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ON THE STABILITY OF THE ONE-STEP EXACT COLLOCATION METHODS FOR THE NUMERICAL SOLUTION OF THE SECOND KIND VOLTERRA INTEGRAL EQUATION

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Abstract

The purpose of this paper is to analyze the stability properties of one-step collocation methods for the second kind Volterra integral equation through application to the basic test and the convolution test equation.
Stability regions are determined when the collocation parameters are symmetric and when they are zeros of ultraspherical polynomials.

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1. Introduction.

The purpose of this paper is to investigate the stability properties of the one-step collocation methods for the second kind Volterra integral equation:

$$y(t) = \int_{t_0}^{t} k(t, s, y(s))\, ds + f(t); \quad t \in [t_0, t]. \tag{1.1}$$

Collocation methods for (1.1) have been discussed by numerous authors (see, for example, [4], [5] and [6]); however, the stability analysis is still very poorly developed.

For the sake of completeness, we recall briefly the basic idea of the one-step collocation method.

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Let

$$\Pi_N = t_0 < t_1 \ldots < t_N = T$$

be a partition of the integration range and let us consider the subintervals:

$$\sigma_0 = [t_0, t_1]; \quad \sigma_i = (t_i, t_{i+1}]; \quad i = 0, \ldots, N-1$$

of size h.

Let π_m be the space of polynomials of degree at most m and let

$$S_m^{(d)} = \{u: u(t)|_{t \in \sigma_i} := u_i(t) \in \pi_m, \quad i = 0, \ldots, N-1;$$
$$u_{i-1}^{(j)}(t_1) = u_i^{(j)}(t_1), \quad j = 0, \ldots, d\}$$

be the space of polynomial splines of degree $m \geq 0$ and continuity class $d \geq -1$.

Define the set of collocation points:

$$X_i = \{\xi_{ij} = t_i + c_j h; \quad 0 \leq c_1 < c_2 \ldots < c_m \leq 1\}$$

where c_j are the collocation parameters.

The collocation method approximates $y(t)$, the true solution of (1.1), by a function $u(t) \in S_{m-1}^{(d)}$, $d \geq -1$, such that:

$$u_i(\xi_{i,j}) = \int_{t_i}^{\xi_{i,j}} k(\xi_{i,j}, s, u_i(s))\, ds + \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} k(\xi_{i,j}, s, u_k(s))\, ds + f(\xi_{i,j}). \tag{1.2}$$

If the integrals in (1.2) are calculated analytically the collocation method is called *exact*, if they are calculated with a quadrature formula the collocation method is called *discrete*.

Moreover if $c_1 = 0$, $c_m = 1$, $u(t)$ is continuous in $[0, T]$.

This paper concerns the stability analysis of the exact collocation method with $c_1 = 0$, $c_m = 1$, following the approach which uses particular test equations.

Among the most frequently used we quote the basic test equation [3], [8], the convolution test equation [7], [14], [19], [20], and the equations with degenerate kernel [8], [11].

In this paper we restrict our attention to the first two above mentioned test equations.

In Section 2 we consider the basic test equation and we prove that, if the collocation parameters are the Lobatto abscissas, the related method is A-stable, and, if the collocation parameters are symmetric, the method is both A_0-stable and I-stable.

Then we consider the convolution test equation; in Section 3 we derive a recurrence relation for a vector containing the values of the numerical solution and we give stability conditions.

In Section 4, using this recurrence relation, we prove that, if the collocation parameters are symmetric, the method has stability regions in the plane $(h\lambda, h^2\mu)$ which are infinite along the direction of the λ-axis but are bounded along the direction of the μ-axis. Moreover, if the collocation parameters are 0,1 and the zeros of the ultraspherical polynomial $P_{m-2}^{(\alpha,\alpha)}(t)$, we give a lower bound, depending on α and m, for the size of the boundary of the stability region along the vertical axis.

In Section 5 we carry out numerical experiments in order to test the sharpness of this bound. Moreover we calculate numerically some stability regions in the plane $(h\lambda, h^2\mu)$ and we present their plots. From the analysis of the theoretical and numerical results, we deduce a conjecture about the stability regions.

2. The basic test equation.

In this section we study the stability of the method applied to the basic test equation:

$$y(t) = y_0 + \lambda \int_0^t y(s)\,ds; \quad t \in [0, T], \quad \operatorname{Re}\lambda \le 0. \tag{2.1}$$

As is known, for this integral equation the exact collocation method is equivalent to the discretized collocation method using an m points interpolatory quadrature formula, based on $c_1, \ldots, c_m$.

