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Unsteady flow of an elastico-viscous liquid

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With 15 figures

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1. Discussion

Ting (1963) has considered certain nonsteady flow problems for the "second order" fluid of *Coleman* and *Noll* (1960). He found that bounded solutions could not be obtained to the problems he attempted in the cases of "physical interest". In this paper consideration is given to the difficulty encountered by *Ting*. It is concluded that the second and third order approximations of *Coleman* and *Nolls* "simple fluid" are unsuitable for use in unsteady flow problems when a *Laplace* Transform technique is used in their solution. It is shown that bounded solutions to problems considered by *Ting* can be obtained for other equations of state.

The equations of state of *Coleman* and *Nolls* "simple fluid" are

$$T = -p\,I + T' \qquad [1]$$

$$T' = \overset{t}{\underset{-\infty}{F}}[C_t(t')]\,. \qquad [2]$$

where T is the stress tensor, I the substitution tensor, p an arbitrary isotropic pressure, F a tensor valued functional and C_t is the history of the right *Cauchy-Green* tensor.

The first approximation used in the reduction of [2] into a manageable form is the assumption that the fluid has a fading memory. This essentially enables us to write [2] as

$$T' = f(A)^{(1)}, A^{(2)}, \ldots, A^{(n)})\,, \qquad [3]$$

where f is a function and the $A^{(n)}$ are related to the successive rate-of-strain tensors $e_{ik}^{(n)}$ by

$$A^{(n)} = 2\,e_{ik}^{(n)}.$$

Thus, we can write

$$A^{(n)} = \left[\frac{d^n}{dt'^n} C_t(t')\right]_{t'=t}. \qquad [4]$$

The second approximation used is the assumption that the fluid flows slowly and so many of the $A^{(n)}$'s (which involve powers of the velocity) can be neglected. In this way, it is possible to obtain *Coleman* and *Nolls* "third-order" fluid and their "second-order" fluid, which was used by *Ting* (1963).

The "second-order" fluid has equations of state

$$T = -p\,I + \alpha_1 A^{(1)} + \alpha_2 A^{(2)} + \alpha_3 A^{(1)2}\,. \qquad [5]$$

Ting found that he could obtain bounded solutions only when $\alpha_1 > 0$ and $\alpha_2 > 0$. This is contrary to the findings of *Coleman* who insists that $\alpha_2 < 0$.

In unsteady flow problems, the time dependence is often separated from the differential equations by the use of *Laplace* transforms. Essentially, this technique replaces all time derivatives by a parameter γ, which can take values from minus infinity to plus infinity. Thus, terms like [4], occurring in the equations of motion via the stress tensor, yield terms of the form

$$\gamma^n\,\bar{C}_t(\gamma)\,, \qquad [6]$$

where $\bar{C}_t(\gamma)$ is the *Laplace* transform of $C_t(t)$. The assumption of slow flow is no longer strong enough to neglect such terms as these, since γ can be infinitely large.

Truesdell (1964) and *Wang* (1965) have shown that the "second- and third-order" equations of state can represent slightly elastic liquids in situations in which the motion is not necessarily slow. However, this is possible only because the "slow flow" approximation is replaced by the assumption that the constants involved are small. When *Laplace* transforms are used, approximations involving the smallness of the constants cannot be used for the same reasons as above.

Hermes and *Fredrickson* (1967) have already questioned the validity of the "order" equations of state. They state that the assumptions made to obtain the "order" equations of state are probably not valid when the kinematic state of the material particle undergoes rapid changes.

It is concluded that the "second- and third-order" equations of state are unsuitable for use in unsteady flow problems, especially when *Laplace* transforms are used.

In this paper, we investigate some of the unsteady flow problems considered by *Ting* (1963) in the light of the previous discussion. We characterize the elastico-viscous liquid by equations of state of the form

$$p_{ik} = -p\, g_{ik} + p'_{ik}, \qquad [7]$$

$$p'^{ik} = 2\int_{-\infty}^{t} \Psi(t-t')\,\frac{\partial x^i}{\partial x'^m}\,\frac{\partial x^k}{\partial x'^r}\, e^{(1)mr}(x',t')\,dt', \qquad [8]$$

$$\Psi(t-t') = \int_0^\infty \frac{N(\tau)}{\tau}\, e^{\frac{-(t-t')}{\tau}}\, d\tau, \qquad [9]$$

where the displacement functions

$$x'^i = x'^i(x,t,t')$$

represent the position at time t' of the element which is instantaneously at the point x^i at time t, p is an arbitrary isotropic pressure, p_{ik} is the stress tensor, $e^{(1)}_{ik}$ is the rate-of-strain tensor and $N(\tau)$ is a distribution function of relaxation times τ. The elastico-viscous liquid represented by [7], [8], and [9] is known as liquid B' (*Walters* 1962).

We shall make use of the *Laplace* transformation. If $\bar{\Phi}$ is the *Laplace* transform of any quantity Φ then

$$\bar{\Phi} = \int_0^\infty \Phi(t)\, e^{-\gamma t}\, dt. \qquad [10]$$

In order to obtain the *Laplace* transformation of the equations of state [7]–[9], it is first noted that for those problems in which the rate of strain is zero for $t < 0$ (i. e. "generation" problems) eq. [8] can effectively be written as (cf. *Thomas* and *Walters*, 1966)

$$p'^{ik} = 2\int_0^{t} t\,\Psi(t-t')\,\frac{\partial x^i}{\partial x'^m}\,\frac{\partial x^k}{\partial x'^r}\, e^{(1)mr}(x',t')\,dt', \qquad [11]$$

where it is assumed that there is no discontinuity in the strain at $t = 0$.

If use is now made of *Duhamels* theorem [see, for example, *Carslaw* and *Jaeger* (1948) and *Thomas* and *Walters* (1966)] the transform of the equation of state given by [11] relevant to the following problems is

$$\bar{p}'_{xy} = 2\,\bar{e}^{(1)}_{xy}\int_0^\infty \frac{N(\tau)}{1+\gamma\tau}\, d\tau. \qquad [12]$$

The reason why *Ting* could not obtain bounded solutions to the problems he considered is that in using the "second-order" fluid he assumed implicity that the integral in [12] can be written as

$$\int_0^\infty N(\tau)\,[1-\gamma\tau]\, d\tau, \qquad [13]$$

which is not valid even for liquids with short relaxation times (*Thomas* and *Walters*, 1966).

