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On heat transfer in a viscoelastic fluid flowing around a steadily rotating and thermally insulated sphere

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With 4 figures in 11 details

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1. Introduction

Recently we have investigated the problem of heat transfer due to steady rotations of a sphere in an infinitely extending non-*Newton*ian fluid which is otherwise at rest (1). Solutions have been obtained for two different types of thermal boundary conditions, i. e. (i) the sphere is maintained at a constant or variable temperature and (ii) it is thermally insulated. It is observed that in both these cases deviations from *Newton*ian behaviour result in some significant changes in the temperature distribution. Later *Giesekus* (2) has experimentally investigated a similar type of problem using a slightly viscoelastic fluid having a relatively high viscosity. With this material a situation is realized where the secondary flow separates into two different zones. It is found that, particularly in this case, the temperature distribution undergoes marked changes. This encouraged us to make a more detailed study of the temperature field for a similar situation, maintaining the sphere thermally insulated and choosing the parameters entering the problem in such a way that the experimental conditions are approximated.

It is found that the parameter responsible for the deviations from *Newton*ian flow behaviour plays an important role in bringing out considerable changes in the temperature field, as will be discussed in detail in section 3.

2. Theory

Since the mathematical calculations for the posed problem have already been provided in (1), we give below only a short outline of the basic theory.

We choose the constitutive equation of a viscoelastic fluid, applying the usual approximation of slow motions, in the form

$$\boldsymbol{s} = -p\,\mathbf{1} + 2\,\eta_0[\boldsymbol{f}^{(1)} + \varkappa_0^{(2)}\boldsymbol{f}^{(2)} + \varkappa_0^{(11)}\boldsymbol{f}^{(1)2} + \cdots], \quad [1]$$

where $\boldsymbol{s}$ represents the stress tensor, p the undetermined pressure, η_0 the *Newton*ian viscosity, $\varkappa_0^{(2)}$ and $\varkappa_0^{(11)}$ two constants having dimensions of time and characterizing viscoelasticity of the fluid in a first approximation; $\boldsymbol{f}^{(n)}$ represent the corotational kinematic tensors defined by the following relations:

$$\boldsymbol{f}^{(1)} = \frac{1}{2}(\boldsymbol{\nabla v} + \boldsymbol{v\nabla}), \quad \boldsymbol{\omega} = \frac{1}{2}(\boldsymbol{\nabla v} - \boldsymbol{v\nabla}),$$

$$\boldsymbol{f}^{(n+1)} = \frac{D\boldsymbol{f}^{(n)}}{Dt} + \boldsymbol{\omega}\cdot\boldsymbol{f}^{(n)} - \boldsymbol{f}^{(n)}\cdot\boldsymbol{\omega}\,. \quad [2]$$

For details of eqs. [1] and [2] cf. *Giesekus* (3, 4)[1].

The solution to the problem is obtained by solving the equations of momentum and continuity

$$\boldsymbol{\nabla}\cdot\boldsymbol{s} = \varrho\frac{D\boldsymbol{v}}{Dt}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \quad [3]$$

and the equation of energy

$$\varrho\, c_p\frac{DT}{Dt} = k\,\boldsymbol{\nabla}^2 T + tr(\boldsymbol{s}\cdot\boldsymbol{f}^{(1)}), \quad [4]$$

where ϱ, c_p and k respectively represent the density, specific heat and heat conductivity. It is understood that the dependence of the material constants on temperature is neglected.

Choose a system of spherical polar coordinates (r, ϑ, φ) with origin at the centre of the sphere and the polar angle ϑ and

[1]) As is well known, in this approximation (often referred to as "*Rivlin-Ericksen* fluid") *Oldroyd* and *Walters* fluids B are included as particular cases, second-order specialization being given by $\varkappa_0^{(11)} = -2\,\varkappa_0^{(2)}$. However, in the present problem $\varkappa_0^{(2)}$ does not enter and thus we find no difference in the predictions by considering second-order effects for the above mentioned fluids. In contrast, differences are found for higher-order effects, cf. footnote 3.