Moreover this is, in turn, equivalent to an extended m stage implicit Pouzet Runge-Kutta method, whose coefficients are:

$$\begin{array}{c|c} c_1 & a_{11} \cdots a_{1m} \\ \vdots & \vdots \quad \cdots \vdots \\ c_m & a_{m1} \cdots a_{mm} \\ \hline & a_{m1} \cdots a_{mm} \end{array}$$

where

$$a_{ji} = [V^{*\prime}(c_i)]^{-1} \int_0^{c_j} [V^*(t)/(t - c_i)]\,dt; \qquad V^*(t) = \textstyle\prod_{j=1}^m (t - c_j).$$

From the known result on the Runge-Kutta method it follows:

THEOREM 2.1: *If c_i are the Lobatto abscissas, then the related collocation method is A-stable.*

Let $R(z)$ be the stability function of the method under consideration. We recall that a method is said I-stable when its stability region contains the imaginary axis, that is when $|R(iy)| \leqslant 1, \quad \forall y \in R$. The following theorems hold:

THEOREM 2.2: *A collocation method with symmetric collocation parameters is I-stable.*

PROOF. In this case the method is its own reflection [9, p. 245] and therefore $|R(iy)| = 1, \forall y \in R$. ■

THEOREM 2.3: *A collocation method with symmetric collocation parameter is A_0-stable.*

PROOF. It can easily be proved that $R(z)$ can be written as [17]:

$$R(z) = N(z)/D(z) \tag{2.2}$$

where $z = (t_{i+1} - t_i)\lambda$ and

$$N(z) = \sum_{k=0}^{m-1} V^{*(k+1)}(1) z^{m-1-k} \tag{2.3}$$

$$D(z) = \sum_{k=0}^{m-1} V^{*(k+1)}(0) z^{m-1-k}. \tag{2.4}$$

On the other hand, if $V^*(t)$ is symmetric in [0, 1], then $V^{*(k)}(0) = (-1)^{m-k} V^{*(k)}(1)$. Let us put:

$$\alpha(z) = \sum_{s=0}^{\lfloor m/2 \rfloor} V^{*(2s)}(1) z^{m-2s}$$

$$\beta(z) = \sum_{s=0}^{\lfloor m/2 \rfloor} V^{*(2s+1)}(1) z^{m-1-2s},$$

then, if z is a real negative number, we find $\alpha(z) > 0$, $\beta(z) > 0$.

Moreover, since

$$N(z) = \alpha(z) + \beta(z)$$

$$D(z) = \alpha(z) - \beta(z)$$

we find

$$|N(z)/D(z)| < 1.$$

Therefore the method is A_0-stable. ■

3. The convolution test equation.

In this section the convolution test equation

$$y(t) = y_0 + \int_0^t [\lambda + \mu(t-s)] y(s)\, ds: \quad \lambda < 0, \quad \mu \leq 0 \tag{3.1}$$

is used for the stability analysis of the collocation method. The method is stable when the numerical solution has the same behavior as the true solution of (3.1), which is a linear combination of exponential functions with negative argument.

In order to determine the behaviour of the numerical solution, we construct a recursive relation for the collocation method applied to (3.1).

First of all, we observe that the collocation method furnishes in each interval σ_i a polynomial $u_i(t)$ of degree $m-1$, satisfying the equation (3.1) in the collocation points, that is solving the equation:

$$(3.2)\qquad u_i(t) = y_0 + \int_{t_i}^{t} [\lambda + \mu(t-s)]u_i(s)\,ds + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} [\lambda + \mu(t-s)]u_k(s)\,ds + [a_0^i + a_1^i(t-t_i)]V_i(t)$$

where $t_0 = 0$

$$V_i(t) = \prod_{j=1}^{m}(t-\xi_{ij})$$

and a_0^i, a_1^i are unknown parameters.

