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Particle Motions in Sheared Suspensions

XVII: Deformation and Migration of Liquid Drops[1])

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With 3 figures in 6 details

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List of Symbols

$A_n^{(m)}, B_n^{(m)}, C_n^{(m)}$	coefficients of $P_n^{(m)}$ in $p_{(n)}, \Phi_{(n)}, \chi_{(n)}$.
$b; b_x, b_y, b_z$	radius of sphere; semiaxes of drop.
C	a point.
D	deformation parameter.
$\boldsymbol{F}$; F	force; surface function.
G	velocity gradient.
h_1, h_2	y-coordinates of upper, lower disc.
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	unit vectors along x, y, z-axes.
$\boldsymbol{I}$	dyadic idemfactor.
k; $K(\lambda)$	gradient in velocity gradient; viscosity ratio factor.
M	$1/(n+1)(2n+3)$.
$\boldsymbol{n}$; n	unit normal vector; an integer.
p, P	outer, inner pressure.
P_n, $P_n^{(m)}$	Legendre function, associated Legendre function.
$p_{(n)}$	spherical harmonic.
$\boldsymbol{r}$, r	radial vector, its magnitude.
S, $\boldsymbol{S}$	drop surface, dilatation tensor.
t	time.
$\boldsymbol{u}$	arbitrary velocity field.
$\boldsymbol{v}$, $\boldsymbol{V}$	outer, inner velocity field.
$\boldsymbol{w}$	migration velocity vector.
x, y, z	Cartesian coordinates.
γ	interfacial tension.
ζ	vorticity vector.
θ	azimuthal angle.
η	outer fluid viscosity.
λ	viscosity ratio.
$\boldsymbol{\pi}$, $\boldsymbol{\Pi}$	stress tensor.
ω, ω_0	radial distance, initial radial distance.
$\chi_{(n)}$	spherical harmonic.
Φ	longitudinal angle.
$\Phi_{(n)}$	spherical harmonic.
ω_1, ω_2	angular velocities of upper, lower disc.

Subscripts

0	referred to origin on axis of rotation; initial value.
1, 2	of upper, lower disc.
b, C, S	value at b, C, S.
(n)	order.
r, θ, Φ	r-, θ-, or Φ-component.

Superscripts

0, 1	zero-order, first-order.
(∞)	undisturbed value.
$'$, $''$	partial, complete.

1. Introduction

A neutrally buoyant particle suspended in a flowing viscous fluid often translates with the local velocity of the fluid in the absence of the particle. However, *Brenner*, investigating the *Stokes* resistance of an arbitrary particle in arbitrary flow fields (1), has shown that this is not true generally, and that certain rigid particles, in some kinds of shear flow, migrate across the undisturbed streamlines. Such a migration of liquid droplets in *Poiseuille* flow through a tube was studied by *Goldsmith* and *Mason* (2), who give references to other work on the subject. *Brenner*'s methods for rigid particles do not involve the detailed calculation of velocity fields and are not applicable to liquid drops, although the limiting behavior of infinitely viscous drops can be inferred.

The radial migration occurring in *Poiseuille* flow was discussed by *Goldsmith* and *Mason* (2) in terms of the non-uniformity of the fluid velocity gradient; they proposed a formula with which they analyzed their data. Here we undertake a detailed hydrodynamic treatment of drop migration in a non-uniform shear flow. Although for the flow to be appreciably non-uniform over the particle, the particle size cannot be very small compared to the size of the apparatus producing the flow, nevertheless we shall not consider here the interaction between the particle and the apparatus walls. Wall effects are, however, taken into account explicitly later (3).

In this investigation we shall find the radial migration of a liquid drop suspended in a viscous fluid contained between counter-rotating discs, by calculating velocity fields outside and inside the drop. Flow between counter-rotating discs is chosen because it is a simple form of nonuniform shear field which is also experimentally realizable. The methods used here may be applicable to flow fields of greater importance. Since a knowledge of the deformed shape of the drop

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is essential to this calculation, we shall discuss *Taylor*'s theory (4) of deformation in detail, presenting new observations which confirm an aspect of the theory not previously tested experimentally.

2. Fundamental Assumptions

Throughout this investigation, the fluid motions will be assumed sufficiently slow that the quasistatic creeping motion equations can be applied. If the velocity and pressure fields are $\boldsymbol{v}$ and p, these equations and the equation of continuity are

$$\eta\, \nabla^2 \boldsymbol{v} = \nabla p\,, \qquad \nabla \cdot \boldsymbol{v} = 0\,, \qquad [1]$$

where η is the fluid viscosity. Suppose the fluid to be contained between two infinite parallel discs that rotate about a common perpendicular axis in opposite directions, with counterclockwise angular velocities ω_1 and $-\omega_2$ as shown in fig. 1a, and the distances from the stationary plane that must exist between them to the corresponding disc are h_1 and h_2. In terms of the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ of a right-handed *Cartesian* coordinate system x_0, y_0, z_0 with its origin in the stationary plane and $\boldsymbol{j}$ directed along the axis of rotation of the discs, the y_0-axis, the velocity $\boldsymbol{v}^{(\infty)''}$ and pressure $p^{(\infty)''}$ between the discs in the absence of suspended particles are

$$\boldsymbol{v}^{(\infty)''} = \boldsymbol{i}\, k\, y_0\, z_0 - \boldsymbol{k}\, k\, x_0\, y_0\,, \quad p^{(\infty)''} = \text{constant}\,, \quad [2]$$

where $k = (\omega_1 + \omega_2)/(h_1 + h_2) = \omega_1/h_1 = \omega_2/h_2$. The boundary condition that the fluid should stick to the discs at $y_0 = h_1$ and $y_0 = -h_2$, as well as [1], is satisfied by $\boldsymbol{v}^{(\infty)''}$ and $p^{(\infty)''}$.

Suppose that a liquid drop of viscosity λ, η and radius b is suspended at $(0, 0, \omega)$. In previous publications (2, 5, 6) the viscosity ratio λ was denoted by p, but here we shall use p for the pressure. We introduce a coordinate system x, y, z with its origin at the drop center, related to x, y, z by $x = x_0$, $y = y_0$, $z = z_0 - \omega$, in which [2] gives for the undisturbed velocity $\boldsymbol{v}^{(\infty)''}$ and pressure $p^{(\infty)''}$

$$\boldsymbol{v}^{(\infty)''} = \boldsymbol{i}\,(k\, y\, z - k\, \omega\, y) - \boldsymbol{k}\, k\, x\, y\,, \quad p^{(\infty)''} = \text{constant}.$$

The actual fields $\boldsymbol{v}''$ and p'' that exist outside the drop and the fields $\boldsymbol{v}''$ and P'' inside it must now be determined to satisfy [1] and also the following boundary conditions, which are analogous to those used by *Taylor* (7). At great distances from the drop, $\boldsymbol{v}''$ and p'' must reduce to $\boldsymbol{v}^{(\infty)''}$ and $p^{(\infty)''}$; both normal velocity components must vanish on the drop surface, and the tangential velocities must be continuous there; the tangential stress must also be continuous across the interface.