2. Flow caused by a tangential surface force

a) Generation of flow

Fielder and *Thomas* (1967) have obtained a solution for the unsteady motion of a lamina starting from rest in a viscoelastic liquid under the influence of a constant force. They found only the velocity of the lamina, and did not consider the velocity profiles of the liquid.

We shall consider the generation of flow in a viscoelastic fluid (at rest at time $t = 0$), contained between the planes $y = 0$ and $y = h$, caused by a constant force $(f, 0, 0)$ acting on the face $y = h$ of the liquid for time $t \geqslant 0$.

All quantities are referred to the cartesian frame of reference (x, y, z).

If we assume a velocity distribution of the form

$$(u(y,t),0,0) \quad 0 < y < h, \quad t > 0, \qquad [14]$$

the initial condition is given by

$$u(y,0) = 0, \quad 0 < y < h, \qquad [15]$$

and the equation of continuity is automatically satisfied.

The equations of motion in the absence of a pressure gradient reduce to

$$\varrho\,\frac{\partial u}{\partial t}(y,t) = \frac{\partial p_{xy}}{\partial y}, \qquad [16]$$

where ϱ is the density of the fluid. The appropriate boundary conditions may be written in the form

$$u(0,t) = 0 \quad t > 0,$$
$$p_{xy}(h,t) = f \quad t > 0, \qquad [17]$$

where f is a constant.

By using [12] the transformed equation of motion [16] becomes

$$\frac{\partial^2 \bar{u}}{\partial y^2} = q^2\,\bar{u}, \qquad [18]$$

where

$$q^2 = \varrho\,\gamma \Big/ \int_0^\infty \frac{N(\tau)}{1+\gamma\tau}\, d\tau, \qquad [19]$$

subject to the boundary conditions

$$\bar{u}(0, \gamma) = 0, \qquad [20]$$

$$\frac{\partial \bar{u}}{\partial y}(h, \gamma) = \frac{f}{\gamma \int_0^\infty \frac{N(\tau)}{1 + \gamma\tau} d\tau}.$$

A solution of [18] and [20] is given by

$$\bar{u}(y, \gamma) = \frac{f q \sinh q y}{\varrho \gamma^2 \cosh q h}. \qquad [21]$$

It is now convenient to introduce an expression for the relaxation spectrum $N(\tau)$ in terms of a limiting viscosity at small rates of shear (η_0), a retardation time (λ_2) and a relaxation time (λ_1), i.e.[1])

$$N(\tau) = \eta_0 \frac{\lambda_2}{\lambda_1} \delta(\tau) + \eta_0 \frac{(\lambda_1 - \lambda_2)}{\lambda_1} \delta(\tau - \lambda_1), \qquad [22]$$

where $\lambda_1 > \lambda_2 > 0$, which reduces liquid B' to *Oldroyds* liquid B [*Oldroyd* (1950), *Walters* (1962)].

Using (22) and (19) we see that

$$q^2 = \frac{\varrho \gamma (1 + \gamma \lambda_1)}{\eta_0 (1 + \gamma \lambda_2)}. \qquad [23]$$

If we now non-dimensionalize in the following manner

$$s = \frac{\varrho h^2}{\eta_0} \gamma, \quad S_1 = \frac{\eta_0 \lambda_1}{\varrho h^2}, \quad S_2 = \frac{\eta_0 \lambda_2}{\varrho h^2}, \quad t_1 = \frac{\eta_0 t}{\varrho h^2}.$$

$$y_1 = \frac{y}{h}, \quad u_0 = \frac{f h}{\eta_0}, \quad q_1^2 = \frac{s(1 + S_1 s)}{(1 + S_2 s)},$$

$$\bar{u}_1 = \int_0^\infty u(t_1) e^{-s t_1} dt_1, \qquad [24]$$

eq. [21] on using [23] becomes

$$\bar{u}_1(y_1, s) = \frac{u_0}{s^{3/2}} \left\{\frac{1 + S_1 s}{1 + S_2 s}\right\}^{1/2} \frac{\sinh q_1 y_1}{\cosh q_1}. \qquad [25]$$

Before evaluating the *Laplace* inversion of $\bar{u}_1$, it is necessary to give some consideration to the singularities of $\bar{u}_1$ with respect to s. These singularities are the zeros of $s\{s(1 + S_2 s)\}^{1/2} \cosh q_1$. Thus, the complete set of singularities is given by $s = 0$, and those given by

$$q_1 = \left(n + \frac{1}{2}\right) \pi i, \quad n = 0, 1, 2, \ldots. \qquad [26]$$

i.e.[2])

$$s_N = \frac{-\alpha_N \mp \beta_N}{2 S_1}, \qquad [27]$$

$$N = (2n + 1)\pi/2,$$

where

$$\alpha_N = 1 + S_2 N^2 \text{ and } \beta_N = \{(1 + S_2 N^2)^2 - 4 S_1 N^2\}^{1/2}. \qquad [28]$$

Since $S_2 > 0$ (eq. [22]) it is obvious that all the singularities lie in that part of the complex plane for which $Re\, s < 0$, except for the singularity at $s = 0$.

In order that we may ascertain the positions of the singularities we consider the function

$$B(m) = (1 + S_2 m)^2 - 4 S_1 m. \qquad [29]$$

Solutions of $B(m) = 0$ are

$$m_1 = \frac{(2 S_1 - S_2) + 2\sqrt{S_1(S_1 - S_2)}}{S_2^2}$$

and

$$m_2 = \frac{(2 S_1 - S_2) - 2\sqrt{S_1(S_1 - S_2)}}{S_2^2}. \qquad [30]$$

By considering the function $B(m)$ and $A(m) = 1 + S_2 m$ with their derivatives and using the fact that $S_1 > S_2 > 0$, we are able to make the following observations concerning the distribution of the singularities

$$s_N^+ = \frac{-\alpha_N + \beta_N}{2 S_1} \text{ and } s_N^- = \frac{-\alpha_N - \beta_N}{2 S_1}. \qquad [31]$$

For $m_2 < N^2 < m_1$ the singularities are conjugate point pairs about points on that portion of the real negative axis between P_1 and P_2 where

$$P_1 = -\frac{1}{S_2}\left\{1 + \sqrt{1 - \frac{S_2}{S_1}}\right\},$$

$$P_2 = -\frac{1}{S_2}\left\{1 - \sqrt{1 - \frac{S_2}{S_1}}\right\}. \qquad [32]$$

By considering those values of N^2 for which s_N is real (i.e. $B(m) > 0$, see fig. 1) we see that

(i) for $N^2 < m_2$,
s_N^+ decreases with increasing N^2 and $P_2 < s_N^+ < 0$
and s_N^- increases with increasing N^2 and $-\frac{1}{S_2} < s_N^- < P_2$;

(ii) for $N^2 > m_1$,
s_N^+ increases with increasing N^2 and $P_1 < s_N^+ < -\frac{1}{S_2}$
and s_N^- decreases with increasing N^2 and $s_N^- < P_1$.

