do not blow up in a finite time (and thus that they are globally defined abstract solutions). The smoothness results of paper 1 will then ensure that the solutions retain the full differentiability of their initial data (hence remain classical solutions) throughout their (global) existence on Minkowski space.

B. An L^∞ Estimate for the Curvature

Suppose p is a point within the domain of local existence of some solution (A_μ, ϕ). Then we can define (as shown in the appendix) the Cronström transform $(\hat{A}_\mu, \hat{\phi})$

of this solution on an open set containing the light cone K_p from p to the initial surface. Since $U(p) = \mathrm{id}$ in this construction it follows that $\hat{F}_{\mu\nu}(p) = F_{\mu\nu}(p)$, $\hat{\phi}(p) = \phi(p)$ and $D_\alpha \hat{\phi}(p) = D_\alpha \phi(p)$.

We can write an integral equation for $\hat{F}_{\mu\nu}(p)$ by using the retarded fundamental solution for the linear wave operator and the covariant wave equation for $\hat{F}_{\mu\nu}$ given by (2.14). Translating the origin of coordinates to p we get, by a standard argument,

$$\begin{aligned}\hat{F}_{\alpha\beta}(0) &= F_{\alpha\beta}(0)\\ &= \hat{F}^{\ell\mathrm{in}}_{\alpha\beta}(0) - \frac{1}{4\pi}\int_{K_p} r\,dr\,d\Omega\{-2\partial_\gamma([\hat{A}^\gamma, \hat{F}_{\alpha\beta}])\\ &\quad + [\partial_\gamma \hat{A}^\gamma, \hat{F}_{\alpha\beta}] - [\hat{A}^\gamma, [\hat{A}_\gamma, \hat{F}_{\alpha\beta}]]\\ &\quad + 2[\hat{F}^\gamma{}_\alpha, \hat{F}_{\gamma\beta}] + ((\hat{F}_{\alpha\beta}\hat{\phi})\cdot\theta_a\hat{\phi})\theta_a\\ &\quad + ((D_\beta\hat{\phi})\cdot\theta_a(D_\alpha\hat{\phi}) - (D_\alpha\hat{\phi})\cdot\theta_a(D_\beta\hat{\phi}))\theta_a\}|_{t=-r},\end{aligned} \tag{2.28}$$

where $\hat{F}^{\ell\mathrm{in}}_{\alpha\beta}(x)$ is that solution of the linear wave equation, $\eta^{\mu\nu}\partial_\mu\partial_\nu \hat{F}^{\ell\mathrm{in}}_{\alpha\beta}(x) = 0$, which has the same Cauchy data as $\hat{F}_{\alpha\beta}$ on the initial surface, i.e.,

$$\begin{aligned}\hat{F}^{\ell\mathrm{in}}_{\alpha\beta}|_{t=t_0} &= \hat{F}_{\alpha\beta}|_{t=t_0},\\ \partial_t \hat{F}^{\ell\mathrm{in}}_{\alpha\beta}|_{t=t_0} &= \partial_t \hat{F}_{\alpha\beta}|_{t=t_0}.\end{aligned} \tag{2.29}$$

In an analogous way we can write an integral equation for

$D_\alpha\hat{\phi}(0) = D_\alpha\phi(0)$ as

$$\begin{aligned}D_\alpha\hat{\phi}(0) &= D_\alpha\phi(0)\\ &= D_\alpha\hat{\phi}^{\ell\mathrm{in}}(0) - \frac{1}{4\pi}\int_{K_p} r\,dr\,d\Omega\left\{-2\partial_\mu(\hat{A}^\mu(D_\alpha\hat{\phi})) + (\partial_\mu\hat{A}^\mu)D_\alpha\hat{\phi} - \hat{A}_\mu\hat{A}^\mu D_\alpha\hat{\phi}\right.\\ &\quad \left.\left. + ((D_\alpha\hat{\phi})\cdot\theta_a\hat{\phi})\theta_a\hat{\phi} + D_\alpha\left(\frac{\partial\hat{P}}{\partial\phi}\right) - 2\hat{F}_\alpha{}^\mu(D_\mu\hat{\phi})\right\}\right|_{t=-r},\end{aligned} \tag{2.30}$$

where $D_\alpha\hat{\phi}^{\ell\mathrm{in}}(x)$ satisfies the linear wave equation and has the same Cauchy data as $D_\alpha\hat{\phi}$.

The solutions $\hat{F}^{\ell\mathrm{in}}_{\mu\nu}$ and $D_\alpha\hat{\phi}^{\ell\mathrm{in}}$ of the linear wave equation can be expressed explicitly in terms of their Cauchy data on the surface $t = t_0$ by the method of spherical means [10].

In particular, we get

$$\begin{aligned}\hat{F}^{\ell\mathrm{in}}_{\alpha\beta}(0) &= \frac{1}{4\pi}\int_{S^2} d\Omega\left.\left\{r_0\frac{\partial\hat{F}_{\alpha\beta}}{\partial t} + r_0\frac{\partial\hat{F}_{\alpha\beta}}{\partial r} + \hat{F}_{\alpha\beta}\right\}\right|_{t=t_0, r=r_0},\\ &= \frac{1}{4\pi}\int_{S^2} d\Omega\{r_0 m^\mu\partial_\mu\hat{F}_{\alpha\beta} + \hat{F}_{\alpha\beta}\}\Big|_{t=t_0, r=r_0},\end{aligned} \tag{2.31}$$

where the integral is over the sphere of radius $r = r_0 = |t_0|$ which is defined by the intersection of K_p with the initial surface.

We express the Cauchy data for $\hat{F}_{\alpha\beta}$ in terms of that for $F_{\alpha\beta}$ by means of the gauge transformation formula. Thus,

$$\hat{F}_{\alpha\beta} = UF_{\alpha\beta}U^{-1} \tag{2.32}$$

and

$$\frac{\partial \hat{F}_{\alpha\beta}}{\partial x^{\mu}} = [\hat{F}_{\alpha\beta}, \hat{A}_{\mu}] + U\frac{\partial F_{\alpha\beta}}{\partial x^{\mu}}U^{-1} - U[F_{\alpha\beta}, A_{\mu}]U^{-1} \tag{2.33}$$