The solution of this problem is facilitated through its separation into two parts, $\boldsymbol{v}''$ and p'' being written as the sum of two terms,

$$\boldsymbol{v}'' = \boldsymbol{v}' + \boldsymbol{v}\,, \qquad p'' = p' + p \qquad [3]$$

where $\boldsymbol{v}'$, p' and $\boldsymbol{v}$, p with the corresponding internal fields $\boldsymbol{V}'$, P' and $\boldsymbol{V}$, P separately satisfy [1] and the boundary conditions at the drop surface, but at great distances $\boldsymbol{v}'$ and $\boldsymbol{v}$ reduce respectively to $\boldsymbol{v}^{(\infty)'}$ and $\boldsymbol{v}^{(\infty)}$ given by

$$\boldsymbol{v}^{(\infty)'} = \boldsymbol{i}\, k\, \omega\, y\,, \quad \boldsymbol{v}^{(\infty)} = \boldsymbol{i}\, k\, y\, z - \boldsymbol{k}\, k\, x\, y\,. \quad [4]$$

Because [1] and the boundary conditions are linear, addition of $\boldsymbol{v}'$ and $\boldsymbol{v}$ yields the solution to the original problem.

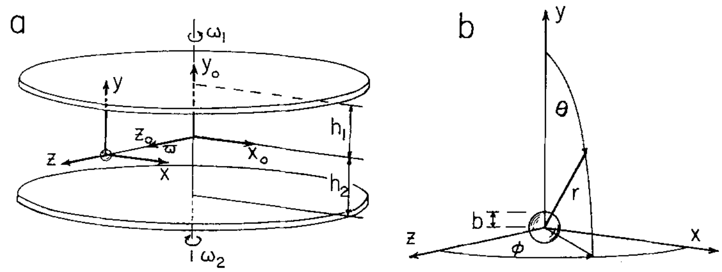

Fig. 1. (a) Counter-rotating discs, with coordinate systems x_0, y_0, z_0 and x, y, z. (b) Spherical polar coordinates r, θ, Φ with the y-axis as polar axis

3. Spherical Drop

We shall first solve this problem for a spherical drop, writing $\boldsymbol{v}^{0\prime}$, $p^{0\prime}$, $\boldsymbol{v}^0$, etc., for the resulting fields. The velocity components outside and inside a spherical drop in uniform shear flow, obtained by *Bartok* and *Mason* (5) from *Taylor*'s hyperbolic-flow solution (7), immediately yield the fields $\boldsymbol{v}^{0\prime}$, $p^{0\prime}$ and $\boldsymbol{V}^{0\prime}$, $P^{0\prime}$, where the velocity gradient $G = k\,\omega$. There is no resulting force on the drop.

To find $\boldsymbol{v}^0$, p^0 a general solution of [1] given by *Lamb* (8) will be used:

$$\boldsymbol{v} = \sum \left\{ \nabla \times (\boldsymbol{r}\, \chi_{(n)}) + \nabla \Phi_{(n)} + \frac{1}{2}\,(n+3)\, M\, r^2\, \nabla p_{(n)}/\eta - n\, M\, \boldsymbol{r}\, p_{(n)}/\eta \right\},$$

$$p = \sum p_{(n)}\,. \qquad [5]$$

Here $M = 1/(n+1)(2n+3)$ and the summation extends over positive and negative integers n; $p_{(n)}$, $\Phi_{(n)}$ and $\chi_{(n)}$ are solid spherical harmonics of degree n; $\boldsymbol{r}$ is the radial vector $\boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$. The radius vector $r = |\boldsymbol{r}|$ and the angles θ and Φ form the spherical polar coordinate system of fig. 1b, the corresponding unit vectors being $\boldsymbol{i}_r$, $\boldsymbol{i}_\theta$ and $\boldsymbol{i}_\Phi$. The $p_{(n)}$, $\Phi_{(n)}$, $\chi_{(n)}$ are conveniently expressed in terms of r, θ, Φ by means of the *Legendre* polynomials P_n, and associated *Legendre* polynomials $P_n^{(m)}$, in which the argument $\cos\theta$ is always understood; *Jahnke* and *Emde* (9) give general formulas for them and tabulate all the ones we shall use. The field $\boldsymbol{v}^{(\infty)}$, $p^{(\infty)}$ of [4] is expressed by [5] when the single harmonic $\chi_{(2)}^{(\infty)}$ is inserted in it, with

$$\chi_{(2)}^{(\infty)} = \frac{1}{3} k r^2 P_2 = \frac{1}{6} k r^2 (3\cos^2\theta - 1) .$$

To obtain the fields $\boldsymbol{v}^0$, p^0 we use [5] with the harmonics $\chi_{(2)}^{(\infty)}$ and $\chi_{(-3)}^0$, which is assumed to be

$$\chi_{(-3)}^0 = -\frac{1}{3} C_{-3} k b r^{-3} P_2 .$$

Similarly, for $\boldsymbol{V}^0$, P^0 we use [5] assuming

$$\chi_{(2)}^0 = \frac{1}{3} C_2 k r^2 P_2 .$$

The constants C_{-3} and C_2 will be evaluated below. The resulting components v_r^0, v_θ^0, v_Φ^0 and V_r^0, V_θ^0, V_Φ^0 are

$$v_r = 0, \quad v_\theta = 0, \quad v_\Phi = k(1 - b^5 r^{-5} C_3) r^2 \sin\theta \cos\theta ,$$
$$V_r = 0, \quad V_\theta = 0, \quad V_\Phi = k C_2 r^2 \sin\theta \cos\theta . \qquad [6]$$

Since $v_r^0 = V_r^0 = 0$, $v_\theta^0 = V_\theta^0$, and $v_\Phi^0 = V_\Phi^0$ if $1 - C_{-3} = C_2$, the boundary conditions for the velocity can be satisfied.