It should be noted that s_N^+ has a limit point at $-\frac{1}{S_2}$ and s_N^- has one at $-\infty$.

[1]) $\delta(x)$ is the *Dirac* delta function defined by $\delta(x) = 0$, $x \neq 0$, and $\int_{-\infty}^{\infty} \delta(x)\, dx = \int_0^\infty \delta(x)\, dx = 1$.

[2]) The singularities of $\bar{u}_1(s, y_1) e^{s t_1}$ given by [27] are simple poles if $\beta_N \neq 0$ and double poles if $\beta_N = 0$.

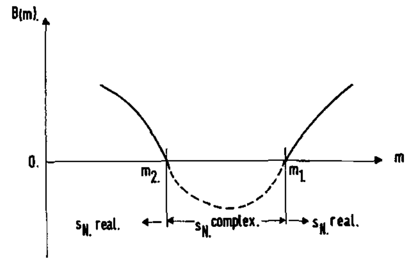

Fig. 1 The variation in sign of B(m) (eq. [29]).

The arrangement of singularities is illustrated in fig. 2.

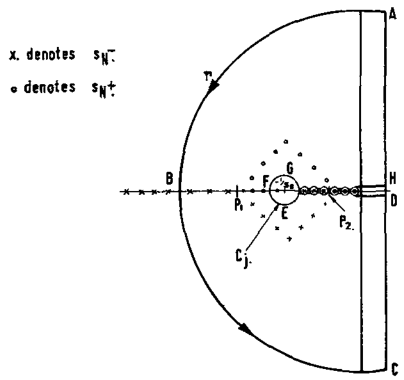

Fig. 2 Contour of integration and the distribution of singularities

We may now proceed to evaluate the complex inversion integral

$$u(y_1, t_1) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \bar{u}_1(y_1, s)\, e^{s t_1}\, ds \quad (\sigma > 0) \qquad [33]$$

by means of *Cauchys* Residue theorem. To this end we define a contour Γ, $ABCDEFGH$ in the direction indicated by the arrows in fig. 2. The curves represented by ABC may be considered as a sequence of semicircles which do not pass through any of the singularities. If we represent ABC by $s = Re^{i\theta}(-\pi < \theta < \pi, R > R_0)$ then the integral around ABC will vanish as $R \to \infty$ provided $t > 0$ and

$$|\bar{u}_1(y_1, s)| < \text{constant.}\ R^{-\varkappa}, \qquad [34]$$

where R_0 and $\varkappa$ are constants such that $\varkappa > 0$. [*Carslaw* and *Jaeger* (1947)].

Since $y_1 \leqslant 1$ it is obvious that

$$\left|\frac{\sinh q_1 y_1}{\cosh q_1}\right| \text{ is finite,} \qquad [35]$$

provided $\cosh q_1 \neq 0$, which is true except at the singularities.

Similarly

$$\left|\frac{1 + S_1 s}{1 + S_2 s}\right|^{1/2} \text{ is bounded, provided } s \neq -\frac{1}{S_2}. \qquad [36]$$

Thus, from eq. [25], we see that

$$|\bar{u}_1(y_1, s)| < \frac{\text{constant}}{R^{3/2}}$$

and the condition (34) is satisfied.

We now construct a sequence of circles $\{C_J\}$ [cf. *Ting* (1963), see fig. 2] whose radii decrease as J increases, around the limit point $\left(s = -\frac{1}{S_2}\right)$ such that

$$\lim_{J \to \infty} \int_{C_J} u_1(y_1, s)\, e^{s t_1}\, ds = 0. \qquad [37]$$

The singularities just to the left of $s = -\frac{1}{S_2}$ are given by s_N^+, where $N = (2n + 1)\pi/2$ and $N^2 > m_1$. We thus define $\{C_J\}$ to be a sequence of circles with centres at $s = -\frac{1}{S_2}$ which pass through the points s_J^+, where $J = n\pi$, and $J > J'$. J' being such that the largest circle $C_{J'}$ does not contain any of the singularities to the right of $s = -\frac{1}{S_2}$ nor any singularities with a non-zero imaginary part. Thus, from fig. 2, we see that the circles $\{C_J\}$, $J > J'$, do not pass through any of the singularities.

The radii ε_J of $\{C_J\}$ are given by

$$\varepsilon_J = \frac{1 + S_2 J^2 - \{(1 + S_2 J^2)^2 - 4 S_1 J^2\}^{1/2}}{2 S_1} - \frac{1}{S_2} \qquad [38]$$

and thus $\lim_{J \to \infty} \varepsilon_J = 0$.

On the circles $\{C_J\}$,

$$s + \frac{1}{S_2} = \varepsilon_J\, e^{i\theta}\ (-\pi \leqslant \theta \leqslant \pi)$$

and

$$\frac{ds}{d\theta} = i\, \varepsilon_J\, e^{i\theta}. \qquad [39]$$

Using [35], [36], [39], and [25] together with the fact that the $\{C_J\}$ do not pass through any singularities we obtain

$$\left|\int_{C_J} \bar{u}_1(y_1, s)\, e^{s t_1}\, ds\right| < \varepsilon_J^{1/2} \left|\int_{-\pi}^{\pi} \frac{\text{constant}\ e^{|s|\, t_1}}{|s|^{3/2}}\, d\theta\right| \qquad [40]$$

From [38] and [40] we see that

$$\lim_{J \to \infty} \int_{C_J} \bar{u}_1(y_1, s)\, e^{s t_1}\, ds = 0.$$

If we now apply *Cauchys* residue theorem to the contour Γ we obtain[1]), as $J \to \infty$ and $R \to \infty$,

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \bar{u}_1(y_1, s)\, e^{s t_1}\, ds$$

$$= \text{the sum of all the residues of } \bar{u}_1(y_1, s)\, e^{s t_1} \text{ in } \Gamma. \quad [41]$$

By calculating the residues and using [33] and [41] we obtain after some reduction

$$\frac{u}{u_0}(y_1, t_1) = y_1 + 2\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{N^2} \sin N y_1\, e^{-\alpha_N t_1/2 S_1}\, G_N(t_1),$$

where

$$G_N(t_1) = \left[\cosh\left(\frac{\beta_N t_1}{2 S_1}\right) + \frac{\{1 + N^2(S_2 - 2 S_1)\}}{\beta_N} \sinh\left(\frac{\beta_N t_1}{2 S_1}\right)\right] \quad [42]$$

and α_N, β_N are defined by [28].