azimuthal angle φ being measured from the axis of rotation and some convenient meridian plane respectively. Let u, v, w represent the components of $\boldsymbol{v}$ in the increasing directions of r, ϑ, φ. If a sphere of radius a rotates with constant angular velocity Ω and is thermally insulated, eqs. [3] and [4] are to be solved under the boundary conditions

$$\left.\begin{aligned} u = v = 0\,, \quad w = a\,\Omega \sin\vartheta \\ \frac{\partial T}{\partial r} = 0\,, \end{aligned}\right\} \text{ at } r = a\,, \qquad [5]$$

$$\left.\begin{aligned} u = v = w = 0\,, \\ T = T_\infty \end{aligned}\right\} \text{ at } r \to \infty\,. \qquad [6]$$

We render all the quantities dimensionless by using a as characteristic length, $a\,\Omega$ as characteristic velocity, T_∞ as characteristic temperature. On doing so the parameters that enter the problem are

$$R = \frac{\varrho\, a^2\, \Omega}{\eta_0} \quad (\textit{Reynolds} \text{ number})\,, \qquad [7]$$

$$\sigma = \frac{\eta_0\, c_p}{k} \quad (\textit{Prandtl} \text{ number})\,, \qquad [8]$$

$$\beta = \frac{\eta_0^2}{\varrho^2\, a^2\, c_p\, T_\infty}\,, \qquad [9]$$

and

$$m = \frac{\eta_0\, \varkappa_0^{(11)}}{2\, \varrho\, a^2}\,. \qquad [10]$$

Assuming *Reynolds* number R to be small, we expand azimuthal component of velocity in a series of the form

$$w = a\,\Omega[w_1 + R^2 w_2 + \cdots]\,, \qquad [11]$$

and introduce the stream function for secondary flow through

$$u = -\frac{1}{r^2 \sin\vartheta}\frac{\partial\psi}{\partial\vartheta}\,, \quad v = \frac{1}{r\sin\vartheta}\frac{\partial\psi}{\partial r}\,, \qquad [12]$$

where

$$\psi = a^3\,\Omega\, R[\psi_1 + R^2 \psi_2 + \cdots]\,. \qquad [13]$$

It may be pointed out that, in the brackets of eqs. [11] and [13], there is no contribution of terms arising from odd powers of R in view of symmetry of the problem.

Substituting from eqs. [11] and [13] into the equation of momentum [3] and making use of the boundary conditions [5] and [6], we get

$$w_1 = \frac{\sin\vartheta}{\xi^2}\,, \qquad [14]$$

$$\psi_1 = \frac{(\xi-1)^2}{8\,\xi^2}\left[1 - 4\,m\left(1 + \frac{2}{\xi}\right)\right]\sin^2\vartheta\cos\vartheta\,, \qquad [15]$$

where

$$\xi = r/a\,. \qquad [16]$$

As may be seen from eq. [15] the nature of the secondary flow depends on the choice of m only. For $m < 1/12$ it resembles to that of a *Newton*ian fluid, $1/12 < m < 1/4$ depicts breaking into two distinct zones, separated by a sphere of radius

$$\xi_0 = \frac{8\,m}{1 - 4\,m}\,, \qquad [17]$$

and $m > 1/4$ depicts complete reversal of the secondary flow field.

As for the velocity components, we expand the temperature in the form

$$T = T_\infty[1 + R^2\, T_1 + R^4\, T_2 + \cdots]\,. \qquad [18]$$

Substitution of the above equations and the respective expressions for the stress components into the energy equation [4] leads to an infinite set of partial differential equations in T_1, T_2, $\cdots$, obtained by equating coefficients of like powers of R on both sides of it[2]). Solving the first two equations in succession and using the boundary conditions [5] and [6] we find:

$$T_1 = \beta\,\sigma\, X_1(\xi, \vartheta)\,, \qquad [19]$$

$$T_2 = \beta\,\sigma[Y_1(\xi, \vartheta, m, \ldots) + \sigma\, Y_2(\xi, \vartheta, m)]\,. \qquad [20]$$