To satisfy the boundary condition for the tangential stress, we use a formula given by *Brenner* (9) for the radial-stress vector $\boldsymbol{\pi}_r$, which is the radial component of the stress tensor $\boldsymbol{\pi}$:

$$\boldsymbol{\pi}_r = \eta r^{-1} \sum \{(n-1) \boldsymbol{\nabla} \times (\boldsymbol{r}\chi_{(n)}) + 2(n-1) \boldsymbol{\nabla}\Phi_{(n)}$$
$$- (2n^2 + 4n + 3) M \boldsymbol{r} p_{(n)}/\eta + n(n+2) M r^2 \boldsymbol{\nabla} p_{(n)}/\eta\} . \quad [7]$$

This yields for the outer and inner radial-stress vectors π_r^0 and Π_r^0,

$$\pi_{rr}^0 = 0, \quad \pi_{r\theta}^0 = 0, \quad \pi_{r\Phi}^0 = \eta r^{-1} k (1 + 4 C_{-3} b^5 r^{-5}) \times r^2 \sin\theta \cos\Phi ,$$

$$\Pi_{rr}^0 = 0, \quad \Pi_{r\theta}^0 = 0, \quad \Pi_{r\Phi}^0 = \lambda \eta r^{-1} k C_2 r^2 \sin\theta \cos\Phi ,$$

so that continuity of tangential stress requires $(1 + 4C_{-3}) = \lambda C_2$. This together with the previously obtained relation $1 - C_{-3} = C_2$, gives

$$C_{-3} = (\lambda - 1)/(\lambda + 4), \quad C_2 = 5/(\lambda + 4) .$$

The force and torque on the sphere, which depend only on $p_{(-2)}$ and $\chi_{(-2)}$, according to *Brenner* (10), are zero because the latter vanish. The pressures p^0 and P^0 are constant. To write down the complete solutions $\boldsymbol{v}^{0''}$, $p^{0''}$ and $\boldsymbol{V}^{0''}$, $P^{0''}$ it is only necessary to add the two partial solutions, as in [3].

4. Drop Deformation

If the interfacial tension γ between the drop and suspending fluid is finite, deformation by the shear field will occur until the discontinuity in π_{rr} at the interface is balanced by the pressure distribution arising from the non-uniform curvature according to the formula of *Laplace*. It has just been shown that $\pi_{rr}^0 = \Pi_{rr}^0$; accordingly the deformation of the drop depends only on the discontinuity between $\pi_{rr}^{0'}$ and $\Pi_{rr}^{0'}$. The result of a velocity field of the form of $\boldsymbol{v}^{0'}$ has been shown by *Taylor* (4) to be a deformation of the drop to a shape expressed by

$$F \equiv r - b(1 + 2Db^{-2}xy) = 0 ,$$
$$D = G b \eta \gamma^{-1} (19\lambda + 16)/(16\lambda + 16) \ll 1 . \qquad [8]$$

D, the deformation, is dimensionless. The longest and shortest axes of the ellipsoid [8] lie in the xy plane making an angle of $\frac{1}{4}\pi$ with the x- and y-axes, the drop being extended in the first and third quadrants and compressed in the second and fourth; the third axis of the drop, lying along the z-axis, equals the radius b of the original sphere.

This result can be generalized to an arbitrary flow field $\boldsymbol{u}^{(\infty)}$ by expanding the original flow in a *Taylor* series about the drop center C:

$$\boldsymbol{u}^{(\infty)} = [\boldsymbol{u}^{(\infty)}]_C + [\boldsymbol{\nabla}\boldsymbol{u}^{(\infty)}]_C \cdot \boldsymbol{r} + \frac{1}{2}\boldsymbol{r}\boldsymbol{r} : [\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{u}^{(\infty)}]_C + \cdots ,$$

where bracketing with subscript C denotes evaluation at C. The $p_{(n)}$, $\Phi_{(n)}$, $\chi_{(n)}$ required in [5] to give $\boldsymbol{u}^0$ satisfying the boundary conditions on a fluid sphere and reducing to the sum of the first two terms of this series are

$$\chi_{(1)}^{(\infty)} = \zeta_c^{(\infty)} \cdot \boldsymbol{r} , \qquad \Phi_{(2)}^{(\infty)} = \frac{1}{2} S_c^{(\infty)} : \boldsymbol{r}\boldsymbol{r} ,$$

$$p_{(-3)}^0 = -\eta b^3 r^{-5} S_c^{(\infty)} : \boldsymbol{r}\boldsymbol{r} (5\lambda - 2)/(\lambda + 1) ,$$

$$\Phi_{(-3)}^0 = -\frac{1}{2} b^5 r^{-5} S_c^{(\infty)} : \boldsymbol{r}\boldsymbol{r} \lambda/(\lambda + 1) ,$$

and

$$p_{(2)} = \frac{21}{2}\lambda\,\eta\, b^{-2}\, \boldsymbol{S}_c^{(\infty)} : \boldsymbol{r}\,\boldsymbol{r}/(\lambda+1)\,,$$

$$\Phi^0_{(2)} = -\frac{3}{4}\,\boldsymbol{S}_c^{(\infty)} : \boldsymbol{r}\,\boldsymbol{r}/(\lambda+1)$$

with $\chi^{(\infty)}_{(1)}$ give the corresponding field $\boldsymbol{U}^0$ inside. Here $\xi_c^{(\infty)}$ and $\boldsymbol{S}_c^{(\infty)}$ are

$$\xi_c^{(\infty)} = \frac{1}{2}[\nabla \times \boldsymbol{u}^{(\infty)}]_c\,,\; \boldsymbol{S}_c^{(\infty)} = \frac{1}{2}[\nabla \boldsymbol{u}^{(\infty)} + (\nabla \boldsymbol{u}^{(\infty)})^{\dagger}]_c\,.$$

The drop deforms to a shape given by

$$r = b\left(1 + \frac{2\,b\,\eta}{\gamma}\cdot\frac{19\,\lambda+16}{16\,\lambda+16}\cdot\frac{\boldsymbol{S}_c^{(\infty)} : \boldsymbol{r}\,\boldsymbol{r}}{b^2}\right)$$

of which [8] is a special case. Since the second term in the parenthesis is small compared to 1, its square and higher powers can be neglected; this allows the equation to be transformed, by squaring, to the standard form for an ellipsoid,

$$\frac{1}{b^2}\left(\boldsymbol{I} - \frac{4\,b\,\eta}{\gamma}\cdot\frac{19\,\lambda+16}{16\,\lambda+16}\,\boldsymbol{S}_c^{(\infty)}\right) : \boldsymbol{r}\,\boldsymbol{r} = 1\,, \quad [9]$$

where $\boldsymbol{I}$ is the dyadic idemfactor. To the order of our approximation, it is evident that the drop deforms into an ellipsoid with its principal axes coincident with those of the local fluid rate-of-strain tensor $\boldsymbol{S}_c^{(\infty)}$, their lengths differing from the undistorted drop radius b by small amounts proportional to the principal rates of strain.