To obtain the corresponding result for a Newtonian fluid it is necessary to put $S_1 = S_2 = 0$ in eqs. [19] and [25], and thus by a similar analysis we obtain

$$\frac{u}{u_0}(y_1, t_1) = y_1 + 2\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{N} \sin N y_1\, e^{-N^2 t_1}. \quad [43]$$

This solution agrees with that given by *Lamb* (1945).

b) *Verification of the solution*

It is necessary to show that our solution satisfies the original differential equation, the initial condition and the boundary conditions. Following *Carslaw* and *Jaeger* (1947) (pages 76, 91) we consider $u(y_1, t_1)$ given by eq. [33].

Using [22] we see that the equation for p'_{xy} given by [11] may be written in the form [cf. *Walters* (1960) (1962)]

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p'_{xy} = 2\eta_0 \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) e^{(1)}_{xy}. \quad [44]$$

From [44], [16], and [24], the differential equation which $u(y_1, t_1)$ must satisfy is

$$\frac{\partial u}{\partial t_1} + S_1 \frac{\partial^2 u}{\partial t_1^2} = \frac{\partial^2 u}{\partial y_1^2} + S_2 \frac{\partial^3 u}{\partial t\, \partial y_1^2}. \quad [45]$$

The corresponding boundary and initial conditions, [15] and [17], should be written in the form (cf. *Carslaw* and *Jaeger* (1947), page 133, and *Doetsch* (1961), page 163):

for fixed y_1 in $0 < y_1 < 1$, $u \to 0$ as $t_1 \to 0$, [46]

for fixed $t_1 > 0$, $(p_{xy})_1 \to 1$ as $y_1 \to 1$ (where $(p_{xy})_1 = p_{xy}/f$), [47]

for fixed $t_1 > 0$, $u \to 0$ as $y_1 \to 0$. [48]

Following *Carslaw* and *Jaeger* (1947) (page 131) we choose a new path of integration L' (see fig. 3), which begins at infinity in the direction $\arg s = -\Phi_1$, where $3\pi/4 > \Phi_1 > \pi/2$, passes to the right of the origin, keeping all singularities to the left, and ends in the direction $\arg s = \Phi_1$. It is possible to define such a curve since the integrand does not have a sequence of singularities extending to infinity along a line parallel to the imaginary axis.

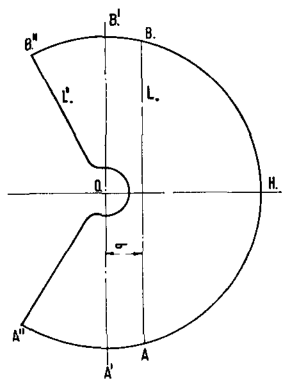

Fig. 3. New path of integration

From [25] and [34] we see that

$$\int \bar{u}_1(y_1, s)\, e^{s t_1}\, ds,$$

taken over the arcs $BB'B''$, $A''A'A$ of the circle of radius R, tends to zero as $R \to \infty$. Since there are no singularities between L and L' we may rewrite our solution [33] as

$$u(y_1, t_1) = \frac{1}{2\pi i}\int_{L'} \frac{u_0}{s^{3/2}} \left\{\frac{1 + S_1 s}{1 + S_2 s}\right\}^{1/2} \frac{\sinh q_1 y_1}{\cosh q_1}\, e^{s t_1}\, ds. \quad [49]$$

We must now evaluate the order of magnitude of the integrand in [49]. On L' we may put $s = r_1 e^{i\Phi_1}$ when $r_1 > R_1$, say. We may assume that

$$1 + S_1 s = r_2 e^{i\Phi_2} \quad \text{and} \quad 1 + S_2 s = r_3 e^{i\Phi_3}.$$

Having chosen Φ_1 we see that $\Phi_3 > \pi/2$, provided $r_1 > -\sec\Phi_1/S_2$.

Thus, when $r_1 > \max(R_1, -\sec\Phi_1/S_2)$,

$$3\pi/4 > \Phi_1 > \Phi_2 > \Phi_3 > \pi/2. \quad [50]$$

[1]) The singularities lying on the real axis to the right of the largest of the $\{C_J\}$ can be regarded as being inside Γ, for the purpose of calculation, since this gives the same result as evaluating the contributions of a sequence of small circles formed by surrounding each singularity separately as shown in fig. 2.

Using [23], [24], we may now write q_1 in the form

$$q_1^2 = r_1\, C(r_1)\, e^{i\,\theta_0}\,, \qquad [51]$$

where $C(r_1) = r_2/r_3$ and $\theta_0 = \Phi_1 + \Phi_2 - \Phi_3$. From [50] we have that $\pi > \theta_0 > \pi/2$ and obviously $C(r_1)$ is always bounded on L'.

If we use eq. [51] together with the inequalities

$$\cosh a > |\cosh(a + i\,b)| > \sinh a,$$

$$\cosh a > |\sinh(a + i\,b)| > \sinh a,$$

we obtain the inequality

$$\frac{\sinh q_1 y_1}{\cosh q_1} < \exp(-(1 - y_1)\,F(r_1)) \times \left[\frac{1 + \exp(-2\,y_1\,F(r_1))}{1 - \exp(-2\,F(r_1))}\right],$$

where $F(r_1) = (r_1\, C(r_1))^{1/2} \cos\frac{\theta_0}{2}$.

But $1 < 1 + \exp(-2\,y_1\,F(r_1)) < 2$ and $1 > 1 - \exp(-2\,F(r_1)) > {}^1/_2$ when $r_1 > R_2$, say. Thus, on L' we obtain the following estimate for the integrand of [49]:

$$|\bar{u}(y_1, s)\, e^{s\,t_1}| < \frac{4\,u_0\, C^{1/2} \exp\left(t_1\, r_1 \cos\Phi_1 - (1 - y_1)\left(C_1\, r_1)^{1/2} \cos\frac{\theta_0}{2}\right)\right.}{r_1^{\,3/2}}, \qquad [52]$$

where $r_1 > R_0 = \max\left(R_1,\, -\frac{\sec\Phi_1}{S_2},\, R_2\right)$ and $C_1 < C(r_1) < C_2$.