The functions X_1, Y_1 and Y_2 expressible in terms of *Legendre* polynominals of even order are given by eqs. (3.15), (3.23) and (3.25)–(3.30) of (1), where r should be replaced by ξ[3]). Eq. [19] takes account of the dissipation arising due to primary flow, given by eq. [14]. However, in eq. [20] there are two different contributions; the first (represented by Y_1) takes account of change in dissipation due to second and third order flows, while the second (represented by Y_2) takes account of heat convection due to second-order flow and is composed of two terms, one being independent of m, whereas the second depending on m linearly[4]):

$$Y_2(\xi, \vartheta, m) = \chi_1(\xi, \vartheta) + m\,\chi_2(\xi, \vartheta)\,. \qquad [21]$$

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[2]) Before doing this Ω is to be substituted by $\dfrac{\eta_0\, R}{\varrho\, a^2}$, i. e. R is to be understood as a nondimensional quantity for Ω.

[3]) In eq. (3.15) read the coefficient of $P_4(\mu)$ as $\dfrac{C_3}{r^5} - \dfrac{M_1}{8\,r^4}$ + etc. while in eq. (3.29) read $Q_1 = \beta\,\sigma^2 \times (2\,\mathrm{m} - 1/2)$.

[4]) In contrast to this, Y_1 has a more complicated structure, containing also terms with m^2 and those arising from third order coefficients, not recorded explicitly in the constitutive eq. [1], cf. *Giesekus* (4) and *Walters* and *Waters* (5). The respective results of (1), therefore, refer only to *Oldroyd* and *Walters* fluid B.

Since the experiments, mentioned earlier in the text, are performed using fluids with high viscosities i. e. with *Prandtl* numbers $\sigma \gg 1$ but showing slight deviations from *Newton*ian behaviour, we consider only the convection effects in T_2 given by eq. [21] and neglect the term Y_1 in eq. [20].

3. Discussion

For the Polysiloxane M 20000, utilized in the experiment by *Giesekus* (2), physical constants have the values

$\varrho = 0.97\ \mathrm{g/cm^3}, \quad \eta_0 = 200$ Poise,

$c_p = 0.37\ \mathrm{cal/g\ grad} = 1.55 \times 10^7\ \mathrm{erg/g\ grad}$,

$k = 0.12\ \mathrm{kcal/m\ h\ grad} = 1.40 \times 10^4\ \mathrm{erg/cm\ sec\ grad}$,

while the radius of the sphere made from teflon was chosen as $a = 1$ cm. For the above values of the physical constants, *Prandtl* number σ and the parameter[5] β become

$$\sigma \approx 2 \times 10^5, \quad \beta \approx 3 \times 10^{-3}. \qquad [22]$$

Taking these, we have calculated the temperature contributions T_1 and T_2 as given by eqs. [19] and [20]. These, in turn, are sufficient to calculate the complete increase of temperature $\Delta T = T - T_\infty$, as given by eq. [18] – cf. footnote 5 – as a function of *Reynolds* number to the order of approximation utilized:

$$\Delta T = R^2 T_1 + R^4 T_2. \qquad [23]$$

We note that the approximation method adopted for the solution of the problem takes heat conduction as the main mechanism and thus treats the mechanism of convection due to second-order flow as perturbation. In view of this, smallness of the product $R^2 \sigma$ restricts the range of validity of the above expression. Presently we find $R \lesssim 2 \times 10^{-2}$ i. e. $\Omega \lesssim 20$ r.p.m. Since in the experiment, cited earlier, the rotational speed is chosen to be about 800 r.p.m., we do not expect to get a close agreement between the observed and the calculated temperature distribution but only to find similarity in the main features.

If it is desired to have solutions which correspond more closely to the experimental situation, we must solve the energy equation, by retaining convection due to secondary flow fully, only neglecting higher-order dissipation terms:

$$\left(\frac{\partial^2 T}{\partial \xi^2} + \frac{2}{\xi}\frac{\partial T}{\partial \xi} + \frac{1}{\xi^2}\frac{\partial^2 T}{\partial \vartheta^2} + \frac{\cot\vartheta}{\xi^2}\frac{\partial T}{\partial \vartheta}\right) - \sigma R^2\left(u_1 \frac{\partial T}{\partial \xi} + \frac{v_1}{\xi}\frac{\partial T}{\partial \vartheta}\right) = -\frac{9\sigma\beta R^2}{\xi^6}\sin^2\vartheta, \qquad [24]$$

where u_1 and v_1 represent nondimensionalized velocity components of secondary flow. Though the solution of the above equation under prescribed boundary conditions can be obtained using standard methods, it is found to lead to very laborious calculations and therefore we have not taken them up here.