Two examples of this of special interest are drop deformation in hyperbolic-radial flow and in plane-hyperbolic flow, these flows being important in connection with extrusion through a nozzle and a slot respectively. The effect of these flows on suspended rigid particles is to orient them, as previously described (11); what happens when the particles are liquid will now be discussed.

Hyperbolic-radial flow is defined by

$$\boldsymbol{u} = \boldsymbol{i}\,G\,x - \frac{1}{2}\boldsymbol{j}\,G\,y - \frac{1}{2}\boldsymbol{k}\,G\,z\,. \quad [10]$$

When $G > 0$, [10] represents flow outwards along the x-axis and radial inflow in the yz-plane; when $G < 0$ the flow is reversed. If the principal semiaxes of the deformed drop are b_x, b_y and b_z, directed along the x-, y-, and z-axes, [9] gives, to order $G\,b\,\eta/\gamma$,

$$b_x = b\,(1+2\,D)\,,\; b_y = b\,(1-D)\,,\; b_z = b\,(1-D)\,, \quad [11]$$

where D is given in [8]. The drop is a spheroid, prolate or oblate according as $D > 0$ or $D < 0$, this ultimately depending on the sign of G in [10].

A particular case of [9] which is especially convenient for experimental verification is drop deformation in plane-hyperbolic flow given by

$$\boldsymbol{u} = \frac{1}{2}\boldsymbol{i}\,G\,x - \frac{1}{2}\boldsymbol{j}\,G\,y\,. \quad [12]$$

In this case, the drop deforms into an ellipsoid with

$$b_x = b\,(1+D)\,,\quad b_y = b\,(1-D)\,,\quad b_z = b\,. \quad [13]$$

Let D_{xy} be the apparent deformation in the xy-plane defined by

$$D_{xy} = (b_x - b_y)/(b_x + b_y)\,. \quad [14]$$

This is not the xy-component of a tensor. For flows [10] and [12], $D_{xy} = \frac{3}{2}D$ and D, respectively. For the latter, a plot of b_x/b, b_y/b and b_z/b against D_{xy} should yield straight lines with slopes 1, -1 and 0, according to [13]. On the other hand, the theory of deformation in hyperbolic flow might not be obeyed, the drop deforming into a prolate spheroid with $b_x = b_y$. This was inferred by *Rumscheidt* and *Mason* (6) from their observations, which were limited by the experimental arrangement to views along the z-axis. In this case, if the volume of the drop is unchanged, the semiaxes will be given by [11], if $\frac{2}{3}D_{xy}$ is substituted for D. The slopes of plots of b_x/b, b_y/b and b_z/b against D_{xy} would thus be $\frac{4}{3}$, $-\frac{2}{3}$ and $-\frac{2}{3}$. Since no experiments in which all three axes of the drop were measured have been reported, we have made these observations and the results are described in the experimental section.

5. Boundary Condition Equations

The velocity fields outside and inside a drop, deformed in this way, suspended between counter-rotating discs, will now be calculated. We shall consider fields $\boldsymbol{v}''$, p'' and $\boldsymbol{V}''$, P'' that satisfy the boundary conditions enumerated earlier, where the drop surface S is not spherical but given by [8]. Separating the problem into parts as before, we observe that the fields $\boldsymbol{v}'$ and $\boldsymbol{V}'$ are those for a deformed drop suspended in the uniform shear flow $\boldsymbol{v}^{(\infty)\prime} = \boldsymbol{i}\,k\,\omega\,y$; because of the symmetry of this configuration, no force can act in the x-, y- or z-direction. Accordingly the fields $\boldsymbol{v}'$, $\boldsymbol{V}'$ are of no interest and will not be considered further. On the other hand, the fields $\boldsymbol{v}$, p and $\boldsymbol{V}$, P must be calculated explicitly. The problem is simplified because D is so small that its square and higher powers are negligible; thus on the surface S the leading terms of the binomial theorem expansion of [8] yield

$$r^n = b^n (1 + 2\,n\,D \sin\theta \cos\theta \sin\Phi)\,. \qquad [15]$$

Since $\boldsymbol{v}$ and $\boldsymbol{V}$ reduce to $\boldsymbol{v}^0$ and $\boldsymbol{V}^0$ for a spherical drop, they can be written

$$\boldsymbol{v} = \boldsymbol{v}^0 + D\boldsymbol{v}^1 + 0\,(D^2)\,, \quad \boldsymbol{V} = \boldsymbol{V}^0 + D\boldsymbol{V}^1 + 0\,(D^2)\,.$$

In the evaluation of $\boldsymbol{v}$ and $\boldsymbol{V}$ on S, since $[\boldsymbol{v}^1]_S = [\boldsymbol{v}^1]_b + 0\,(D)$, we find

$$[\boldsymbol{v}]_S = [\boldsymbol{v}^0]_S + D[\boldsymbol{v}^1]_b + 0\,(D^2)\,,$$
$$[\boldsymbol{V}]_S = [\boldsymbol{V}^0]_S + D[\boldsymbol{V}^1]_b + 0\,(D^2)\,, \qquad [16]$$

where bracketing with subscript S or b denotes evaluation on S with [8] or on the sphere $r = b$. That the unknown fields $\boldsymbol{v}^1$, $\boldsymbol{V}^1$ are required to satisfy boundary conditions on a sphere and not S simplifies the problem considerably.