Using [52] it is easy to show that we have the uniform convergence that allows differentiation under the integral sign in [49], for all the derivatives required in [45] when $t_1 > 0$ and $0 < y_1 < 1$. It follows immediately that the differential eq. [45] is satisfied by $u(y_1, t_1)$ as given by [49] and therefore by [33].

We now show that the initial condition [46] is satisfied. For a fixed y_1 in $0 < y_1 < 1$ the integral [49] is uniformly convergent for $t_1 < 0$ and is thus a continuous function of t_1 in $t_1 \geqslant 0$. Hence,

$$\lim_{t_1 \to 0} u(y_1, t_1) = \frac{1}{2\pi i}\int_{L'} \frac{u_0}{s^{3/2}} \left\{\frac{1 + S_1 s}{1 + S_2 s}\right\}^{1/2} \frac{\sinh q_1 y_1}{\cosh q_1}\, ds\,.$$

But the same integral, taken over the arc $B''B'HAA''$ of fig. 3, tends to zero when $r_1 \to \infty$. Since there are no poles within the closed contour of fig. 2, it follows that

$$\lim_{t_1 \to 0} u(y_1, t_1) = 0\,.$$

Lastly, we show that the boundary conditions [47] and [48] are satisfied. We know that

$$\frac{\partial u}{\partial y_1}(y_1, t_1) = \frac{1}{2\pi i}\int_{L'} \frac{u_0}{s}\left\{\frac{1 + S_1 s}{1 + S_2 s}\right\} \frac{\cosh q_1 y_1}{\cosh q_1}\, e^{s\,t_1} ds\,.$$

From [52] we see that this integral converges uniformly for fixed $t_1 > 0$ in $0 \leqslant y_1 \leqslant 1$. It is thus a continuous function of y_1 in this interval. Therefore

$$\lim_{y_1 \to 1} \frac{\partial u}{\partial y_1}(y_1, t_1) = \frac{1}{2\pi i}\int_{L'} \frac{u_0}{s}\left\{\frac{1 + S_1 s}{1 + S_2 s}\right\} e^{s\,t_1} ds$$

$$= u_0\left\{1 + \left(\frac{S_1}{S_2} - 1\right) e^{-t_1/S_2}\right\} \text{ when } t_1 > 0\,. \qquad [53]$$

Using this expression for $\frac{\partial u}{\partial y_1}(y_1, t_1)$ it may be shown that the boundary condition [47] is satisfied.

From eq. [53] we see that

$$\lim_{t_1 \to 0^+} \frac{\partial u}{\partial y_1}(1, t_1) = \frac{S_1\, u_0}{S_2} \text{ and } \lim_{t_1 \to \infty} \frac{\partial u}{\partial y_1}(1, t_1) = u_0\,.$$

This shows that as $t_1 \to \infty$ the velocity (at $y_1 = 1$) approaches that for steady simple shearing flow, but since $\frac{\partial u}{\partial y_1}(1, 0) = 0$ there is a discontinuity in $\frac{\partial u}{\partial y_1}(1, t_1)$ at $t_1 = 0$ equal to $S_1\, u_0/S_2$. (The corresponding value for this discontinuity in the Newtonian case is u_0).

The phenomenon of a discontinuity in the rate of strain, without an instantaneous change in deformation, when a stress is suddenly applied may well be possible in some fluids including the Newtonian fluid [*Oldroyd* (1965)]. *Ting* showed that for the "second-order" fluid of *Coleman* and *Noll* $\frac{\partial u}{\partial y_1}(y_1, t_1)$ is continuous at $t_1 = 0$. This was to be expected since *Oldroyd* (1965) has already pointed out that the *Coleman* and *Noll* "simple" fluid did not allow for the possibility of the above phenomenon.

We know that $u(y_1, t_1)$ is given by [49] for $0 < y_1 < 1$. From [52] we see that the integral in [49] converges uniformly for a fixed $t_1 (> 0)$ in $0 \leqslant y_1 \leqslant 1$. Therefore, $u(y_1\, t_1)$ is a continuous function of y_1 in [0,1], and is zero when $y_1 = 0$, and so

$$\lim_{y_1 \to 0} u(y_1, t_1) = 0 \quad \text{for a fixed } t_1\, (> 0)\,.$$

This completes the verification that $u(y_1, t_1)$ is in fact a solution of the differential eq. [45] with the boundary conditions [47] and [48] and the initial condition [46].

c) Decay of the steady flow

In this section we consider the decay of the steady state obtained in the previous section when the tangential surface force ceases to act. Thus, by assuming a velocity distribution of the form [14], the initial condition becomes

$$u(y, t) = \frac{f\, y}{\eta_0}, \quad 0 < y < h, \quad t \leqslant 0.$$

The boundary conditions may be written as

$$\left.\begin{aligned} u(0, t) &= 0 \\ p_{xy}(h, t) &= 0 \end{aligned}\right\} t > 0.$$

The corresponding transformed equation of motion becomes[1])

$$\frac{\partial^2 \bar{u}}{\partial y^2} - q^2\, \bar{u} + \frac{f\, y\, q^2}{\eta_0\, \gamma} = 0, \qquad [54]$$

where q^2 is given by [23], subject to the boundary conditions

$$\bar{u}(0, \gamma) = 0,$$

$$\frac{\partial \bar{u}}{\partial y}(h, \gamma) = \frac{f(\lambda_2 - \lambda_1)}{\eta_0(1 + \gamma\, \lambda_2)}. \qquad [55]$$

A solution of [54] and [55], in terms of the dimensionless variables [24] is given by

$$\bar{u}_1 = \frac{u_0\, y_1}{s} - \frac{u_0}{s^{3/2}} \left\{\frac{1 + s\, S_1}{1 + s\, S_2}\right\}^{1/2} \frac{\sinh q_1\, y_1}{\cosh q_1}. \qquad [56]$$

The *Laplace* inversion of [56] is obtained by a method analogous to that used in the previous section and the final solution is given by

$$\frac{u}{u_0}(y_1, t_1) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n \sin N\, y_1}{N^2}\, e^{-\frac{\alpha N^{t_1}}{2 S_1}}\, G_N(t_1). \qquad [57]$$

This solution may be verified by a similar method to that used in the previous section 2 (b). The corresponding Newtonian solution is given by

$$\frac{u}{u_0}(y_1, t_1) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{N^2} \sin N\, y_1\, e^{-N^2 t_1}. \qquad [58]$$

3. Parallel flow through a channel under a constant pressure gradient

a) Generation of steady flow

We now consider the generation of flow through a channel, bounded by the fixed planes $y = 0$ and $y = h_1$, under a constant pressure gradient $\frac{\partial p}{\partial x} (= k)$.