The terms in the integral expression for $\hat{F}^{\ell\text{in}}_{\alpha\beta}(0)$ involving U may be estimated at $t = t_0$ even though U is not explicitly known. This follows from noting that

$$\begin{aligned} |\hat{F}^{(c)}_{\alpha\beta}|^2 &\leqq \sum_c \hat{F}^{(c)}_{\alpha\beta}\hat{F}^{(c)}_{\alpha\beta} = \operatorname{Tr}(\hat{F}_{\alpha\beta}\hat{F}_{\alpha\beta}) \\ &= \operatorname{Tr}(F_{\alpha\beta}F_{\alpha\beta}), \end{aligned} \tag{2.34}$$

and similarly that

$$|(U(m^{\mu}\partial_{\mu}F_{\alpha\beta})U^{-1})^{(c)}|^2 \leqq \operatorname{Tr}(m^{\mu}\partial_{\mu}F_{\alpha\beta})^2 \tag{2.35}$$

and

$$|(U[F_{\alpha\beta}, m^{\mu}A_{\mu}]U^{-1})^{(c)}|^2 \leqq \operatorname{Tr}([F_{\alpha\beta}, m^{\mu}A_{\mu}])^2. \tag{2.36}$$

The first integral on the right hand side of Eq. (2.28), namely

$$I^1_{\alpha\beta} \equiv \frac{1}{2\pi}\int_{K_p} rdrd\Omega\{\partial_{\gamma}([\hat{A}^{\gamma}, \hat{F}_{\alpha\beta}])\}|_{t=-r} \tag{2.37}$$

may be evaluated explicitly in terms of the initial data. To see this one need only write out the divergence explicitly and make use of the gauge condition $x^{\mu}\hat{A}_{\mu} = 0$ to simplify a step in the integration by parts. The result is[3]

$$I^1_{\alpha\beta} = \frac{1}{4\pi}\int_{S^2} d\Omega\{r_0[m^{\mu}\hat{A}_{\mu}, \hat{F}_{\alpha\beta}]\}|_{t=t_0, r=r_0}, \tag{2.38}$$

which precisely cancels a term in the expression for $\hat{F}^{\ell\text{in}}_{\alpha\beta}(0)$. Thus we have shown that

$$\begin{aligned} \hat{F}^{\ell\text{in}}_{\alpha\beta}(0) + I^1_{\alpha\beta} = \frac{1}{4\pi}\int_{S^2} d\Omega\{&UF_{\alpha\beta}U^{-1} \\ &+ r_0(U(m^{\mu}\partial_{\mu}F_{\alpha\beta})U^{-1} - U[F_{\alpha\beta}, m^{\mu}A_{\mu}]U^{-1})\}|_{t=t_0} \end{aligned} \tag{2.39}$$

and that each term on the right hand side may be estimated in terms of the (temporal gauge) initial data.

Next consider the integrals

$$I^2_{\alpha\beta} \equiv -\frac{1}{4\pi}\int_{K_p} rdrd\Omega\{[\partial_{\gamma}\hat{A}^{\gamma}, \hat{F}_{\alpha\beta}]\}|_{t=-r} \tag{2.40}$$

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3 In evaluating this and similar integrals it is important to note that $\hat{F}_{\alpha\beta}$ stands for the projection $\frac{\partial}{\partial x^{\alpha}} \cdot {}^{(4)}\hat{F} \cdot \frac{\partial}{\partial x^{\beta}}$ and thus transforms as a scalar

and

$$I^3_{\alpha\beta} \equiv +\frac{1}{4\pi}\int_{K_p} r dr d\Omega \{[\hat{A}^\gamma, [\hat{A}_\gamma, \hat{F}_{\alpha\beta}]]\}|_{t=-r}, \tag{2.41}$$

which occur on the right hand side of Eq. (2.28).

Making use of Eqs. (2.26) and (2.27) to substitute for the potential and its divergence and reexpressing the integrals somewhat we find that

$$\begin{aligned} I^2_{\alpha\beta} + I^3_{\alpha\beta} = &\frac{1}{4\pi}\int_{K_p} r dr d\Omega \\ &\left\{ 2\int_0^1 d\lambda \int_0^\lambda d\mu [\mu x^\nu \hat{F}_{\nu\gamma}(\mu x), \eta^{\gamma\delta}[\lambda x^\sigma \hat{F}_{\sigma\delta}(\lambda x), \hat{F}_{\alpha\beta}(x)]] \right. \\ &\left. -\left[\hat{F}_{\alpha\beta}(x), \int_0^1 d\lambda \lambda^2 x^\nu((D_\nu\hat{\phi}(\lambda x))\cdot\theta_a\hat{\phi}(\lambda x))\theta_a\right]\right\}. \end{aligned} \tag{2.42}$$

Let us write $I^F_{\alpha\beta}$ for the integral involving the cubic term in ${}^{(4)}F$ and $I^H_{\alpha\beta}$ for the integral involving the Higgs field and its gradient. By reexpressing the integral over μ and λ in $I^F_{\alpha\beta}$, making use of the fact that $x^\sigma \hat{F}_{\sigma\gamma}(\lambda x) = r\ell^\sigma \hat{F}_{\sigma\gamma}(\lambda x)$ for $x \in K_p$, etc., one can show that

$$\begin{aligned} |I^F_{\alpha\beta}| &\leqq \frac{C}{2}\int_0^{r_0} dr \, \|{}^{(4)}\hat{F}(-r)\|_{L^\infty} \int_{S^2} d\Omega \int_0^r d\bar{r}\bar{r}^2 \\ &\quad \cdot\sum_a \left\{ \tfrac{1}{4}(\hat{\ell}\cdot{}^{(4)}F^{(a)}\cdot\hat{m}(-\bar{r},\bar{r},\theta,\phi))^2 + \sum_A (\hat{\ell}^{(4)}\cdot F^{(a)}\cdot\hat{e}_A(-\bar{r},\bar{r},\theta,\phi))^2 \right\} \\ &\leqq CE_0 \int_0^{r_0} dr \, \|{}^{(4)}\hat{F}(-r)\|_{L^\infty}, \end{aligned} \tag{2.43}$$

where

$$\begin{aligned} \|{}^{(4)}\hat{F}(t)\|_{L^\infty} &\equiv \left| \sum_{\alpha\beta a} \hat{F}^{(a)}_{\alpha\beta}\hat{F}^{(a)}_{\alpha\beta}(t) \right|^{1/2}_{L^\infty} \\ &= \|{}^{(4)}F(t)\|_{L^\infty}, \end{aligned} \tag{2.44}$$

and where we have used the conservation of energy equation (2.21) in the last step. Notice that since $\sum_a F^{(a)}_{\alpha\beta}(x)F^{(a)}_{\mu\nu}(x)$ is gauge invariant, the L^∞ norm of ${}^{(4)}F$, as defined above, is also gauge invariant. This justifies the last equality in Eq. (2.44).