The first requirement is that the tangential velocity is continuous at S. Since both outer and inner normal velocities vanish at S, the entire velocity vector will be continuous there; this condition is easily formulated from [16]:

$$[\boldsymbol{v} - \boldsymbol{V}]_S = [\boldsymbol{v}^0 - \boldsymbol{V}^0]_S + D[\boldsymbol{v}^1 + \boldsymbol{V}^1]_b = 0\,.$$

Substitution of [6] into this gives

$$[\boldsymbol{v} - \boldsymbol{V}]_S = \boldsymbol{i}_\Phi [k\,\{(1 - b^5 r^{-5} C_{-3}) - C_2\}\, r^2 \sin\theta\cos\theta]_S + D\,[\boldsymbol{v}^1 - \boldsymbol{V}^1]_b = 0\,.$$

Using [15], we obtain from this after collecting terms and substituting for C_2:

$$[\boldsymbol{v}^1 - \boldsymbol{V}^1]_b = -\frac{5}{4}\,\boldsymbol{i}_\Phi\, k\, b^2\, C_{-3}\,(1 - \cos 4\,\theta)\sin\Phi\,. \qquad [17]$$

Vanishing of both normal velocities will now be assured by making one of them zero; thus we shall require $[\boldsymbol{v}\cdot\boldsymbol{n}]_S = 0$, where $\boldsymbol{n}$ is a unit vector normal to S. Since the vector ∇F is normal to the surface $F = 0$, as shown by *Gibbs* and *Wilson* (12), we form ∇F using F given by [8]:

$$\nabla F = \boldsymbol{i}_r - 2\,b^{-1} D\,\{\boldsymbol{i}_r\,(2\,r\sin\theta\cos\theta\sin\Phi) + \boldsymbol{i}_\theta\, r\,(\cos^2\theta - \sin^2\theta)\sin\Phi + \boldsymbol{i}_\Phi\, r\cos\theta\,\cos\Phi\}\,.$$

The unit vector in the same direction is found by dividing this by its magnitude. To first order in D we have

$$\boldsymbol{n} = \nabla F/(\nabla F\cdot\nabla F)^{1/2} = \boldsymbol{i}_r - 2\,D\,(\boldsymbol{i}_\theta\cos 2\,\theta\sin\Phi + \boldsymbol{i}_\Phi\cos\theta\cos\Phi)\,, \qquad [18]$$

$$[\boldsymbol{V}\cdot\boldsymbol{n}]_S = [\boldsymbol{V}^0\cdot\boldsymbol{i}_r]_S - 2\,D\,[\boldsymbol{V}^0\cdot(\boldsymbol{i}_\theta\cos 2\,\theta\sin\Phi + \boldsymbol{i}_\Phi\cos\theta\cos\Phi)]_b + D\,[\boldsymbol{V}^1\cdot\boldsymbol{i}_r]_b = 0\,.$$

After substitution of [6] and [15] this yields the boundary condition

$$[\boldsymbol{V}^1\cdot\boldsymbol{i}_r]_b = 2\,k\,b^2\,C_2\sin\theta\cos^2\theta\cos\Phi\,. \qquad [19]$$

To express the third condition to be satisfied at S, that the tangential stress is continuous, we must find the normal-stress vector $\boldsymbol{\pi}\cdot\boldsymbol{n}$ and take its tangential component. This is accomplished by operating on $\boldsymbol{\pi}\cdot\boldsymbol{n}$ with the dyadic $\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}$: as described by *Gibbs* and *Wilson* (12), $\boldsymbol{I} - \boldsymbol{n}\,\boldsymbol{n}$ annihilates all vectors parallel to $\boldsymbol{n}$, leaving those perpendicular to $\boldsymbol{n}$ unchanged; thus $(\boldsymbol{I} - \boldsymbol{n}\,\boldsymbol{n})\cdot\boldsymbol{\pi}\cdot\boldsymbol{n}$ is $\boldsymbol{\pi}\cdot\boldsymbol{n}$ with its normal component removed, that is, the required tangential stress vector. $\boldsymbol{I} - \boldsymbol{n}\,\boldsymbol{n}$ is a generalization of the dyadic $\boldsymbol{I} - \boldsymbol{i}_r\,\boldsymbol{i}_r$ used by *Brenner* in the Appendix of (10) to find the tangential stress on a sphere. The boundary condition of continuity of tangential stress at S then becomes

$$[(\boldsymbol{I} - \boldsymbol{n}\,\boldsymbol{n})\cdot(\boldsymbol{\pi} - \boldsymbol{\Pi})\cdot\boldsymbol{n}]_S = [(\boldsymbol{I} - \boldsymbol{n}\,\boldsymbol{n})\cdot(\boldsymbol{\pi}^0 - \boldsymbol{\Pi}^0)\cdot\boldsymbol{n}]_S + D\,[(\boldsymbol{I} - \boldsymbol{i}_r\,\boldsymbol{i}_r)\cdot(\boldsymbol{\pi}^1 - \boldsymbol{\Pi}^1)\cdot\boldsymbol{i}_r]_b + 0\,(D^2) = 0\,,$$

from which we find

$$D\,[\boldsymbol{i}_\theta\,(\pi^1_{r\theta} - \Pi^1_{r\theta}) + \boldsymbol{i}_\Phi\,(\pi^1_{r\Phi} - \Pi^1_{r\Phi})]_b = [\boldsymbol{n}\,\boldsymbol{n}\cdot(\boldsymbol{\pi}^0 - \boldsymbol{\Pi}^0)\cdot\boldsymbol{n}]_S - [(\boldsymbol{\pi}^0 - \boldsymbol{\Pi}^0)\cdot\boldsymbol{n}]\,. \qquad [20]$$

To obtain the stress tensors $\boldsymbol{\pi}^0$ and $\boldsymbol{\Pi}^0$, we apply equations for finding $\boldsymbol{\pi}$ from $\boldsymbol{v}$, p in spherical polar coordinates, given, for example, by *Milne-Thomson* (13), using [6] and noting that the pressures p^0, P^0 are constant. The $r\,r$-, $\theta\,\theta$-, $\Phi\,\Phi$- and $r\,\theta$-components vanish, leaving

$$\pi^0_{\theta\Phi} = -\,\eta\,k\,(1 - C_{-3}\,b^5 r^{-5})\,r\sin^2\theta\,,$$
$$\pi^0_{r\Phi} = \eta\,k\,(1 + 4\,C_{-3}\,b^5 r^{-5})\,r\sin\theta\cos\theta\,,$$
$$\Pi^0_{\theta\Phi} = -\,\lambda\,\eta\,k\,C_2\,r\sin^2\theta\,,$$
$$\Pi^0_{\theta\Phi} = \lambda\,\eta\,k\,C_2\,r\sin\theta\cos\theta\,.$$

Then, from [18] we get

$$(\boldsymbol{\pi}^0 - \boldsymbol{\Pi}^0)\cdot\boldsymbol{n} = \boldsymbol{i}_\Phi\,(\pi^0_{r\Phi} - \Pi^0_{r\Phi}) - 2\,\boldsymbol{i}_\Phi D\,(\pi^0_{\theta\Phi} - \Pi^0_{\theta\Phi})\cos 2\,\theta\sin\Phi - 2\,\boldsymbol{i}_\theta D\,(\pi^0_{\theta\Phi} - \Pi^0_{\theta\Phi})\cos\theta\cos\Phi - 2\,\boldsymbol{i}_r D\,(\pi^0_{r\Phi} - \Pi^0_{r\Phi})\cos\theta\cos\Phi\,,$$

and

$$\boldsymbol{n}\,\boldsymbol{n}\cdot(\boldsymbol{\pi}^0 - \boldsymbol{\Pi}^0)\cdot\boldsymbol{n} = -\,2\,\boldsymbol{i}_r D\,(\pi^0_{r\theta} - \Pi^0_{r\theta})\cos\theta\cos\Phi\,.$$