[1]) When account is taken of the rate of strain for $t < 0$ an extra term appears in [12].

By assuming a velocity distribution and initial condition of the form [14] and [15], the relevant equation of motion is

$$\varrho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial p_{xy}}{\partial y},$$

subject to the boundary conditions

$$u(0, t) = u(h, t) = 0 \quad (t > 0). \qquad [59]$$

As in the previous problem we use a *Laplace* transformation and obtain the transformed equation of motion,

$$\frac{\partial^2 \bar{u}}{\partial y^2} - q^2\, \bar{u} = \frac{k\, q^2}{\varrho\, \gamma^2}. \qquad [60]$$

where q^2 is given by [23], subject to

$$\bar{u}(0, \gamma) = \bar{u}(h, \gamma) = 0. \qquad [61]$$

A solution of [60] and [61] in terms of the dimensionless variables [24] is given by

$$\frac{\bar{u}_1}{u_0} = -\frac{8}{s^2} \left\{\frac{\sinh q_1\, y_1 - \sinh q_1 + \sinh (q_1(1 - y_1))}{\sinh q_1}\right\}, \qquad [62]$$

where $u_0 = -\frac{k\, h^2}{8\, \eta_0}$.

The singularities of $\bar{u}_1(y_1, s)\, e^{s t_1}$ with respect to s are given by $s = 0$ and $\sinh q_1 = 0$. Thus, in this case, the complete set of singularities is given by the simple poles $s = 0$ and $s_{N_1}^{\pm} = \frac{-\alpha_{N_1} \pm \beta_{N_1}}{2\, S_1}$, where

$$\alpha_{N_1} = 1 + S_2\, N_1^2, \quad \beta_{N_1} = \{(1 + S_2\, N_1^2)^2 - 4\, S_1\, N_1^2\}^{1/2} \qquad [63]$$

and $N_1 = n\, \pi$, $n = 0, 1, 2 \ldots$
Thus, the arrangement of singularities is essentially the same as for the previous problem. We find the inversion of [62] by choosing a similar contour to fig. 2. The sequences of semicircles ABC and small circles $\{C_J\}$ are chosen not to pass through any of the singularities, and thus we obtain after some reduction[2])

$$\frac{u}{u_0}(y_1, t_1) \qquad [64]$$

$$= -4\, y_1(y_1 - 1) - 32 \sum_{n=1}^{\infty} \frac{\sin N\, y_1}{N^3}\, e^{-\frac{\alpha N^{t_1}}{2 S_1}}\, G_N(t_1),$$

where $N = (2\, n - 1)\, \pi$.

The justification of this solution can be carried out by a method similar to that in 2(b). The corresponding Newtonian solution is

$$\frac{u}{u_0}(y_1, t_1) = -4\, y_1(y_1 - 1) - 32 \sum_{n=1}^{\infty} \frac{\sin N\, y_1}{N^3}\, e^{-N^2 t_1} \qquad [65]$$

[cf. *Bromwich* (1930)].

[2]) N_1 even gives a zero residue.

b) Decay of the steady flow

As in the first problem we now consider the decay of the steady flow that has been produced by the constant pressure gradient when the pressure gradient ceases to act.

We assume a velocity distribution of the form [14] and the initial condition is then given by

$$u(y,t) = \frac{k\,y}{2\,\eta_0}(y-h) \quad 0 < y < h\,, \quad t \leqslant 0\,.$$

The boundary conditions are given by [59]. The transformed equation of motion is

$$\frac{\partial^2 u}{\partial y^2} - q^2 \bar{u} + \frac{q^2 k\, y}{2\,\eta_0\,\gamma}(y-h) + \frac{q^2 k}{\varrho\,\gamma}\,\frac{(\lambda_1-\lambda_2)}{(1+\gamma\,\lambda_1)} = 0\,, \qquad [66]$$

subject to

$$\bar{u}(0,\gamma) = \bar{u}(h,\gamma) = 0\,. \qquad [67]$$

A solution of [66] and [67] in terms of the dimensionless variables [24] is given by

$$\frac{\bar{u}_1}{u_0} = \frac{-8}{s}\left[\frac{y_1(y_1-1)}{2}\right. + \left.\left\{\frac{\sinh q_1 - \sinh q_1 y_1 + \sinh\left(q_1(y-1)\right)}{s\,\sinh q_1}\right\}\right].$$

On inversion, by a similar method to 2(a), we obtain the solution

$$\frac{u}{u_0}(y_1,t_1) = 32\sum_{n=1}^{\infty}\frac{\sin N\,y_1}{N^3}\,e^{-\frac{\alpha\,N^{t_1}}{2\,S_1}}\,G_N(t_1)\,, \qquad [68]$$

with $N = (2\,n-1)\,\pi$, which can also be verified in a manner similar to 2(b). The corresponding Newtonian solution is

$$\frac{u}{u_0}(y_1,t_1) = 32\sum_{n=1}^{\infty}\frac{\sin N\,y_1}{N^3}\,e^{-N^2 t_1}. \qquad [69]$$

4. Conclusions

As expected, the presence of elasticity does not affect the terminal velocity profiles, which are the same as for a Newtonian fluid. Fig. 4 and 5 show the effect of S_1 and S_2 for the generation of flow due to a constant tangential force. Fig. 6 shows the velocity at $y_1 = 1$ for a fixed value of S_1 and various values of S_2. The corresponding decay curves are shown in figs. 7, 8 and 9. A similar set of figures for the generation and decay of flow in a channel due to a constant pressure gradient is given in figs. 10 to 15.