Making use of Hölder's inequality with exponents (6, 2, 3) applied to the integral of

$$(|\hat{\phi}(\lambda x)||\ell^\nu D_\nu\hat{\phi}(\lambda x)|\cdot 1), \tag{2.45}$$

we can estimate $I^H_{\alpha\beta}$ via

$$\begin{aligned} |I^H_{\alpha\beta}| &\leqq C\int_0^{r_0} dr \, \|{}^{(4)}\hat{F}(-r)\|_{L^\infty} \left\{ \left(\int_{S^2} d\Omega \int_0^r \bar{r}^2 d\bar{r} |\hat{\phi}(-\bar{r},\bar{r},\theta,\phi)|^6 \right)^{1/6} \right. \\ &\quad \left. \cdot\left(\int_{S^2} d\Omega' \int_0^r r'^2 dr' \tfrac{1}{2}(\ell^\nu D_\nu\hat{\phi}(-r',r',\theta,\phi))^2 \right)^{1/2} \right\} \\ &\leqq CE_0^{1/2}\int_0^{r_0} dr \, \|{}^{(4)}\hat{F}(-r)\|_{L^\infty} \left(\int_{S^2} d\Omega \int_0^r \bar{r}^2 d\bar{r} |\hat{\phi}(-\bar{r},\bar{r},\theta,\phi)|^6 \right)^{1/6}, \end{aligned} \tag{2.46}$$

where we have again used the conservation of energy. The L^6 norm of $\hat{\phi}$ which occurs in the final expression above is defined over the subcone $K_p(r)$ of K_p having $\bar{r} \leqq r \leqq r_0$. We can bound this norm by using the gauge invariant Sobolev estimate of Jaffe and Taubes [11]. Regard the integral as defined over a solid sphere of radius r with Riemannian metric $d\ell^2 = d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2)$ and orthonormal basis fields $\left\{\frac{\partial}{\partial \bar{r}}, \hat{e}_A\right\}$ and note that

$$\frac{\partial}{\partial \bar{r}}\hat{\phi}(-\bar{r}, \bar{r}, \theta, \phi) = \ell^\nu \partial_\nu \hat{\phi}(t, \bar{r}, \theta, \phi)|_{t=-\bar{r}}. \tag{2.47}$$

The gauge invariant Sobolev estimate gives

$$\|\hat{\phi}\|_{L^6} \leqq K_0 \|{}^{(3)}D\hat{\phi}\|_{L^2} + K_1 \|\hat{\phi}\|_{L^2} \tag{2.48}$$

where $({}^{(3)}D\hat{\phi}) = (\ell^\nu D_\nu \hat{\phi}, D_{\hat{e}_A}\hat{\phi})$ and where the norms are defined through integration over the solid sphere. The last term on the right (which could be replaced by $\|\hat{\phi}\|_{L^q}$ for any $q > 0$) is necessary because of the compactness of the region of integration [12].

We shall show in the appendix that the term $\|\hat{\phi}\|_{L^2}$ can always be bounded by an expression involving the energy. However, this result is immediate if the Higgs potential $P(\phi)$ has a suitable form. The requirements of positivity of P and finiteness of energy for $\phi \in H_s$ imply that $P(\phi)$ cannot have a non-zero constant term or a term linear in ϕ. Let us assume that P has the form

$$P(\phi) = \tfrac{1}{2} m_{\kappa\lambda}\phi_\kappa\phi_\lambda + P^{(4)}{}_{(\phi)}, \tag{2.49}$$

where $m_{\kappa\lambda}$ is a positive definite (mass) matrix and where $P^{(4)}$ is a positive quartic term. Then from Eq. (2.21) it is clear that we can bound the L^2 norm of $\hat{\phi}$ on the light cone by the square root of the energy. Since $\|{}^{(3)}D\hat{\phi}\|_{L^2} = \|(D_\ell\hat{\phi}, D_{\hat{e}_A}\hat{\phi})\|_{L^2}$ is already so bounded we get from Eq. (2.48) that

$$\|\hat{\phi}\|_{L^6} \leqq K E_0^{1/2}, \tag{2.50}$$

and thus from Eqs. (2.43) and (2.46) that

$$|I^2_{\alpha\beta} + I^3_{\alpha\beta}| \leqq C E_0 \int_0^{r_0} dr \, \|{}^{(4)}\hat{F}(-r)\|_{L^\infty}. \tag{2.51}$$

Returning to Eq. (2.28) we now define

$$I^4_{\alpha\beta} = -\frac{1}{4\pi}\int_{K_p} r\,dr\,d\Omega(\Xi_{\alpha\beta})|_{t=-r}, \tag{2.52}$$

where

$$\Xi_{\alpha\beta} \equiv 2[\hat{F}^\gamma{}_\alpha, \hat{F}_{\gamma\beta}] + ((D_\beta\hat{\phi})\cdot\theta_a(D_\alpha\hat{\phi}) - (D_\alpha\hat{\phi})\cdot\theta_a(D_\beta\hat{\phi}))\theta_a. \tag{2.53}$$

A remarkable feature of $\Xi_{\alpha\beta}$ is that it can be expressed as a sum of products of projections of ${}^{(4)}\hat{F}$ and $D_\alpha\hat{\phi}$ such that each term in the sum has (at least) one factor whose square integral over K_p is boundable by the total energy E_0.

To see this let us introduce the spatial orthonormal basis $\{\hat{e}_i\}$ where

$$\{\hat{e}_i\} = \{\hat{e}_1, \hat{e}_A\}, \quad \hat{e}_1 = \frac{\partial}{\partial r}. \tag{2.54}$$

This basis is related to the Cartesian orthonormal basis $\left\{\frac{\partial}{\partial x^j}\right\}$ by an orthogonal transformation

$$\frac{\partial}{\partial x^j} = O_{j\hat{\ell}}\hat{e}_{\ell}, \quad \hat{e}_{\ell} = O_{j\hat{\ell}}\frac{\partial}{\partial x^j}, \tag{2.55}$$

where $O_{j\hat{\ell}} = O_{j\hat{\ell}}(x)$ satisfies the orthogonality relations

$$O_{i\hat{\ell}}O_{j\hat{\ell}} = \delta_{ij}, \quad O_{j\hat{\ell}}O_{j\hat{m}} = \delta_{\hat{\ell}\hat{m}}. \tag{2.56}$$

We also define $\hat{e}_0 = \frac{\partial}{\partial t}$ and note that (recalling Eq. (2.19))

$$\hat{e}_0 = \tfrac{1}{2}(\hat{m} - \hat{\ell}), \quad \hat{e}_1 = \tfrac{1}{2}(\hat{m} + \hat{\ell}). \tag{2.57}$$