Differentiating the equation (3.2) and putting

$$(3.3)\qquad q_i(t) = \mu\int_{t_i}^{t} u_i(s)\,ds + \mu\sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} u_k(s)\,ds$$

we obtain the following system of first order differential equations:

$$(3.4^{\mathrm{I}})\qquad u_i'(t) = q_i(t) + \lambda u_i(t) + a_0^i V_i'(t) + a_1^i[(t-t_i)V_i'(t) + V_i(t)]$$

$$(3.4^{\mathrm{II}})\qquad q_i'(t) = \mu u_i(t)$$

$$(3.4^{\mathrm{III}})\qquad u_i(t_i) = u_{i-1}(t_i)$$

$$(3.4^{\mathrm{IV}})\qquad q_i(t_i) = q_{i-1}(t_i).$$

$$(3.5)\qquad \Gamma(t,x) = \sum_{k=0}^{m} [(1+k)V^{*(k)}(t) + tV^{*(k+1)}(t)]x^k$$

$$(3.6)\qquad \phi(t,x) = \sum_{k=0}^{m-1} V^{*(k+1)}(t)x^k.$$

Let α_i, $i = 1,2$ be the roots of the equation $\alpha^2 - \lambda\alpha - \mu = 0$ and $z_i = h\alpha_i$. Last let $S = (s_{ij})$, $i,j = 1,2$, be the matrix whose elements are:

$$(3.7^{\mathrm{I}})\qquad s_{11} = \frac{1}{\Delta(z_2 - z_1)}\{[z_1\Gamma(0,1/z_2) - z_2\Gamma(0,1/z_1)][\phi(1,1/z_1) - \phi(1,1/z_2)] + [z_2\phi(0,1/z_1) - z_1\phi(0,1/z_2)][\Gamma(1,1/z_2)]\}$$

$$(3.7^{\mathrm{II}})\qquad s_{12} = \frac{h}{\Delta(z_2 - z_1)}\{[\Gamma(0,1/z_2) - \Gamma(0,1/z_1)][\phi(1,1/z_1) - \phi(1,1/z_2)] + [\phi(0,1/z_1) - \phi(0,1/z_2)][\Gamma(1,1/z_1) - \Gamma(1,1/z_2)]\}$$

$$(3.7^{\text{III}})\quad s_{21} = -\frac{z_1 z_2}{h\Delta(z_2 - z_1)}\left\{[z_1\Gamma(0,1/z_2) - z_2\Gamma(0,1/z_1)]\left[\frac{\phi(1,1/z_1)}{z_1} - \frac{\phi(1,1/z_2)}{z_2}\right] + [z_2\phi(0,1/z_1) - z_1\phi(0,1/z_2)]\left[\frac{\Gamma(1,1/z_1)}{z_1} - \frac{\Gamma(1,1/z_2)}{z_2}\right]\right\}$$

$$(3.7^{\text{IV}})\quad s_{22} = -\frac{z_1 z_2}{\Delta(z_2 - z_1)}\left\{[\Gamma(0,1/z_2) - \Gamma(0,1/z_1)]\left[\frac{\phi(1,1/z_1)}{z_1} - \frac{\phi(1,1/z_2)}{z_2}\right] + [\phi(0,1/z_1) - \phi(0,1/z_2)]\left[\frac{\Gamma(1,1/z_1)}{z_1} - \frac{\Gamma(1,1/z_2)}{z_2}\right]\right\}$$

with $\Delta = \Gamma(0,1/z_1)\phi(0,1/z_2) - \Gamma(0,1/z_2)\phi(0,1/z_1)$.

Then the following theorem holds:

THEOREM 3.1: *The exact collocation method applied to equation* (3.1) *leads to the recurrence relation:*

$$\begin{pmatrix} u_i(t_{i+1}) \\ q_i(t_{i+1}) \end{pmatrix} = S\begin{pmatrix} u_{i-1}(t_i) \\ q_{i-1}(t_i) \end{pmatrix}.$$

PROOF. The proof is computationally laborious and we give here only a short outline of the logical steps.

We solve the system (3.4) by variation of constants, and calculate by repeated integration by parts the integrals involved. Then we calculate a_0^i, a_1^i solving a linear system obtained by imposing that $u_i(t)$, $q_i(t)$ are polynomials.

Then the result follows substituting the expressions of a_0^i, a_1^i into those of $u_i(t)$, $q_i(t)$ and calculating those functions in the point t_{i+1}. ■

Now we can prove the following:

THEOREM 3.2: *The stability region of the method is the set of values* $\{h\lambda, h^2\mu\}$ *such that the eigenvalues of the matrix S, given in* (3.7), *have modulus less than* 1.

PROOF. From the above theorem and the theory of linear difference equations with matrix coefficients it follows that $u_i(t_{i+1})$ satisfies the recursion:

$$(3.8)\qquad \det(EI - S)u_i(t_{i+1}) = 0$$

where E is the forward shift operator and I is the identity matrix.