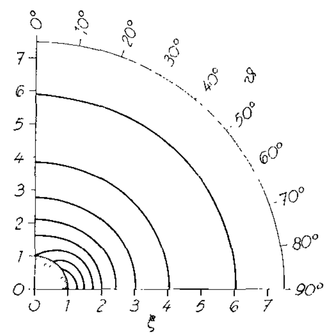

Fig. 1. Isotherms depicting the temperature distribution T_1; $2 \times 10^2 \leqslant T_1 \leqslant 9 \times 10^2$, two neighbouring isotherms differing by $\Delta T_1 = 10^2$

The numerical evaluations of T_1 as well as T_2 for various values of parameter m have been performed using an IBM 360–44 computer. The results of these are depicted by the plots of isotherms in figs. 1–3. Moreover, we have plotted in fig. 4 the temperature distribution on the axis of rotation and the equatorial plane respectively.

The temperature field T_1 arising from dissipation due to primary flow (fig. 1) shows a nearly radial decay, as expected. Only in the immediate surrounding of the sphere we observe some deviations originating in that the intensity of dissipation is maximum on the equatorial plane.

Contribution to temperature T_2 arising from heat convection due to secondary flow for a *Newton*ian fluid ($m = 0$) shows negative values in some neighbourhood of the axis of rotation, but positive values near the equatorial plane (fig. 2a). This may be easily understood if we take account of the direction of secondary flow, in which the fluid comes in at the axis and is thrown out near the equator. Since this direction is reversed in the whole field for highly viscoelastic fluids (e. g. $m = 100$) we observe corresponding reverse in the temperature distribution also (fig. 2b).

When the fluid is only very slightly viscoelastic ($m \lesssim 1/12$), the temperature distribution once again resembles that for a Newtonian fluid, but the temperature profile as a whole flattens markedly as a result of reduction of the velocity of second-order flow due to counteracting viscoelastic forces (fig. 3a). In the case when m exceeds the

[5]) It is evident from the eqs. [18], [19] and [20] that T_∞ does not really enter in the problem. Because of this reason, we have chosen $T_\infty = 1$ to calculate β.

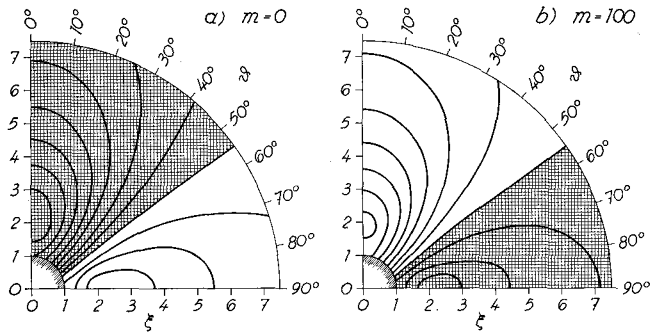

Fig. 2. Isotherms depicting the temperature distribution T_2: (a) for $m = 0$ with two neighbouring isotherms differing by $\Delta T_2 = 0.50 \times 10^5$ and (b) for $m = 100$ with $\Delta T_2 = 0.40 \times 10^8$; the marked region signifies the negative values of T_2