Substitution of these expressions into [20] yields the last boundary condition:

$$[\boldsymbol{\pi}^1_r - \boldsymbol{\Pi}^1_r]_b = -\,10\,\boldsymbol{i}_\Phi\,\eta\,k\,b\,C_{-3}\sin^2\theta\cos\theta\cos\Phi + 10\,\boldsymbol{i}_\Phi\,\eta\,k\,b\,C_{-3}\sin^2\theta\,(6\cos^2\theta - 1)\sin\Phi\,. \qquad [21]$$

6. Harmonics in Lamb's Solution

We now must find $\boldsymbol{v}^1$, p^1 and $\boldsymbol{V}^1$, P^1 to satisfy [17], [19] and [21]. Again using [5], we select the $p_{(n)}$, $\Phi_{(n)}$, $\chi_{(n)}$ for $\boldsymbol{v}^1$, p^1 by assuming, subject to *a posteriori* verification, that they will be the same functions of r, θ, Φ as the $p_{(n)}$, $\Phi_{(n)}$, $\chi_{(n)}$ that appear in the solution of the problem of an infinitely viscous deformed drop, $\lambda = \infty$. For this limiting case the boundary conditions are the same as if the drop was a rigid body, being completely specified by the limiting form of [19]. Since when $\lambda = \infty$ there is no internal motion, we obtain this form by setting $\boldsymbol{V} = 0$ and inserting the limiting value $\dot{C}_{-3} = 1$ into [19]:

$$[\boldsymbol{v}^1]_b = -10\, \boldsymbol{i}_\Phi\, k\, b^2 \sin^2\theta \cos^2\theta \sin\Phi\,. \qquad [22]$$

A boundary-value problem expressed in this way can be solved using a general method given by *Brenner* in his investigation of the *Stokes* resistance of a slightly deformed sphere (10). Thus, the $p_{(n)}$, $\Phi_{(n)}$ and $\chi_{(n)}$ that must be inserted in [5] to yield $\boldsymbol{v}^1$ are found to be:

$$p^1_{(-4)} = \frac{5}{3} A^{(1)}_{-4}\, \eta\, k\, b^5 r^{-4} P^{(1)}_3 \cos\Phi\,,$$

$$p^1_{(-2)} = A^{(1)}_{-2}\, \eta\, k\, b^3 r^{-2} P^{(1)}_1 \cos\Phi\,,$$

$$\Phi^1_{(-4)} = \frac{1}{6} B^{(1)}_{-4}\, k\, b^7 r^{-4} P^{(1)}_3 \cos\Phi\,,$$

$$\Phi^1_{(-2)} = \frac{1}{2} B^{(1)}_{-2}\, k\, b^5 r^{-2} P^{(1)}_1 \cos\Phi\,,$$

$$\chi^1_{(-5)} = -\frac{1}{7} C^{(1)}_{-5}\, k\, b^7 r^{-5} P^{(1)}_4 \sin\Phi\,,$$

$$\chi^1_{(-3)} = -\frac{5}{63} C^{(1)}_{-3}\, k\, b^5 r^{-3} P^{(1)}_2 \sin\Phi\,. \qquad [23]$$

The coefficients $A^{(m)}_n$, $B^{(m)}_n$ and $C^{(m)}_n$, introduced for generality, are equal to unity when $\lambda = \infty$. By inserting $p^1_{(n)}$, $\Phi^1_{(n)}$, $\chi^1_{(n)}$ in [5] and setting $r = b$, one can verify that these yield $\boldsymbol{v}^1$ satisfying [22]. To obtain the complete solution $\boldsymbol{v}$, [5] is used with the $p^1_{(n)}$, $\Phi^1_{(n)}$, $\chi^1_{(n)}$ listed above multiplied by D, together with $\chi^{(\infty)}_{(2)}$ and $\chi^0_{(-3)}$.

To solve the problem when λ is finite, we shall use the $p^1_{(n)}$, $\Phi^1_{(n)}$, $\chi^1_{(n)}$ given by [23] to find $\boldsymbol{v}^1$, p^1, but the coefficients $A^{(m)}_n$, $B^{(m)}_n$, $C^{(m)}_n$ are at present unknown. For $\boldsymbol{V}^1$, P^1, we select $p_{(n)}$, $\Phi_{(n)}$, $\chi_{(n)}$ analogous to those of [23], but regular at the origin, and also with coefficients to be determined, namely,

$$p^1_{(3)} = \frac{5}{3} A^{(1)}_3\, \lambda\, \eta\, k\, b^{-2} r^3 P^{(1)}_3 \cos\Phi\,,$$

$$p^1_{(1)} = A^{(1)}_1\, \lambda\, \eta\, k\, r\, P^{(1)}_1 \cos\Phi\,,$$

$$\Phi^1_{(3)} = \frac{1}{6} B^{(1)}_3\, k\, r^3 P^{(1)}_3 \cos\Phi\,,$$

$$\Phi^1_{(1)} = \frac{1}{2} B^{(1)}_1\, k\, b^2 r\, P^{(1)}_1 \cos\Phi\,,$$

$$\chi^1_{(4)} = -\frac{1}{7} C^{(1)}_4\, k\, b^{-2} r^4 P^{(1)}_4 \sin\Phi\,,$$

$$\chi^1_{(2)} = -\frac{5}{63} C^1_{(2)}\, k\, r^2 P^{(1)}_2 \sin\Phi\,. \qquad [24]$$