It is straightforward to show that

$$\begin{aligned}\Xi_{tj} = & -2O_{k\hat{B}}[\hat{F}_{tk}, (O_{j\hat{A}}\hat{e}_B \cdot {}^{(4)}\hat{F}\cdot\hat{e}_A + O_{j\hat{1}}\,\hat{e}_B \cdot {}^{(4)}\hat{F}\cdot\hat{\ell})] - O_{k\hat{1}}[\hat{m}\cdot{}^{(4)}\hat{F}\cdot\hat{\ell}, \hat{F}_{kj}] \\ & + \tfrac{1}{2}O_{j\hat{1}}[(D_{\hat{e}_0}\hat{\phi})\cdot\theta_a(D_{\hat{\ell}}\hat{\phi}) - (D_{\hat{\ell}}\hat{\phi})\cdot\theta_a(D_{\hat{e}_0}\hat{\phi})]\theta_a \\ & + \tfrac{1}{2}O_{j\hat{1}}\,O_{i\hat{1}}[(D_i\hat{\phi})\cdot\theta_a(D_{\hat{\ell}}\hat{\phi}) - (D_{\hat{\ell}}\hat{\phi})\cdot\theta_a(D_i\hat{\phi})]\theta_a \\ & + O_{j\hat{A}}[(D_{\hat{e}_0}\hat{\phi})\cdot\theta_a(D_{\hat{e}_A}\hat{\phi}) - (D_{\hat{e}_A}\hat{\phi})\cdot\theta_a(D_{\hat{e}_0}\hat{\phi})]\theta_a.\end{aligned} \tag{2.58}$$

The analogous expression for Ξ_{ij} and for the corresponding terms in the Higgs field equation are given in the appendix. It follows from inspection of these formulas that $\Xi_{\alpha\beta}$ has the special property mentioned above[4]. Since $|O_{j\hat{\ell}}(x)| \leqq 1$ the factors involving $O_{j\hat{\ell}}$ may be bounded by constants in making the estimates below.

From these considerations it follows that

$$|I^4_{\alpha\beta}| \leqq CE_0^{1/2}\left(\int_0^{r_0} dr(\|{}^{(4)}\hat{F}(-r)\|^2_{L^\infty} + \|D\hat{\phi}(-r)\|^2_{L^\infty})\right)^{1/2}, \tag{2.59}$$

where

$$\begin{aligned}\|D\hat{\phi}(t)\|_{L^\infty} &\equiv \left\|\sum_\mu (D_\mu\hat{\phi}\cdot D_\mu\hat{\phi})(t)\right\|^{1/2}_{L^\infty} \\ &= \|D\phi(t)\|_{L^\infty}\end{aligned} \tag{2.60}$$

Finally the integral

$$I^5_{\alpha\beta} \equiv -\frac{1}{4\pi}\int_{K_p} r\,dr\,d\Omega((\hat{F}_{\alpha\beta}\hat{\phi})\cdot\theta_a\hat{\phi})\theta_a \tag{2.61}$$

is boundable via

$$|I^5_{\alpha\beta}| \leqq C\left(\int_0^{r_0} dr\|{}^{(4)}\hat{F}(-r)\|^2_{L^\infty}\right)^{1/2}\left(\int_{K_p} r^2\,dr\,d\Omega|\hat{\phi}|^4\right)^{1/2} \tag{2.62}$$

and the gauge invariant Sobolev estimate (for norms defined over K_p)

$$\|\hat{\phi}\|_{L^4} \leqq K_0\|D\hat{\phi}\|^{3/4}_{L^2}\|\hat{\phi}\|^{1/4}_{L^2} + K_1\|\hat{\phi}\|_{L^2}. \tag{2.63}$$

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4 i.e., the algebraic property mentioned following Eq. (2.53)

This estimate follows from the usual Sobolev estimate on smoothly bounded regions in R^3 and the Jaffe–Taubes invariance argument [12, 11].

It follows that

$$\|\hat{\phi}\|_{L^4} \leqq K E_0^{1/2} \tag{2.64}$$

and thus, from Eq. (2.62) that

$$|I^5_{\alpha\beta}| \leqq C' E_0 \left(\int_0^{r_0} dr \, \|{}^{(4)}\hat{F}(-r)\|^2_{L^\infty} \right)^{1/2}. \tag{2.65}$$

Recalling Eq. (2.39) and Eqs. (2.34)–(2.36) we also have that

$$\begin{aligned} |\hat{F}^{\ell in}_{\alpha\beta}(0) + I^1_{\alpha\beta}| \leqq C\{ &\|{}^{(4)}F(t_0)\|_{L^\infty} + r_0 \|m^\mu \partial_\mu {}^{(4)}F(t_0)\|_{L^\infty} \\ &+ r_0 \|[{}^{(4)}F, m^\mu A_\mu](t_0)\|_{L^\infty}\}, \end{aligned} \tag{2.66}$$

where the norms on the right hand side involve only temporal gauge initial data.

We may now combine Eqs. (2.51), (2.59), (2.65) and (2.66) to obtain

$$\begin{aligned} |\hat{F}_{\alpha\beta}(0)| = |F_{\alpha\beta}(0)| \leqq\, & C_1 E_0 \int_0^{r_0} dr \, \|{}^{(4)}F(-r)\|_{L^\infty} \\ & + C_2 E_0^{1/2} \left(\int_0^{r_0} dr (\|{}^{(4)}F(-r)\|^2_{L^\infty} + \|D\phi(-r)\|^2_{L^\infty}) \right)^{1/2} \\ & + C_3 E_0 \left(\int_0^{r_0} dr \, \|{}^{(4)}F(-r)\|^2_{L^\infty} \right)^{1/2} + (K_1(t_0) + r_0 K_2(t_0)), \end{aligned} \tag{2.67}$$

where we have used the gauge invariance of $\|{}^{(4)}F\|_{L^\infty}$ and $\|D\phi\|_{L^\infty}$ to reexpress the result and where $K_1(t_0)$ and $K_2(t_0)$ are finite constants which depend on the temporal gauge initial data only.