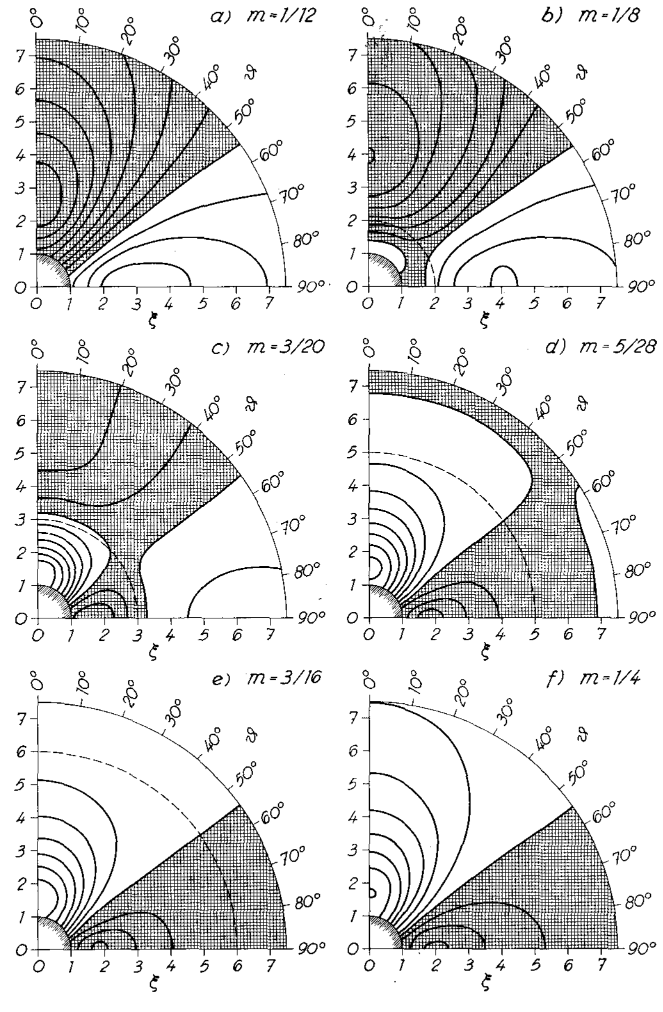

Fig. 3. Isotherms depicting the temperature distribution T_2: $m = {}^1/_{12}$, $\Delta T_2 = 0.20 \times 10^5$; (b) $m = {}^1/_8$, $\Delta T_2 = 0.10 \times 10^5$; (c) $m = {}^3/_{20}$, $\Delta T_2 = 0.50 \times 10^4$; (d) $m = {}^5/_{28}$, $\Delta T_2 = 0.20 \times 10^5$; (e) $m = {}^3/_{16}$, $\Delta T_2 = 0.25 \times 10^5$; (f) $m = {}^1/_4$, $\Delta T_2 = 0.50 \times 10^5$ The marked regions signify the negative values of T_2 while the dashed curves represent the sphere of separation for secondary flow