When $\lambda = \infty$, $B^{(1)}_3$, $B^{(1)}_1$, $C^{(1)}_4$ and $C^{(1)}_2$ vanish, but the behavior of $A^{(1)}_3$ and $A^{(1)}_1$ cannot be predicted. We then insert [23] and [24] into [5] and [7] using spherical polar coordinates, to evaluate the left-hand sides of [17], [19] and [21]. As an example, we give the result for $[v^1_\Phi - V^1_\Phi]_b$:

$$\begin{aligned}
&[v^1_\Phi - V^1_\Phi]_b \\
&= k\, D\, b^2 \Big\{ \Big(\frac{1}{24} A^{(1)}_{-4} - \frac{1}{8} B^{(1)}_{-4} + \frac{5}{48} A^{(1)}_3 + \frac{1}{8} B^{(1)}_3 \Big) \\
&\times (3 + 5\cos 2\theta) + \Big(-\frac{1}{2} A^{(1)}_{-2} - \frac{1}{2} B^{(1)}_{-2} + \frac{1}{5} A^{(1)}_1 + \frac{1}{2} B^{(1)}_1 \Big) \\
&\qquad + \frac{5}{28} \big(C^{(1)}_{-5} - C^{(1)}_4 \big) (\cos 2\theta + 7\cos 4\theta) \\
&\qquad + \frac{5}{21} \big(C^{(1)}_{-3} - C^{(1)}_2 \big) \cos 2\theta \Big\} \sin\Phi\,.
\end{aligned}$$

This is compared with the Φ-component of the right-hand side of [17] and the coefficients of $\sin\Phi$, $\cos 2\theta \sin\Phi$, and $\cos 4\theta \sin\Phi$, which are linearly independent, are equated. In this way twelve independent equations for the $A^{(m)}_n$, $B^{(m)}_n$, $C^{(m)}_n$ are obtained; the solution is:

$$A^{(1)}_{-4} = (7\lambda^2 + 34\lambda + 22)/7(\lambda + 1)(\lambda + 4)\,,$$

$$A^{(1)}_{-2} = (3\lambda^2 + 14\lambda - 2)/3(\lambda + 1)(\lambda + 4)\,,$$

$$B^{(1)}_{-4} = (7\lambda^2 + 20\lambda + 8)/7(\lambda + 1)(\lambda + 4)\,,$$

$$B^{(1)}_{-2} = (3\lambda^2 + 8\lambda - 8)/3(\lambda + 1)(\lambda + 4)\,,$$

$$C^{(1)}_{-5} = (\lambda - 1)/(\lambda + 2)\,,$$

$$C^{(1)}_{-3} = (\lambda - 1)(\lambda - 16)/(\lambda + 4)^2\,,$$

$$A^{(1)}_3 = -12(3\lambda + 4)/7(\lambda + 1)(\lambda + 4)\,,$$

$$A^{(1)}_1 = 10(5\lambda - 8)/3(\lambda + 1)(\lambda + 4)\,,$$

$$B^{(1)}_3 = 4(29\lambda + 34)/21(\lambda + 1)(\lambda + 4)\,,$$

$$B^{(1)}_1 = 2(\lambda + 14)/3(\lambda + 1)(\lambda + 4)\,,$$

$$C^{(1)}_4 = 2(\lambda - 1)/(\lambda + 2)(\lambda + 4)\,,$$

$$C^{(1)}_2 = 20(\lambda - 1)/(\lambda + 4)^2\,. \qquad [25]$$

7. Migration Velocity

Brenner (10) shows that the force and torque acting on a particle can be expressed respectively in terms of the harmonics $p_{(-2)}$ and $\chi_{(-2)}$ appearing in [5], the force being $\boldsymbol{F} = -4\pi \nabla (r^3 p_{(-2)})$. Here the torque vanishes since $\chi_{(-2)} = 0$; however, we find

$$\boldsymbol{F} = -4\pi \boldsymbol{k}\, k\, b^3 D = \boldsymbol{k} \frac{k^2 b^4 \omega \eta^2}{\gamma} \times \frac{\pi (3\lambda^2 + 14\lambda - 2)(19\lambda + 16)}{12(\lambda+1)^2(\lambda+4)}. \quad [26]$$

An independent check of [26] for the limiting case $\lambda = \infty$ was made from *Brenner*'s theory (1) for a rigid particle in an arbitrary flow.

Thus, a stationary drop must be held in place by an external force equal and opposite to the hydrodynamic force $\boldsymbol{F}$, which tends to impel it in the z-direction, towards or away from the axis of revolution of the discs. Conversely, a free drop will migrate along the z-axis; to find its velocity, one should in principle superpose on the system a uniform streaming velocity $-\boldsymbol{w}$, to be determined by making $\boldsymbol{F}$ vanish. Because of the linearity of the boundary conditions and the equations of motion [1], $\boldsymbol{w}$ will be the same as the velocity imparted to a drop by $\boldsymbol{F}$ in a quiescent fluid. Moreover, the difference between the velocity of the deformed drop and that of a spherical drop will be proportional to D^2, because $\boldsymbol{F}$ contains D – indeed, the neglected term in the velocity may even be of order D^3, this being certainly the case for a rigid body whose shape is given by [8] moving along the z-axis, as can be seen from *Brenner*'s equation 4.15 (10) for the translation of a slightly deformed sphere. We are therefore justified in applying the result of *Hadamard* and *Rybczynski*, given by *Lamb* (8), for the velocity of a fluid sphere whose motion is restrained by a hydrodynamic force $\boldsymbol{F}$. This is then used with [26].

$$\boldsymbol{F} = -6\pi b \eta \boldsymbol{w} (3\lambda + 2)(3\lambda + 3),$$
$$\boldsymbol{w} = -\frac{19}{24} \boldsymbol{k}\, k^2 b^3 \omega \eta \gamma^{-1} K(\lambda), \quad [27]$$

where the viscosity-ratio factor $K(\lambda)$ is

$$K(\lambda) = (3\lambda^2 + 14\lambda - 2)(19\lambda + 16)/19(3\lambda + 2) \times (\lambda + 1)(\lambda + 4).$$

Now $\boldsymbol{w}$ is simply $\boldsymbol{k}\, d\omega/dt$. If the deformation of the drop changes with ω according to [8], then the trajectory of the droplet can be found by integration; it is

$$\ln(\omega/\omega_0) = -k^2 b^3 \eta \gamma^{-1} K(\lambda)\, t,$$

where t is the time and ω_0 is the value of ω at $t = 0$.

The direction of migration depends on the algebraic sign of $K(\lambda)$ and on λ; $K(\lambda)$ is plotted in fig. 2. For large λ, inward migration, towards the axis of rotation of the discs, is indicated; however, at $\lambda = 0.139$ K vanishes and the drop should not migrate, and for $0 < \lambda < 0.139$, outward migration is predicted.