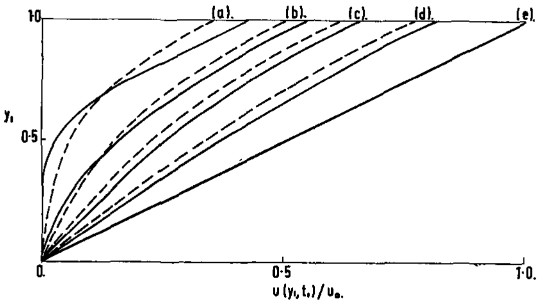

Fig. 4

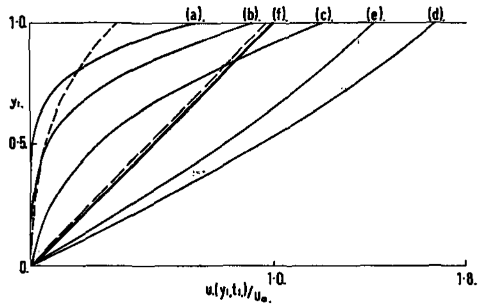

Fig. 5

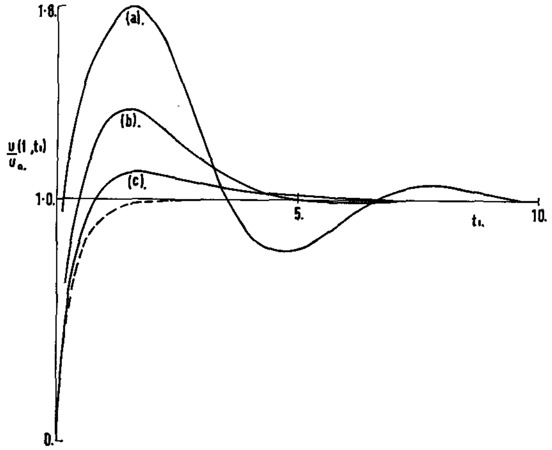

Fig. 6

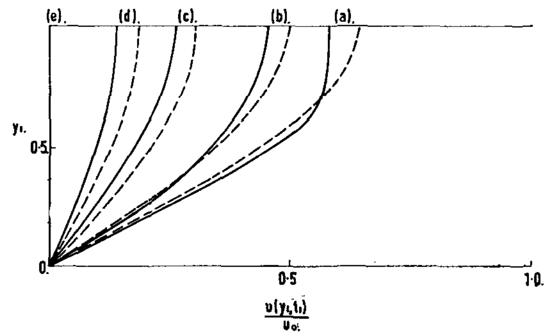

Fig. 7

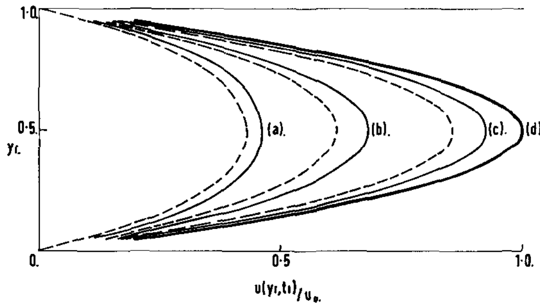

Fig. 10

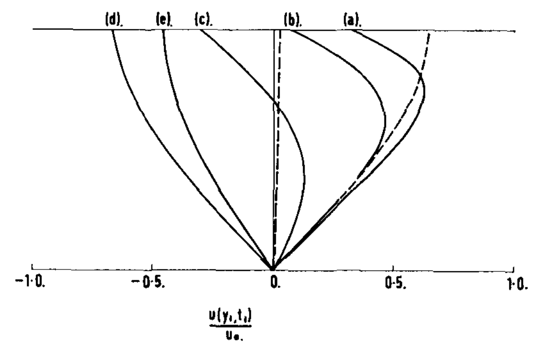

Fig. 8

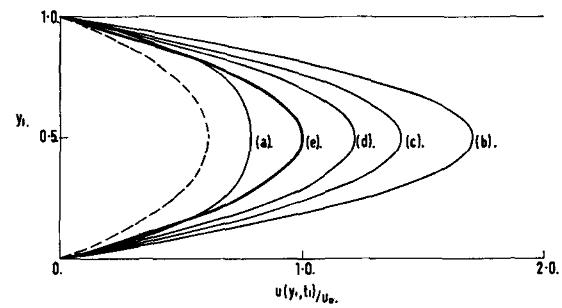

Fig. 11

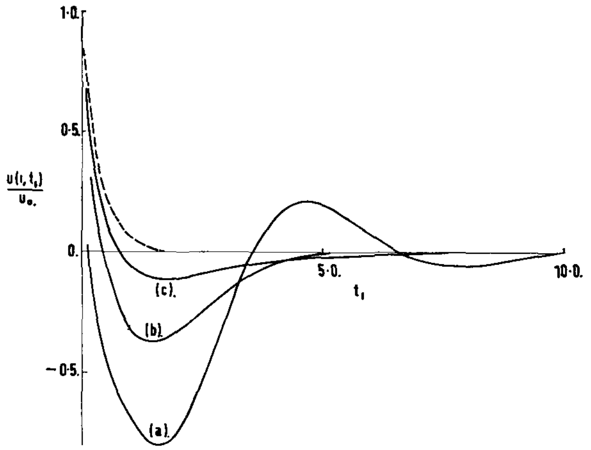

Fig. 9

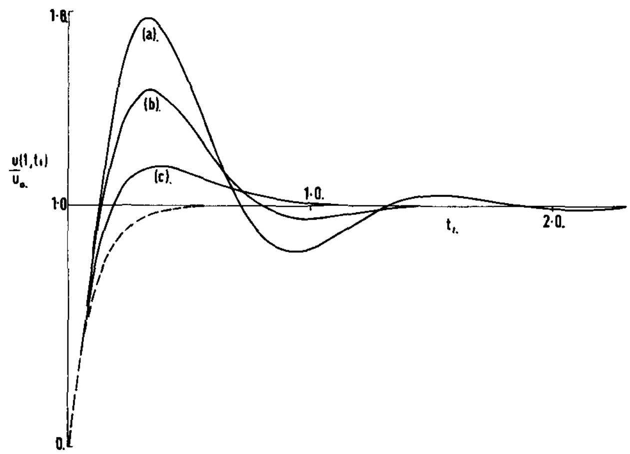

Fig. 12

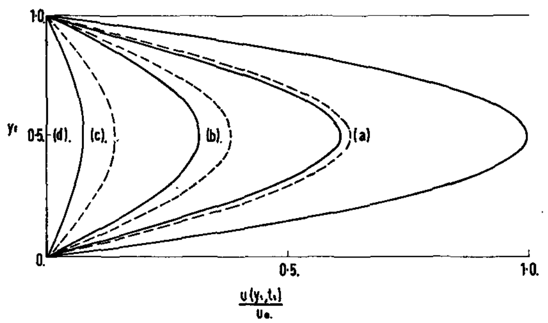

Fig. 13

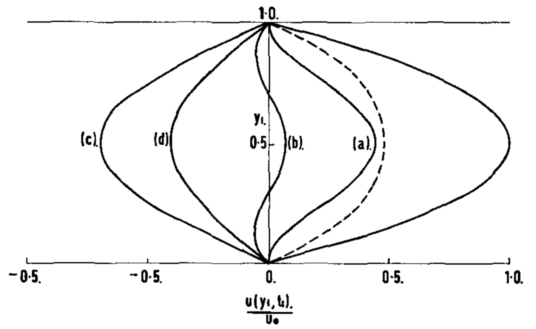

Fig. 14

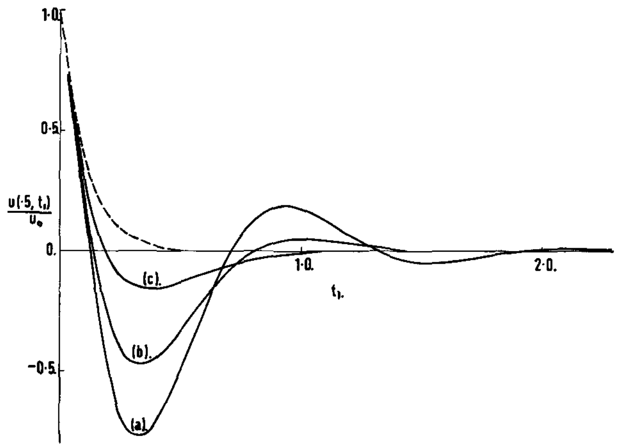

Fig. 15

From fig. 4 it may be observed that, for small time, the effect of a relatively small amount of elasticity tends to increase the velocity of the fluid near $y_1 = 1$, but decrease it near $y_1 = 0$. For larger time the effect of elasticity is to increase the velocity for all y_1, until the terminal velocity is reached. The effect of a small amount of elasticity on the decay of the flow when the tangential force ceases to act (fig. 7) is seen, for small time, to decrease the velocity more quickly near $y_1 = 1$ and more slowly near $y_1 = 0$. For larger time the effect of elasticity is to decrease the velocity for all y_1 before decaying to zero.

The effect of a similarly small amount of elasticity on the generation of flow through a channel under a constant pressure gradient is seen to increase the velocity for all y_1 and t_1 (fig. 10), and when the pressure gradient ceases to act the resulting decay of the steady flow is accelerated by the presence of the small amount of elasticity (fig. 13).

It is of interest to note that in both the "generation of flow" and "decay" problems the effect of a larger amount of elasticity is such that the velocity profiles overshoot the terminal velocity before eventually tending to it with increasing time (figs. 5, 11, 8, 14). In fact, when the velocities are plotted against time for a particular y_1 it is found that the velocity oscillates about the terminal velocity before tending to that terminal velocity for sufficiently large time (figs. 6, 9, 12, 15). In all cases increasing S_2 for a particular value of S_1 tends to dampen the oscillation.

The oscillatory behaviour of viscoelastic liquids in unsteady flow problems has been noted before, theoretically by *Fielder* and *Thomas* (1967) and experimentally by *Hermes* and *Fredrickson* (1967).

The four problems considered in this paper could be very easily set up experimentally and so conceivably may be of some use in determining S_1 and S_2 for specific liquids. It may be more convenient, experimentally, to consider the velocity development (or degeneration to, for the decay) from a slower but steady state instead of from rest. The above analysis may be easily adapted in this case.

The more difficult problems of the generation and decay of steady flows in a pipe of circular cross-section, which may be more easily set up experimentally, are at present under consideration.

