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SUM OF MOMENTS OF CONVEX POLYGONS

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In a plane of constant curvature, let D be a domain and O a point. Let $f(x)$ be a function defined for $x \geqq 0$. The integral

$$M_f(D, O) = \int_D f(OX)\,dX$$

is called the moment of D with respect to O, where dX is the area-element at the variable point X.

The following known theorem ([2], [3] pp. 80—84, 137—141, [5] pp. 219—224, [9], [11]) has various applications in the theory of circle-packing and circle-covering ([3] p. 80, 140, [7], [11]), in the theory of convex polyhedra ([1], [2], [3] pp. 139—140, [4], [5] pp. 308—312, [6]) in the location theory [13], and has points of contact with information theory [12] and biology ([5] p. 233).

THEOREM. *In a plane of constant curvature let O be a point and $P_1, \ldots, P_n$ convex polygons of total number of sides N and total area A. If $f(x)$ is a decreasing function defined for $x > 0$, then*

$$\sum_{i=1}^{n} M_f(P_i, O) \leqq N M_f(\bar{\Delta}, O),$$

where $\bar{\Delta} = \bar{A}O\bar{B}$ is a triangle of area A/N such that $O\bar{A} = O\bar{B}$ and $\sphericalangle AOB = 2\pi n/N$. Equality is attained whenever the P_i's are congruent regular polygons centred at O. If f satisfies the additional condition

$$f(r-0) > f(R+0), \quad r = O\bar{M}, \quad R = O\bar{A},$$

where $\bar{M}$ is the midpoint of the side $\bar{A}\bar{B}$, then equality holds only in the regular case.

In what follows we shall give a simple new proof of this theorem.

We shall restrict ourselves to functions defined by

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 < x \leqq a, \\ 0 & \text{for} \quad x > a, \end{cases}$$

where a is a positive constant. In this case $M_f(D, O)$ is equal to the area of the part of D covered by the circle of radius a centred at O. Having proved the theorem in this case, the general case can easily be settled by approximating a monotonous function by step-functions ([8], [9], [10]).

First we consider the case when

$$r < a < R < \infty.$$

In the remaining cases the theorem is either void, or, as it will be seen later, can easily be proved. To prove the theorem in the above mentioned main case we shall need the following

LEMMA. *Let U and V be two circles of radii a and b $(b>a)$ with the common centre O, respectively. Let $u(x)$ and $v(x)$ be the areas of the segments cut off from U and V by a straight line of distance x $(0\leqq x<a)$ from O. Then $u'(x)/v'(x)$ is a strictly decreasing, positive function of x, less than* 1.

Let L be a point such that $OL=x$. Let one of the halflines, erected perpendicularly to OL in L, cut U and V in M and N. Then $-\frac{1}{2}u'(x)dx$ and $-\frac{1}{2}v'(x)dx$ are the areas swept over by the segment LM and LN while L is moving on the line OL through the infinitesimal distance dx.

Observe that $-\frac{1}{2}u'(x)=\sin LM$ or LM or $\operatorname{sh} LM$ according to the three geometries. In the same way, we have $-\frac{1}{2}v'(x)=\sin LN$ or LN or $\operatorname{sh} LN$. Using these relations, we obtain by a simple computation

$$\frac{u'(x)}{v'(x)}=\begin{cases}\sqrt{\dfrac{\cos^2 x-\cos^2 a}{\cos^2 x-\cos^2 b}}\\[2ex] \sqrt{\dfrac{a^2-x^2}{b^2-x^2}}\\[2ex] \sqrt{\dfrac{\operatorname{ch}^2 a-\operatorname{ch}^2 x}{\operatorname{ch}^2 b-\operatorname{ch}^2 x}}\end{cases}$$

which is, in fact, in all three cases a positive, decreasing function of x, less than 1.

In what follows we shall denote a domain and its area with the same symbol. In agreement with the above notations, let U be a circle of radius a centred at O. Let $\Delta=AOB$ be an arbitrary triangle, S the sector cut out from U by the angular region spanned by Δ at O and, finally, $F=\Delta\cap U$ the part of Δ covered by U. In the proof of the theorem a central part will be played by the quantity

$$w = F-pS-q\Delta,$$

where p and q are fixed real numbers such that $p+q<1$, $0<q<1$. We are looking for the triangle with the maximum value of w.

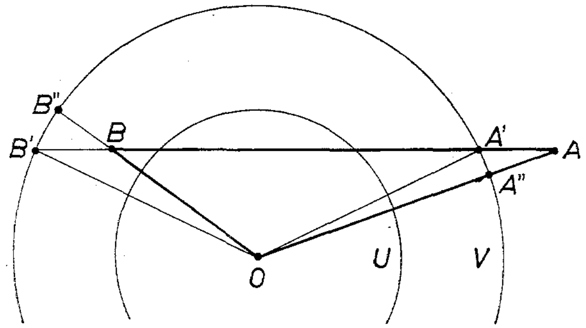
Obviously, we may suppose that non of A and B lies in the interior of U. For, supposing that $OA<a$ and $OB<a$ we obtain a better triangle by replacing A and B by their central projection onto the boundary of U. Assuming now that $OB<a$ and $OA\geqq a$, we obtain a better triangle by replacing B by its central projection B' onto the boundary of U and A by the point A' of the side OA such that AB and $A'B'$ intersect each other on the boundary of U.

Furthermore, if $OB=a$, then we may suppose that $\sphericalangle OBA\leqq 90°$. Otherwise we replace A by the point cut out from OA by the tangent of U at B.