Then the result follows observing that the zeros of the characteristic polynomial of (3.8) are the eigenvalues of S.

4. Stability regions.

In this section a characterization of the stability regions for some choices of the collocation parameters c_j, $j = 1, \ldots, m$, is given. Here we report only a brief indication of the proofs, that can be found in [10].

The first result we have is:

THEOREM 4.1: *The stability region of the one-step collocation method whose collocation parameters are $c_1 = 0$, $c_2 = 1$ is the following strip of the plane $\{h\lambda, h^2\mu\}$:*

$$\begin{cases} -\infty < h\lambda \le 0 \\ -12 < h^2\mu \le 0. \end{cases} \tag{4.1}$$

PROOF. (4.1) follows directly from the characteristic equation.

REMARK 1: The above collocation method applied to the convolution test equation is equivalent to the trapezoidal direct quadrature method of product type. Therefore also this method has the strip (4.1) as stability region.

Hereafter we made the hypothesis that the collocation parameters are symmetric in [0, 1].

From theorem 2.3 we can immediately deduce the following:

REMARK 2: If the collocation parameters are symmetric in [0, 1], the horizontal semiaxis $\lambda < 0$, $\mu = 0$ always belongs to the stability region.

Let us now consider the intersection of the boundary of the stability region with the vertical axis $\lambda = 0$.

We define the polynomials:

$$d_{11}(t, z) = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} V^{*(2k+2)}(t) z^{\lfloor (m-1)/2 \rfloor - k} \tag{4.2}$$

$$d_{12}(t, z) = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} [(2k+2)V^{*(2k+1)}(t) + tV^{*(2k+2)}(t)] z^{\lfloor (m-1)/2 \rfloor - k} \tag{4.3}$$

$$d_{21}(t, z) = \sum_{k=0}^{\lfloor m/2 \rfloor} V^{*(2k+1)}(t) z^{\lfloor m/2 \rfloor - k} \tag{4.4}$$

$$d_{22}(t, z) = \sum_{k=0}^{\lfloor m/2 \rfloor} [(2k+1)V^{*(2k)}(t) + tV^{*(2k+1)}(t)] z^{\lfloor m/2 \rfloor - k} \tag{4.5}$$

$$R(z) = d_{11}(1, z) d_{21}(1, z)[2d_{12}(1, z) - d_{11}(1, z)][2d_{22}(1, z) - d_{21}(1, z)]. \tag{4.6}$$

The the following theorems hold:

THEOREM 4.2: *Let r be the largest negative zero of $R(z)$ with odd multiplicity. If the* *collocation parameters are symmetric, the boundary of the stability region contains the range:*

$$\begin{cases} h\lambda = 0 \\ r \le h^2\mu \le 0. \end{cases}$$

PROOF. Putting $\lambda = 0$ in the stability matrix (3.7) and using the symmetry hypothesis, it follows, with elementary algebraic calculations, that the characteristic polynomial of S is:

$$x^2 + 2(-1)^{m-1}\frac{d_{11}(1,z)d_{22}(1,z) + d_{12}(1,z)d_{21}(1,z) - d_{11}(1,z)d_{21}(1,z)}{d_{11}(1,z)d_{22}(1,z) - d_{12}(1,z)d_{21}(1,z)}x + 1 = 0.$$

The roots have modulus 1 if and only if the discriminant of the above equation is less than or equal to zero, and this, in turn, is true if and only if $R(z) \leqq 0$.

As $R(z)$ is negative in a neighbourhood of the origin, the theorem is proved. ■

An analogous theorem has been proved by Kramarz [15] for the study of the stability of the collocation methods for periodic differential problems. However, his result is not applicable to the collocation methods for Volterra integral equations.

THEOREM 4.3: *If the collocation parameters are symmetric, the boundary of the stability region cannot contain the whole vertical semiaxis $\lambda = 0$, $\mu < 0$.*

PROOF. The coefficient of the leading term of $R(z)$ is positive, and therefore, for z large enough, $R(z)$ is positive and the result follows: ■

Now let us impose the further hypothesis that the collocation parameters are $c_1 = 0, c_m = 1$, and $c_j, j = 2, \dots, m-1$, are the zeros of the ultraspherical polynomial $P_{m-2}^{*(\alpha,\alpha)}(t)$, $\alpha > -1$, shifted in $[0, 1]$.

In this case the result of the theorem 4.3 can be improved, if we derive an upper bound of r, depending on m and α.