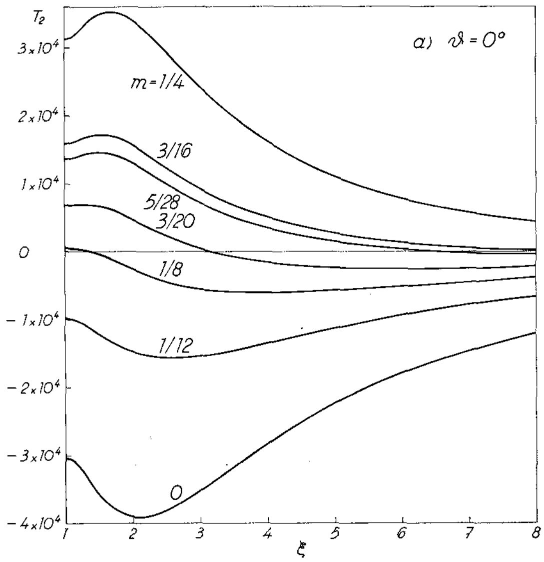

Fig. 4a

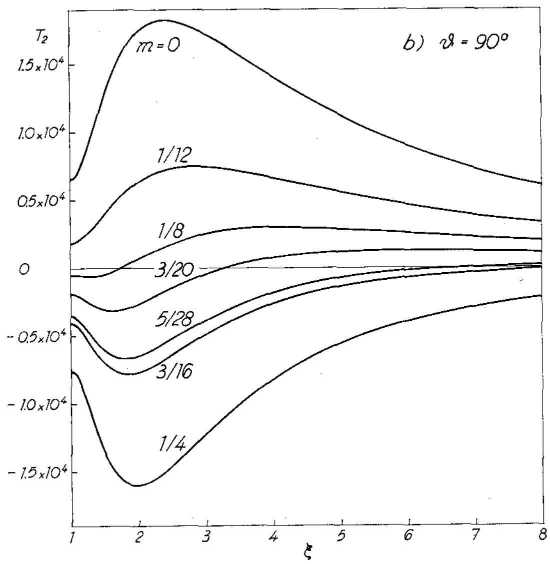

Fig. 4b

Fig. 4. Profiles of temperature T_2 for $m = 0$, ${}^1/_{12}$, ${}^1/_8$, ${}^3/_{20}$, ${}^5/_{28}$, ${}^3/_{16}$ and ${}^1/_4$: (a) on the axis of rotation and (b) on the equatorial plane

value ${}^1/_{12}$ and the separation of the secondary flow occurs, the temperature field T_2 is influenced in such a way that three different regions, separated by two zero-isotherms, occur. One of these, lying in the outer flow zone, is similar to that of *Newton*ian case but is moved away from the surface of the sphere, whereas the other originating on the sphere extends up to the axis of rotation and encloses a certain region with positive values of T_2. As m increases this region becomes larger and larger and even extends beyond the inner flow zone. This development is demonstrated in figs. 3b–e. When $m = {}^1/_4$, i. e. the inner flow zone covers the whole field (fig. 3f), one of the zero-isotherms vanishes and thus the temperature distribution resembles that of highly viscoelastic fluids, cf. fig. 2b.

Because the variations in temperature are most pronounced on the axis of rotation ($\vartheta = 0°$) and on the equatorial plane ($\vartheta = 90°$) we have plotted the profiles of temperature T_2 for both of these cases in fig. 4. These profiles bring out a more detailed picture of the temperature variations with increase of m.

For all cases under consideration it is seen that the temperature attains the absolute extremum away from the sphere. This may be explained as a result of velocity distribution of secondary flow, which also has maximum intensity at some distance from the surface. Further, it is observed that the descent or ascent is steepest near the separating sphere when its relative radius ξ_0 is not very large (as e. g. for $m = {}^1/_8$ and ${}^3/_{20}$). This is in agreement with the experimental observations made for which $\xi_0 \approx 2$.

From figs. 4a and 4b it is seen that the absolute maximum of temperature T_2 on the axis of rotation is about double to that on the equatorial plane. This is due to the typical character of secondary flow, which is also doubly intensive on the axis than on the equatorial plane provided equal distances on them are taken for the sake of comparison.

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Summary

In continuation of an earlier investigation, where the problem of heat transfer due to a sphere steadily rotating in an infinitely extending non-*Newton*ian fluid is discussed under various types of thermal boundary conditions, we presently reconsider this problem for the case in which the sphere is thermally insulated in some more details. The main attention is given to a situation where the secondary flow breaks down into two distinct zones and in which heat convection is strongly dominating over dissipation effects due to viscoelasticity. It is found that, for flow situations from beginning of separation up to complete reversal, the temperature distribution also undergoes marked changes. The chosen conditions correspond to those used in a recently reported experiment performed with polysiloxane. The theoretical and the observed results provide a fairly good qualitative agreement.

Zusammenfassung

Das in einer früheren Untersuchung behandelte Problem der Wärmeübertragung in einer unendlich ausgedehnten nicht-*Newton*schen Flüssigkeit infolge der stationären Rotation einer Kugel bei verschiedenen thermischen Randbedingungen wird für den Fall der thermisch isolierten Kugel hier ausführlich diskutiert. In erster Linie werden Bedingungen betrachtet, bei denen die Sekundärströmung in zwei getrennte Zonen zerfällt und die Konvektion von erheblich stärkerem Einfluß ist als die Dissipationseffekte infolge Viskoelastizität. Man findet, daß die Änderung der Sekundärströmung vom Beginn der Trennung in zwei Zonen bis zur vollständigen Richtungsumkehr von einer ausgeprägten Änderung der Temperaturverteilung begleitet ist. Der hier untersuchte Fall entspricht genähert den Bedingungen eines vor kurzem beschriebenen Experiments, bei dem Silikonöl verwendet wurde. Die vorhergesagten Ergebnisse stimmen mit den beobachteten in qualitativer Hinsicht recht gut überein.

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