Reversing the steps which shifted the origin of coordinates to the point p we can reexpress the above result as

$$\begin{aligned} |F_{\alpha\beta}(t, x)| \leqq\, & (C_1 E_0 t^{1/2} + C_2 E_0^{1/2} + C_3 E_0) \\ & \cdot \left(\int_0^t ds (\|{}^{(4)}F(s)\|^2_{L^\infty} + \|D\phi(s)\|^2_{L^\infty}) \right)^{1/2} \\ & + K_1(0) + t K_2(0), \end{aligned} \tag{2.68}$$

where $(x^\mu_p) = (t, x)$ and $t = 0$ is the initial data surface. Since the right hand side of Eq. (2.68) is independent of the spatial coordinates of p it follows that

$$\begin{aligned} \|{}^{(4)}F(t)\|^2_{L^\infty} \leqq\, & (C_1' E_0^2 t + C_2' E_0 + C_3' E_0^2) \\ & \cdot \int_0^t ds (\|{}^{(4)}F(s)\|^2_{L^\infty} + \|D\phi(s)\|^2_{L^\infty}) \\ & + C_0' (K_1(0) + t K_2(0))^2, \end{aligned} \tag{2.69}$$

where the C_i' are positive constants.

We can treat the integral equation (2.30) for $D_\alpha \hat{\phi}(0)$ in a completely analogous way. Following steps essentially identical to those above one finds that

$$\begin{aligned}|D_\alpha\hat{\phi}(0)| &= |D_\alpha\phi(0)|\\ &\leqq \frac{1}{4\pi}\left|\int d\Omega\{r_0 U(m^\mu\partial_\mu(D_\alpha\phi) + m^\mu A_\mu(D_\alpha\phi)) + U(D_\alpha\phi)\}(-r_0, r_0, \theta, \phi)\right|\\ &\quad + |I_\alpha^2 + I_\alpha^3 + I_\alpha^4 + I_\alpha^5| + |I_\alpha^6|\\ &\leqq C_0\{\|D_\alpha\phi(t_0)\|_{L^\infty} + r_0\|m^\mu\partial_\mu(D_\alpha\phi)(t_0)\|_{L^\infty} + r_0\|m^\mu A_\mu(D_\alpha\phi)(t_0)\|_{L^\infty}\}\\ &\quad + (C_1E_0r_0^{1/2} + C_2E_0^{1/2} + C_3E_0)\\ &\quad \cdot\left(\int_0^{r_0} dr(\|^{(4)}F(-r)\|_{L^\infty}^2 + \|D\phi(-r)\|_{L^\infty}^2)\right)^{1/2} + |I_\alpha^6|,\end{aligned} \tag{2.70}$$

where

$$\begin{aligned}I_\alpha^2 + I_\alpha^3 + I_\alpha^4 + I_\alpha^5 = -\frac{1}{4\pi}\int_{K_p} r\,dr\,d\Omega\{&(\partial_\mu\hat{A}^\mu)(D_\alpha\hat{\phi}) - \hat{A}_\mu\hat{A}^\mu(D_\alpha\hat{\phi})\\ &- 2\hat{F}_\alpha{}^\mu(D_\mu\hat{\phi}) + ((D_\alpha\hat{\phi})\cdot\theta_a\hat{\phi})\theta_a\hat{\phi}\}\end{aligned} \tag{2.71}$$

and

$$I_\alpha^6 = -\frac{1}{4\pi}\int_{K_p} r\,dr\,d\Omega\left(D_\alpha\left(\frac{\partial\hat{P}}{\partial\phi}\right)\right). \tag{2.72}$$

The terms in the first bracket on the right hand side of Eq. (2.70) involve only the temporal gauge initial data.

Only the integral I_α^6 has no direct analogue in the curvature integral equation. However, we can easily estimate it by first noting that

$$\left(D_\alpha\left(\frac{\partial P}{\partial\phi}\right)\right)_\kappa = \frac{\partial^2 P}{\partial\phi_\kappa\partial\phi_\lambda}(D_\alpha\phi)_\lambda. \tag{2.73}$$

This formula follows from the gauge invariance of P which implies that

$$\frac{\partial P}{\partial\phi_\kappa}(\theta_a\phi)_\kappa \equiv 0. \tag{2.74}$$

From Eq. (2.72) and the foregoing estimates on the norms $\|\hat{\phi}\|_{L^2}$ and $\|\hat{\phi}\|_{L^4}$ (defined over K_p as before) one can easily show that

$$|I_\alpha^6| \leqq (C_0 + C_1r_0^3 + C_2E_0^2)^{1/2}\cdot\left(\int_0^{r_0} dr\|D\phi(-r)\|_{L^\infty}^2\right)^{1/2}. \tag{2.75}$$

Thus, reverting to the original notation, we get the estimate

$$\begin{aligned}\|D\phi(t)\|_{L^\infty}^2 &\leqq (C_1'E_0^2t + C_2'E_0 + C_3'E_0^2 + C_4' + C_5't^3)\\ &\quad\cdot\int_0^t ds(\|^{(4)}F(s)\|_{L^\infty}^2 + \|D\phi(s)\|_{L^\infty}^2)\\ &\quad + C_0'(K_1'(0) + tK_2'(0))^2\end{aligned} \tag{2.76}$$

in which $K_1'(0)$ and $K_2'(0)$ depend only on the temporal gauge initial data and are always finite for the class of solutions considered.

Defining

$$N(t) = \|{}^{(4)}F(t)\|_{L^\infty}^2 + \|D\phi(t)\|_{L^\infty}^2, \tag{2.77}$$

we see, from Eqs. (2.69) and (2.76), that

$$N(t) \leqq f(t) + g(t) \int_0^t ds N(s), \tag{2.78}$$

where $f(t)$ and $g(t)$ are positive polynomials in t with coefficients which depend only on the energy E_0 and the temporal gauge initial data for ${}^{(4)}F$ and $D_\alpha\phi$. To apply Gronwall's lemma to get a bound on $N(t)$ we need only show that $N(t)$ is continuous. However, continuity of $\|{}^{(4)}F(t)\|_{L^\infty}$ follows from the triangle inequality and the Sobolev estimate

$$\|f\|_{L^\infty} \leqq K\|f\|_{H_2}, \tag{2.79}$$

since these give

$$\begin{aligned} \left| \|{}^{(4)}F(t+\varepsilon)\|_{L^\infty} - \|{}^{(4)}F(t)\|_{L^\infty} \right| &\leqq \|{}^{(4)}F(t+\varepsilon) - {}^{(4)}F(t)\|_{L^\infty} \\ &\leqq \|{}^{(4)}F(t+\varepsilon) - {}^{(4)}F(t)\|_{H_2} \xrightarrow[\varepsilon\to 0]{} 0, \end{aligned} \tag{2.80}$$

where the last step follows from continuity of ${}^{(4)}F(t)$ as a curve in H_2. The same argument obviously applies to $\|D\phi(t)\|_{L^\infty}$.