To this purpose, let us premise some properties, which will be useful later:

PROPERTY I.

$$P_m^{*(\alpha,\alpha)(k)}(1) = P_m^{*(\alpha,\alpha)}(1)\prod_{j=0}^{k-1}\frac{m(m+2\alpha+1) - j(j+2\alpha+1)}{j+\alpha+1}. \tag{4.7}$$

This follows, using induction on k, from the differential equation satisfied by $P_m^{*(\alpha,\alpha)}(t)$.

By algebraic calculation it follows:

PROPERTY II.

$$\frac{P_m^{*(\alpha,\alpha)(k+2)}(1)}{P_m^{*(\alpha,\alpha)(k+1)}(1)} < \frac{P_m^{*(\alpha,\alpha)(k+1)}(1)}{P_m^{*(\alpha,\alpha)(k)}(1)}. \tag{4.8}$$

Now let us put

$$g(\alpha, m) = \begin{cases} 12 & \text{if} \quad m = 2 \\ 9.6 & \text{if} \quad m = 3 \\ \dfrac{8m^3 + m^2(8\alpha - 12) + m(24\alpha + 52) + 40\alpha}{m^3 + m^2(\alpha - 2) + m(3\alpha + 6) + 2 - 2\alpha} & \text{if} \quad m \geq 4. \end{cases} \tag{4.9}$$

Then the following theorem holds:

THEOREM 4.4: *If the collocation parameters are* $c_1 = 0$, $c_m = 1$ *and* c_j, $j = 2, \ldots, m-1$, *zeros of* $P_{m-2}^{*(\alpha,\alpha)}(t)$, *the boundary of the stability region of the one-step collocation method contains at least the range:*

$$\begin{cases} h\lambda = 0 \\ -g(\alpha, m) \leq h^2\mu \leq 0. \end{cases}$$

PROOF. From property II we derive that the ratio between successive coefficients of each polynomial of $R(z)$ is decreasing. Applying a theorem on the localization of the roots [2], and using the properties I and II, the theorem follows. ∎

THEOREM 4.5: *If the collocation parameters are* $c_1 = 0$, $c_m = 1$ *and* c_j, $j = 2, \ldots, m-1$ *zeros of* $P_{m-2}^{*(\alpha,\alpha)}(t)$, *the boundary of the stability region of the one-step collocation method contains at least the range*

$$\begin{cases} h\lambda = 0 \\ -8 \leq h^2\mu \leq 0. \end{cases}$$

PROOF. For every α, $g(\alpha, m)$ is a sequence decreasing with respect to m, and its limit is 8.

5. Concluding remarks and numerical results.

From the results of the previous section we deduce that, if the collocation parameters are symmetric, the stability regions of the related one-step collocation methods are regions of the third quadrant of the plane $(h\lambda, h^2\mu)$ which are infinite in the direction of the horizontal axis, but bounded on the vertical axis.

We have also computed numerically these regions for some collocation methods

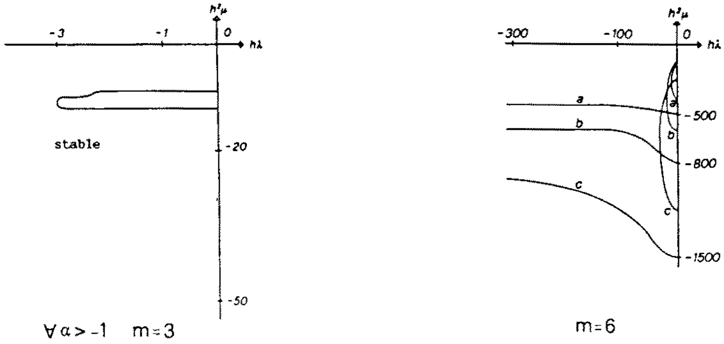

Fig. 1. Fig. 2.