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Summary

Consideration is given to the reasons why the *Coleman* and *Noll* second order fluids are unsuitable for use in the solution of unsteady flow problems for elastico-viscous liquids, especially when *Laplace* transforms are used. A number of unsteady flow problems are then solved using a constitutive equation of the „integral" type. The presence of elasticity in the liquid is shown to have quite a dramatic effect on the velocity profiles.

Zusammenfassung

Es werden die Gründe betrachtet, aus denen die *Coleman*- und *Noll*-Flüssigkeiten nicht geeignet sind, um Probleme der instationären Strömung elastoviskoser Flüssigkeiten zu lösen, besonders wenn *Laplace*-Transformationen benutzt werden. Einige Probleme des instationären Fließens werden dann mit Hilfe einer Zustandsgleichung vom Integraltyp gelöst. Es wird abschließend gezeigt, daß eine in der Flüssigkeit vorhandene Elastizität einen starken Einfluß auf die Geschwindigkeitsprofile hat.

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A theoretical study on fiber spinnability

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With 6 figures

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I. Introduction

In preparing polymer solutions or melts for manufacturing synthetic fibers, one of the fundamental questions one faces is the criterion for spinnability. It is commonly understood that only a certain class of liquids is spinnable under the set of spinning conditions. It is also possible that the liquid which is spinnable at one set of spinning conditions may not be spinnable at another set of spinning conditions. It is generally known that spinnability depends, among many other things, on (a) the rheological properties of liquids to be spun, (b) jet stretch, (c) the hole size and shape, and (d) the rate of mass and heat transfer between the extruded filament and the coagulation medium (in the case of wet spinning) or the cooling medium (in the case of melt spinning). Here jet stretch is defined by the ratio of the velocity of the filament at the take-up device to the average velocity of the spinning solution at the exit of a spinnerette hole.

Nitschmann and *Schrade* (24) appear to be the first who attempted to explain the problem of spinnability in terms of material properties. *Thiele* and *Lamp* (37) investigated the technique of measuring spinnability by devising a suitable apparatus and determined the spinnability of colloidal solutions by means of high speed photography. And in their later study (38) the same authors reported that maximum spinnability was obtained at intermediate values of viscosity and elasticity.

Today it is generally accepted, from practical industrial experience, that almost all spinnable liquids being used for manufacturing synthetic fibers exhibit normal stress effect, i. e., elastic effect. Very recently,