We have thus proven that the norms $\|{}^{(4)}F(t)\|_{L^\infty}$ and $\|D_\alpha\phi(t)\|_{L^\infty}$ cannot blow up in a finite time. Another estimate which we shall need below follows from the (temporal gauge) calculation

$$\frac{d}{dt}\int_{R^3} \phi\cdot\phi = 2\int_{R^3} \pi\cdot\phi, \quad \left|\int_{R^3} \pi\cdot\phi\right| \leqq (2E_0)^{1/2}\|\phi\|_{L^2}. \tag{2.81}$$

A straightforward argument shows that, for $t \geqq 0$,

$$\|\phi(t)\|_{L^2} \leqq \|\phi(0)\|_{L^2} + (2E_0)^{1/2}t. \tag{2.82}$$

We now have the key ingredients to complete the global existence proof.

C. Energy Estimates and the Global Existence Theorem

To complete the global existence proof we need only show that the $(H_2 \times H_1)^2$ norm of (A_i, E_i, ϕ, π) cannot blow up in a finite time. From conservation of energy (see expression (2.24)) we know that $\|F_{\mu\nu}(t)\|_{L^2}$ and $\|D_\mu\phi(t)\|_{L^2}$ are bounded by a constant. From the results of the previous section we know that $\|{}^{(4)}F(t)\|_{L^\infty}$, $\|D\phi(t)\|_{L^\infty}$ and $\|\phi(t)\|_{L^2}$ cannot blow up in a finite time.

Writing the equation of motion $\partial_t A_i = E_i$ in integral form,

$$A_i(t,x) = A_i(0,x) + \int_0^t ds E_i(s,x) \tag{2.83}$$

we see that

$$\|A_i(t)\|_{L^\infty} \leqq \|A_i(0)\|_{L^\infty} + \int_0^t ds \|E_i(s)\|_{L^\infty} \tag{2.84}$$

and, in a similar way, that

$$\|\phi(t)\|_{L^\infty} \leqq \|\phi(0)\|_{L^\infty} + \int_0^t ds \|\pi(s)\|_{L^\infty}. \tag{2.85}$$

Thus the bounds on $\|^{(4)}F(t)\|_{L^\infty}$ and $\|D\phi(t)\|_{L^\infty}$ imply that $\|A_i(t)\|_{L^\infty}$ and $\|\phi(t)\|_{L^\infty}$ cannot blow up in a finite time.

Let us define the "energies" $\mathscr{E}_0$ and $\mathscr{E}_1$ by

$$\begin{aligned}\mathscr{E}_0 = \tfrac{1}{2} \int_{R^3} \{&E_i^{(a)} E_i^{(a)} + (\partial_j A_i^{(a)})(\partial_j A_i^{(a)}) + m^2 A_i^{(a)} A_i^{(a)} \\ &+ \pi \cdot \pi + (\partial_i \phi)\cdot(\partial_i \phi) + m^2 \phi \cdot \phi\},\end{aligned} \tag{2.86}$$

and

$$\begin{aligned}\mathscr{E}_1 = \tfrac{1}{2} \int_{R^3} \{&(\partial_j E_i^{(a)})(\partial_j E_i^{(a)}) + (\partial_j \partial_k A_i^{(a)})(\partial_j \partial_k A_i^{(a)}) \\ &+ (\partial_i \pi)\cdot(\partial_i \pi) + (\partial_i \partial_j \phi)\cdot(\partial_i \partial_j \phi)\},\end{aligned} \tag{2.87}$$

where $m > 0$ is an arbitrary positive constant and where $(\partial_i \phi)\cdot(\partial_i \phi) \equiv (\partial_i \phi_\kappa)(\partial_i \phi_\kappa)$, etc. Clearly $(\mathscr{E}_0 + \mathscr{E}_1)^{1/2}$ is equivalent to the $(H_2 \times H_1)^2$ norm of the solution considered so that it suffices to prove that $\mathscr{E}_0$ and $\mathscr{E}_1$ cannot blow up.

Computing the time derivative of $\mathscr{E}_0$ and using the (conserved) constraint, $\partial_i E_i = [E_i, A_i] - (\pi \cdot \theta_a \phi)\theta_a$, to reexpress a term in $\partial_i E_i$ we get, after a straightforward application of Hôlder's inequality,

$$\begin{aligned}\left|\frac{d\mathscr{E}_0}{dt}\right| \leqq [&C_0 + C_1 \|^{(4)}F(t)\|_{L^\infty} + C_2 \|D\phi(t)\|_{L^\infty}]\mathscr{E}_0 \\ &+ \left|\int_{R^3} \left(\pi \cdot \frac{\partial P}{\partial \phi}\right)\right|,\end{aligned} \tag{2.88}$$

where the C_i are positive constants. Making use of the form of $\dfrac{\partial P}{\partial \phi}$ discussed in Sect. (II B) and using the Sobolev estimate

$$\|\phi\|_{L^3} \leqq K \|\partial \phi\|_{L^2}^{1/2} \|\phi\|_{L^2}^{1/2} \leqq K' \mathscr{E}_0^{1/4} \|\phi\|_{L^2}^{1/2}, \tag{2.89}$$

we find that

$$\left|\int_{R^3} \left(\pi \cdot \frac{\partial P}{\partial \phi}\right)\right| \leqq C_3 \mathscr{E}_0 + C_4 \|\pi\|_{L^\infty} \mathscr{E}_0 + C_5 \|\pi\|_{L^\infty} \|\phi\|_{L^2} \mathscr{E}_0, \tag{2.90}$$

and thus that

$$\begin{aligned}\left|\frac{d\mathscr{E}_0}{dt}\right| \leqq [&C_0' + C_1' \|^{(4)}F(t)\|_{L^\infty} + C_2' \|D\phi(t)\|_{L^\infty} \\ &+ C_3' \|\pi(t)\|_{L^\infty} \|\phi(t)\|_{L^2}]\mathscr{E}_0.\end{aligned} \tag{2.91}$$

Using the *a priori* bounds derived above for the quantities in brackets and applying Gronwall's lemma we see that $\mathscr{E}_0(t)$ cannot blow up in a finite time.