Stability regions; a: $\alpha = 1$; b: $\alpha = 0$; c: $\alpha = -.5$

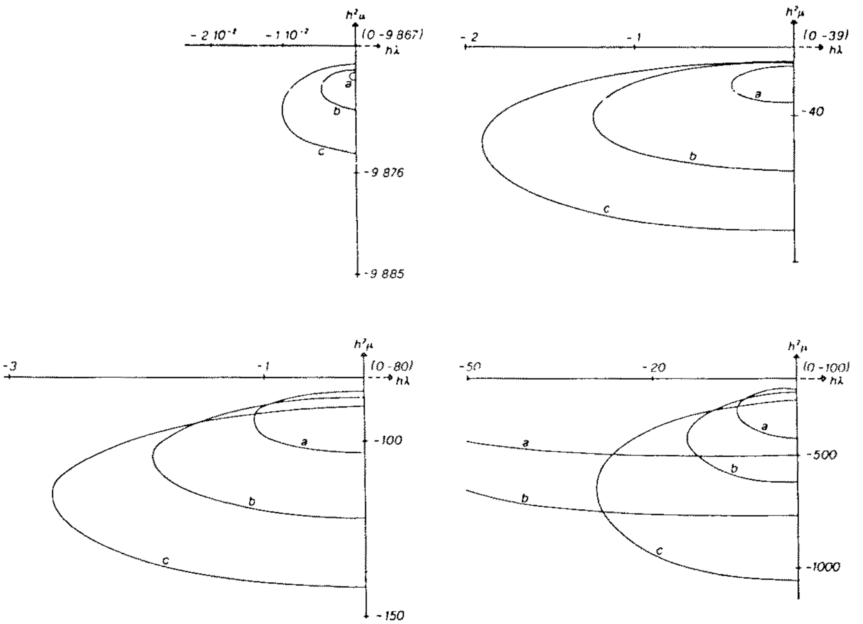

Fig. 3. Enlargements of the gaps in the stability region of figure 2.

and we present their plots for the following cases (figures 1–3):

$m = 3$ $c_1 = 0$, $c_2 = 1/2$, $c_3 = 1$

$m = 6$ $c_1 = 0$, $c_6 = 1$, c_j, $j = 2, \ldots, 5$ zeros of the Legendre polynomial $P_4^*(x)$.

$m = 6$ $c_1 = 0$, $c_6 = 1$, c_j, $j = 2, \ldots, 5$ zeros of the Chebyshev polynomial $T_4^*(x)$.

$m = 6$ c_j, $j = 1, \ldots, 6$ Lobatto abscissas.

We recall that in this case the one-step collocation method is superconvergent.

From a numerical observation, also of cases not reported in this paper, we conjecture that all the gaps of the stability region touch the vertical axis, and that the stability regions are bounded in the lower part.

Moreover, in order to test the sharpness of the expression (4.9) of $g(\alpha, m)$, we have evaluated numerically the zeros of the polynomial $R(z)$ given in (4.5). We found that $R(z)$ can have multiple zeros, so that the expression (4.9) of $g(\alpha, m)$, which is a bound for the first zero, is pessimistic.

Therefore, we report in Table 1 the numerical values of the largest negative odd multiplicity zero of $R(z)$ in the cases of most interest, that is $\alpha = -1/2, 0, 1/2, 1$ and for $m = 3$ to 20.

Table 1. *Largest negative odd multiplicity zero of the polynomial* $R(z)$ *given in* (4.6).

m \ α	$-1/2$	0	1/2	1
3	− 9.60000	− 9.60000	− 9.60000	− 9.60000
4	− 9.89136	− 9.85309	− 9.83356	− 9.82171
5	− 9.85895	− 9.86260	− 9.86509	− 9.86690
6	− 9.86869	− 9.86895	− 9.86924	− 9.86952
7	− 9.86954	− 9.86957	− 39.44486	− 39.46418
8	− 9.86960	− 39.47276	− 39.47582	− 39.47754
9	− 39.47736	− 39.47802	− 39.47833	− 39.47837
10	− 39.87435	− 39.47839	− 88.82146	− 88.82425
11	− 39.47841	− 88.82526	− 88.82604	− 88.82663
12	− 88.82621	− 88.82635	− 88.82639	− 88.82643
13	− 88.82642	− 88.82643	−157.91277	− 88.82643
14	− 88.82643	−157.91349	−157.91360	−156.91365
15	−157.91363	−157.91365	−246.73906	−156.91366
16	−157.91366	−157.91366	−246.74005	−246.74007
17	−246.74004	−246.74008	−246.74010	−246.74010
18	−246.74010	−246.74010	−246.74010	−246.74010
19	−246.74010	−246.74010	−355.30574	−355.30575
20	−246.74010	−355.30575	−483.61046	−483.61057

The numerical values have been obtained using the routine RPOLY [12], based on a three stage algorithm of Jenkins and Traub [13] on a computer VAX 750; we have considered two zeros equal when the first five decimal digits are equal.

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