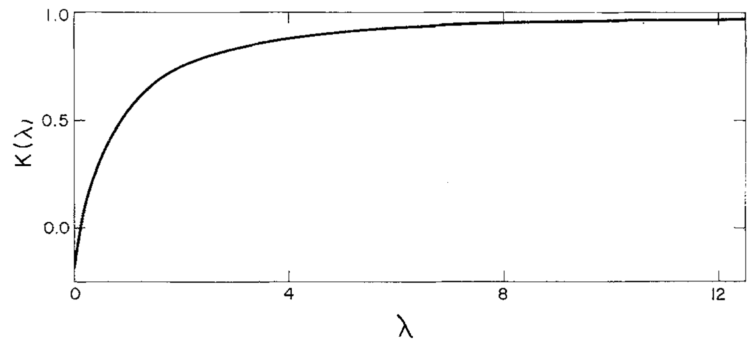

Fig. 2. Calculated viscosity-ratio factor $K(\lambda)$ plotted against λ

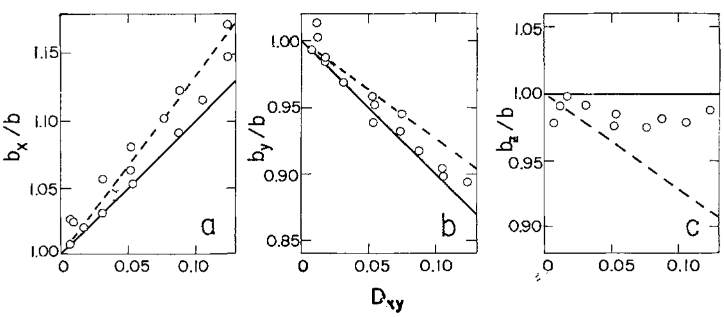

Fig. 3. Plots against D_{xy} of (a) b_x/b, (b) b_y/b, (c) b_z/b. The points are experimental; the full lines calculated from [13] with $D_{xy} = D$; the dashed lines, from [11] with $D_{xy} = 2/3\, D$

8. Experimental

Experiments to measure the three axes of deformed drops in hyperbolic flow were performed as described by *Rumscheidt* and *Mason* (6), with the four-roller apparatus used by them, to produce the flow (12). Photographs were taken viewing along the z- and x- (or y-) axes, windows having been put in the sides of the apparatus to allow this. Drops of silicone fluid (Dow Corning, 200 series, 5000 centistoke grade) or water, suspended in Pale 4 oxidized castor oil (Baker Castor Oil Co., Bayonne, N. J.), and water drops suspended in silicone fluid were used. Measurement of the projected photographs of the drops yielded the three principal axes. The results of all the experiments were similar; we present in detail observations of a silicone drop, $b = 0.128$ cm, suspended in castor oil and subjected to hyberbolic flow [12] with velocity gradients G varying from 0.0088 sec^{-1} to 0.136 sec^{-1}. The room temperature was 24.6 °C. In fig. 3, the observed b_x/b, b_y/b and b_z/b are plotted against D_{xy} obtained from [14]. The full lines are theoretical, calculated from [13] with $D = D_{xy}$; the dashed lines are calculated from [11] with $D = D_{xy}$. The experimental b_x/b and b_y/b plots are insufficiently sensitive to distinguish whether [11] or [13] is obeyed; however, the b_z/b plot shows definitely that b_z/b is nearly equal to unity, in good agreement with the theory. Some observations made of highly deformed drops show that when D is not small $b_x/b > 1 + D_{xy}$, $b_y/b \doteq 1 - D_{xy}$ and $b_z/b < 1$; however, even with drops deformed into threads, $b_z < b_y$.

The experiments on drop migration between counter-rotating discs gave no conclusive results. They were done in the apparatus described by *Anczurowski* and *Mason* (14), which was fitted with two horizontal transparent discs 1 cm apart, of radius 15 cm; the disc surfaces deviated from flatness by 0.001 cm. The space between the discs was filled with a mixture of silicone fluid (Dow Corning, 200 series, 10000 centistoke grade) and tetrachlorodifluoroethane made up to have the same density as the particles used in each experiment. Rigid spheres and water, glycerol and corn syrup drops were used; the rigid particles migrated in an erratic way (presumably because of imperfections in flow) and the drops migrated inwards at all λ at comparable velocities to those of the rigid spheres. Since no conclusive results were being obtained, the experiments were stopped. The principal difficulty was finding a suitable pair of liquids with equal densities. From [25], it is seen that high interfacial tension is desirable so that the velocity gradient $k\omega$ can be increased without the drop deforming excessively. No system was found in which the migration velocity, calculated from [27], is greater than the erratic drift velocities observed with rigid particles.

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Summary

When a liquid drop is suspended in a viscous fluid undergoing shear flow, it deforms; if the deformation is small, the drop becomes an ellipsoid with its principal axes directed along the principal directions of strain of the fluid. In general, the lengths of the axes are all different; this is the case for hyperbolic flow, for which explicit theoretical formulas are given. Experimental observations of all three axes of deformed drops in hyperbolic flow agree with the theory.

The migration of a liquid drop in the non-uniform shear field between counter-rotating discs is calculated by finding in detail velocity fields that satisfy the creeping motion equations. If the drop shape is only slightly different from spherical, it is possible to find the velocity and pressure fields by a perturbation scheme in the small parameter characterizing the deformation, using *Lamb*'s general solution in spherical harmonics (8). The harmonics required in the solution are found by first solving the problem of an infinitely viscous deformed drop, which is the same as a rigid body; the solution for drops of any viscosity is then determined by using the same harmonics but with different coefficients. A force is found to act on a fixed drop along the line joining the drop center to the axis of rotation of the discs. The velocity at which a free drop migrates along this line is then found by using the solution of *Hadamard* and *Rybczinski* for a sedimenting liquid sphere. Experiments in a counter-rotating disc apparatus gave inconclusive results.

Zusammenfassung

Dieser Beitrag enthält eine Theorie des Verhaltens eines Flüssigkeitstropfens, der in einer viskosen Flüssigkeit suspendiert ist, die einer ungleichförmigen Scherströmung unterworfen wird und sich zwischen zwei parallelen Scheiben, die langsam um eine gemeinsame Achse gegeneinander rotieren, befindet. Es wird vorausgesagt, daß der Tropfen auf die Achse zuwandert, wenn seine Viskosität 13,9% der Viskosität der suspendierenden Flüssigkeit überschreitet; andernfalls entfernt er sich von der Achse mit einer Geschwindigkeit proportional seinem Verformungsparameter.

Eine Verallgemeinerung der Theorie der Verformung eines Tropfens zu einem Ellipsoid, die für willkürliche Scherfelder gilt, zeigt, daß seine drei Hauptachsen alle verschieden sein können. Dieses Ergebnis wird bestätigt durch Experimente beim hyperbolischen Fließen.

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