Finally, computing $\dfrac{d\mathscr{E}_1}{dt}$ and proceeding as above with the use of the constraint and the Sobolev estimate (valid for functions $f \in L^2(R^3)$)

$$\|f\|_{L^6} \leqq K\|\partial f\|_{L^2}, \tag{2.92}$$

we get

$$\begin{aligned}\left|\frac{d\mathscr{E}_1}{dt}\right| &\leqq [C_0\|A(t)\|_{L^\infty} + C_1\|\phi(t)\|_{L^\infty} + C_2\|^{(4)}F(t)\|_{L^\infty} \\ &\quad + C_3\|\pi(t)\|_{L^\infty} + C_4\mathscr{E}_0(t)]\mathscr{E}_1 \\ &\quad + [C_5 + C_6\|\phi(t)\|_{L^\infty} + C_7\|\phi(t)\|^2_{L^\infty} \\ &\quad + C_8\|^{(4)}F(t)\|_{L^\infty} + C_9\|\pi(t)\|_{L^\infty}]\mathscr{E}_0.\end{aligned} \tag{2.93}$$

Making use of the foregoing estimates for $\|\phi(t)\|_{L^\infty}$, etc. and applying Gronwall's lemma we thus find that $\mathscr{E}_1(t)$ cannot blow up in a finite time.

We have thus proven the global existence

Theorem: *If $u_0 = (A_i, E_i, \phi, \pi)$ is initial data lying in $(H_{2+k} \times H_{1+k})^2$ for $k \geqq 3$ and satisfying the initial value constraint,*

$$\partial_i E_i = [E_i, A_i] - (\pi \cdot \theta_a \phi)\theta_a,$$

then there is a unique solution $u(t) \in (H_{2+k} \times H_{1+k})^2$ of the temporal gauge YMH *equations defined for all $t \in (-\infty, \infty)$ and having $u(0) = u_0$. The corresponding fields $(A_\mu(x), F_{\mu\nu}(x), \phi(x), D_\alpha\phi(x))$ are globally defined on Minkowski space, lie in $(C^k \times C^{k-1} \times C^k \times C^{k-1})$ and satisfy Eq.* (2.10), (2.11), (2.14) *and* (2.16) *in the classical sense.*

Appendix

A. Cronström's Gauge Condition

Suppose that $A_\mu(x)$ is a C^r potential (for $r \geqq 1$) on some open set S_p containing the point p. Assume further that S_p is "star-shaped" relative to p in the sense that it may be completely covered by connected geodesics through p. We want to show that there exists a unique C^r gauge transformation $U(x)$ defined on S_p (with $U(x_p) = \text{id}$) which transforms A_μ to Cronström's gauge. Recall that the Cronström transforms $\hat{A}_\mu, \hat{F}_{\mu\nu}$ are required to satisfy

$$\begin{gathered}(x^\mu - x^\mu_p)\hat{A}_\mu(x) = 0, \quad \hat{A}_\mu(x_p) = 0, \\ \hat{F}_{\mu\nu}(x_p) = F_{\mu\nu}(x_p)\end{gathered} \tag{A.1}$$

on S_p. For convenience we may translate the origin of coordinates to p.

Consider the linear system of ordinary differential equations (depending upon a parameter x^μ) given by

$$\frac{dW}{d\lambda} + x^\mu A_\mu(\lambda x) W = 0, \tag{A.2}$$

where W is a real $d \times d$ matrix and take, as initial condition for $W(\lambda, x)$,

$$W(0, x) = \text{id}. \tag{A.3}$$

From the antisymmetry of A_μ it follows that $\frac{d}{d\lambda}$ (trace $(W^T W)) = 0$ and thus that the Euclidean "length" of W is a constant of the motion. The standard existence theory for linear systems assures us that there exists a unique solution $W(\lambda, x)$, defined throughout the region $\mathscr{R} = [0, 1] \times S_p$, which is a C^r function of both λ and x^μ.

Furthermore, a simple scaling argument (letting $x^\alpha \to \mu x^\alpha, \lambda \to \frac{1}{\mu}\lambda$ with μ a positive constant) applied to the differential equation shows that $W(\lambda, x)$ is a function of λx^α alone. Therefore, with a slight abuse of notation, we can write

$$W(\lambda, x) = W(\lambda x), W(0) = \mathrm{id}. \tag{A.4}$$

We shall now show that $W(\lambda, x)$ lies in the group G for all $(\lambda, x) \in \mathscr{R}$.

First note that since the matrix group G preserves the inner product $\phi \cdot \phi = \phi_\kappa \phi_\kappa$, G must either be the orthogonal group $O(d)$ or a subgroup thereof. It follows at once from the differential equation and its transpose (again using the antisymmetry of A_μ) that

$$\frac{d}{d\lambda}(W^T W) = 0, \tag{A.5}$$

where W^T is the transpose of W. Thus $W(\lambda x)$ remains in $O(d)$ for all $(\lambda, x) \in [0, 1] \times S_p$. We need only show that W cannot leave the subgroup G.

Fix $x \in S_p$ and suppose that $W(\lambda_0 x) \in G$ for some $\lambda_0 \in [0, 1]$. By introducing local charts for G on a neighbourhood of $W(\lambda_0 x)$, and recalling that A_μ takes values in the Lie algebra $\mathscr{g}$ of G, we can reexpress the differential equation (A.2) as a non-linear system (of class C^r) for curves in G. The standard existence theory for non-linear systems assures us that a solution exists on same neighborhood of λ_0. However, the solution curve (viewed as a curve in the linear space of $d \times d$ matrices) also satisfies the original differential equation (A.2) and thus coincides (on the common interval of existence) with the solution $W(\lambda x)$. It follows that $W(\lambda x)$ cannot leave G for any $x \in S_p$ and any $\lambda \in [0, 1]$.

Now, since $W(\lambda, x) = W(\lambda x)$ we have

$$\left.\frac{dW}{d\lambda}\right|_{\lambda=1} = x^\mu \frac{\partial W}{\partial x^\mu}(x) \tag{A.6}$$

and thus, from Eq. (A.2), putting $\lambda = 1$ and writing $U^{-1}(x)$ for $W(x)$, we get

$$0 = x^\mu \partial_\mu U^{-1}(x) + x^\mu A_\mu(x) U^{-1}(x) \tag{A.7}$$

for $x \in S_p$. Thus, recalling the gauge transformation formula, we find that

$$\hat{A}_\mu \equiv U A_\mu U^{-1} + U \partial_\mu U^{-1} \tag{A.8}$$

is a C^{r-1} potential on S_p satisfying

$$x^\mu \hat{A}_\mu(x) = 0. \tag{A.9}$$

The corresponding transform of ${}^{(4)}F, \hat{F}_{\alpha\beta} \equiv U F_{\alpha\beta} U^{-1}$, is a C^{r-1} curvature on S_p since $F_{\alpha\beta}$ is (in general) C^{r-1} and U is C^r. The initial condition $U(0) = \mathrm{id}$ shows that $\hat{F}_{\alpha\beta}(0) = F_{\alpha\beta}(0)$.