Let V be a circle of radius b concentric to U such that

$$(1-p)U = qV.$$

We claim that if not both A and B lie on the boundary of V, we can construct a triangle with a greater value of w with vertices A and B such that $OA=OB=b$. Let A', B', A'', B'' be the points cut out from the boundary of V by the line AB and the half-lines OA, OB, choosing the notations so that the cyclic order of the points on V should be of type $AABB$ (see the figure). Comparing the triangles



AOB and $A'OB'$, we see that w is greater for the latter, the increment being q times the total area of the "triangles" $AA'A''$ and $BB'B''$, where the sides $A'A''$ and $B'B''$ are the corresponding circular arcs of V.

For a triangle AOB with $OA=OB=b$ the value of w depends only on the distance x of AB from O. For this triangle we write $w=w(x)$. Let T be the intersection of V and the angular region AOB. Using the notations of the lemma and extending the definition of $u(x)$ for $x\geqq a$ by $u(x)=0$, we have

$$w(x) = S-u(x)-pS-q(T-v(x)) = (1-p)S-qT+qv(x)-u(x) = qv(x)-u(x).$$

This shows that $w(x)$ attains its maximum in the closed interval $[0, a]$.

In view of the lemma the equation $w'(x)=0$ has at most one root in $[0, a]$ and if $w'(x_0)=0$, then $w(x)$ has a maximum at x_0. In this case we have for any triangle AOB $w\leqq w(x_0)$ with equality only for an isosceles triangle with $OA=OB=b$ and altitude x_0.

Keeping in mind the assumption that $r<a<R<\infty$, we choose p and q so that the radius b of the circle V defined by $(1-p)U = qV$ should be equal to R. Furthermore, we choose q so that $q=\frac{u'(r)}{v'(r)}$. In view of the lemma we have $0<q<1$. On the other hand, we have, by $a<R<\infty$, $p+q<1$. Thus, with this choise of p and q, we have for an arbitrary triangle

$$w = F-pS-q\Delta \leqq \bar{F}-p\bar{S}-q\bar{\Delta} = \bar{w},$$

where the bars refer to the respective values in $\bar{\Delta}$.

After these preliminaries the proof of the theorem will be very easy. We obviously may suppose that $O\in P_i$ $(i=1, \ldots, n)$. Let $\Delta_1, \ldots, \Delta_N$ be the triangles based on the sides of the polygons $P_1, \ldots, P_n$ and having O as common apex. (Some of these triangles may degenerate into segments.) Denoting the values of F, S and w in

Δ_i by F_i, S_i and w_i, we have

$$\sum_{i=1}^{N} F_i = p\sum_{i=1}^{N} S_i + q\sum_{i=1}^{N} \Delta_i + \sum_{i=1}^{N} (F_i - pS_i - q\Delta_i) =$$

$$= pN\bar{S} + qN\bar{\Delta} + \sum_{i=1}^{N} w_i \leqq pN\bar{S} + qN\bar{\Delta} + N\bar{w} = N\bar{F}.$$

Equality holds only if all Δ_i's are congruent to $\bar{\Delta}$.

This completes the proof of the theorem in the case when $r<a<R<\infty$.

If $a<r$ or $a>R$ the statement of the theorem is void. Furthermore, it is obvious that the inequality $\sum_{i=1}^{N} F_i \leqq N\bar{F}$ holds also if $a=r$ or $a=R$. Now equality holds only if each P contains the circle of radius r with centre O, and is contained in the circle of radius R centred at O, respectively. In the first case we have

$$A \geqq 2Na\left(\frac{\pi n}{N}\right) = A,$$

where A is, in accordance with the notations of the theorem, the total area of the polygons, and $a(\varphi)$ is the area of a triangle OMA such that $\sphericalangle OMA = \pi/2$, $\sphericalangle MOA = \varphi$ and $OM = r$. This follows immediately [7] from the convexity of $a(\varphi)$ and Jensen's inequality. Equality holds only if all of the P_i's are congruent regular polygons. Since, on the other hand, equality must hold, the case of equality is settled if $a=r$. The case of equality when $a=R$ can be settled in a similar way [7] referring to the concavity of the function $b(\varphi)$, where $b(\varphi)$ denotes the area of a triangle OMA such that $\sphericalangle OMA = \pi/2$, $\sphericalangle MOA = \varphi$ and $OA = R$.

The only case left is the case when $R=\infty$. Now, necessarily, all P_i's are completely asymptotic polygons.

Let $\Delta = AOB$ be a triangle such that $OA = OB = \infty$ and $\sphericalangle AOB = 2\varphi$. Using the above notations, we have

$$F = S - u(x),$$

where the relation between x and φ is given by $\operatorname{ch} x \sin\varphi = 1$. We claim that F is a concave function of φ. Since S is a linear function of φ, all we need to show is that $u(x(\varphi))$ is a convex function of φ $(0<\varphi<\pi)$. Let the value of φ belonging to $x=a$ be $\varphi_0 = \arcsin\frac{1}{\operatorname{ch} a}$. Obviously, $u(x(\varphi))$ is a continuous function equal to 0 for $0<\varphi\leqq\varphi_0$ and positive for $\varphi_0<\varphi<\pi/2$. On the other hand, we have for $\varphi_0<\varphi<\pi/2$, by a simple computation,

$$\frac{du}{d\varphi} = \sqrt{\operatorname{ch}^2 a - \frac{1}{\sin^2\varphi}}.$$

This being an increasing function of φ, u is convex in the whole interval $(0, \pi/2)$. In view of the concavity of $F = F(\varphi)$, the validity of the theorem in the case when $R=\infty$ follows immediately by Jensen's inequality.

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