If we recall the defining equation for $\hat{F}_{\alpha\beta}$ in terms of $\hat{A}_\alpha$ we can easily show, using $x^\mu \hat{A}_\mu(x) = 0$, that

$$x^\mu \hat{F}_{\mu\nu}(x) = x^\mu(\partial_\mu \hat{A}_\nu(x)) + \hat{A}_\nu(x). \tag{A.10}$$

It follows that

$$\frac{d}{d\lambda}(\lambda \hat{A}_\nu(\lambda x)) = \lambda x^\mu \hat{F}_{\mu\nu}(\lambda x) \tag{A.11}$$

and thus that

$$\hat{A}_\nu(x) = \int_0^1 d\lambda\, \lambda x^\mu \hat{F}_{\mu\nu}(\lambda x), \tag{A.12}$$

which is Cronström's formula for the potential in terms of the curvature. Note that it follows immediately from Eq. (A.12) that

$$\hat{A}_\nu(0) = 0. \tag{A.13}$$

Using Eq. (A.12) and the field Eq. (2.10) one can easily derive Eq. (2.27) for $\partial_\nu \hat{A}^\nu(x)$.

The Cronström transforms of ϕ and $D_\alpha\phi$ are of course defined by $\hat{\phi} = U\phi$ and $D_\alpha\hat{\phi} = UD_\alpha\hat{\phi}$ (see Eq. (2.7)) and are respectively C^r and C^{r-1} maps on S_p if ϕ and $D_\alpha\phi$ are C^r and C^{r-1}. From the gauge covariance of the field equations it follows that if $({}^{(4)}A, \phi)$ are C^r potentials on S_p with $r \geqq 3$ which satisfy the field equations (2.10) and (2.11) then $({}^{(4)}F, D\phi)$ are C^{r-1} curvatures which satisfy the field equations (2.14) and (2.16) and that $({}^{(4)}\hat{A}, \hat{\phi})$ are C^{r-1} potentials and $({}^{(4)}\hat{F}, D\hat{\phi})$ are C^{r-1} curvatures which satisfy the corresponding equations in the Cronström gauge.

Thus we can always transform the fields to Cronström's gauge on any star-shaped region within the domain of local existence of a given solution.

B. A Bound for $\|\hat{\phi}\|_{L^2}$ on the Light Cone

In Sect. II B we made a special assumption about the quadratic term in the potential $P(\phi)$ in order to be able to bound the L^2 norm of $\hat{\phi}$ on the light cone K_p. We here remove that assumption by deriving an *a priori* bound. Since $\hat{\phi}\cdot\hat{\phi} = \phi\cdot\phi$ it suffices to bound the L^2 norm in temporal gauge.

Define a vector field

$$V^\alpha = -X^\alpha \phi\cdot\phi = -\delta^\alpha_t \phi\cdot\phi, \tag{A.14}$$

where $\hat{X} = \dfrac{\partial}{\partial t}$ and integrate the divergence $\partial_\alpha V^\alpha$ over the region bounded by the light cone K_p and the solid sphere B_p in the initial surface (see Sect. II A for definitions). The result is

$$\int_{K_p} r^2 dr\, d\Omega(\phi\cdot\phi) = \int_{B_p} r^2 dr\, d\Omega(\phi\cdot\phi) + \int_{I_p} d^4x(2\phi\cdot\pi), \tag{A.15}$$

where I_p is the interior of the region bounded by $K_p \cup B_p$. This last integral may be estimated via

$$\left|\int_{I_p} d^4x(2\phi\cdot\pi)\right| \leqq 2(2E_0)^{1/2} \int_0^t ds\, \|\phi(s)\|_{L^2}. \tag{A.16}$$

Using the bound on $\|\phi(t)\|_{L^2}$ derived at the end of Sect. II B, we can easily show that

$$\|\phi\|_{L^2}|_{K_p} \leqq \|\phi(0)\|_{L^2} + (2E_0)^{1/2}t, \tag{A.17}$$

where the right hand side depends only on the initial data and the energy. Using this bound in place of the previous one makes only a slight difference in the form of the subsequent estimates.

C. Algebraic Terms in the Curvature Equations

For completeness we include here the remaining components of $\Xi_{\alpha\beta}$ (defined in Sect. II B) and the corresponding terms for the $D_\alpha\phi$ equation given by

$$\Lambda_\alpha \equiv \hat{F}_\alpha{}^\mu(D_\mu\hat{\phi}). \tag{A.18}$$

The components are

$$\begin{aligned}
\Xi_{ij} &= 2\{O_{i\hat{1}}O_{j\hat{B}}O_{k\hat{1}}(D_k\hat{\phi})\cdot\theta_a(D_{\hat{e}_B}\hat{\phi}) + O_{j\hat{1}}O_{i\hat{B}}O_{k\hat{1}}(D_{\hat{e}_B}\hat{\phi})\cdot\theta_a(D_k\hat{\phi})\\
&\quad + O_{i\hat{A}}O_{j\hat{B}}O_{k\hat{A}}(D_k\hat{\phi})\cdot\theta_a(D_{\hat{e}_B}\hat{\phi})\}\theta_a,\\
\Lambda_t &= \tfrac{1}{2}(\hat{m}\cdot{}^{(4)}\hat{F}\cdot\hat{\ell})O_{k\hat{1}}(D_k\hat{\phi}) + O_{k\hat{A}}\hat{F}_{tk}(D_{\hat{e}_A}\hat{\phi}),\\
\Lambda_i &= O_{j\hat{A}}\hat{F}_{ij}(D_{\hat{e}_A}\hat{\phi}) + \tfrac{1}{2}O_{i\hat{1}}(\hat{m}\cdot{}^{(4)}\hat{F}\cdot\hat{\ell})(D_{\hat{e}_0}\hat{\phi})\\
&\quad + \tfrac{1}{2}O_{i\hat{A}}O_{j\hat{A}}\hat{F}_{jt}(D_{\hat{\ell}}\hat{\phi}) + \tfrac{1}{2}O_{i\hat{A}}O_{j\hat{A}}O_{k\hat{1}}\hat{F}_{jk}(D_{\hat{\ell}}\hat{\phi})\\
&\quad + \tfrac{1}{2}O_{i\hat{A}}(\hat{e}_A\cdot{}^{(4)}\hat{F}\cdot\hat{\ell})\cdot(D_{\hat{e}_0}\hat{\phi} + O_{j\hat{1}}(D_j\hat{\phi})).
\end{aligned} \tag{A.19}$$

Each term in these expressions contains one projection of $D_\alpha\hat{\phi}$ or $\hat{F}_{\alpha\beta}$ whose square integral over K_p is boundable by the energy.

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