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A 2-D Systems Approach to River Pollution Modelling

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Abstract. The paper presents some applications of 2-D systems theory to the problem of modeling the river pollution processes and the associated selfpurification phenomena. The dynamical evolution of the biological oxygen demand (BOD) and the dissolved oxygen (DO) in a one-dimensional river model is discussed under various physical assumptions.

1. Introduction

The unquestioned success of the estimation and regulation procedures in 1-D theory mainly relies on state space methods, that allow for efficient and explicit synthesis algorithms. Along the same lines, it is expected that the introduction of state space models that depend on two independent variables will eventually display concrete applications of the rich body of 2-D theory.

The aim of this paper is to point out how 2-D state space models apply in representing the process of pollution and selfpurification of a river. The results we present have a preliminary character. Further research will, it is to be hoped, do much to clarify advantages and drawbacks of different 2-D models, but we may feel confident that the outlines at least are broadly visible. Many results, already available in 2-D literature, offer promising applications in monitoring and control of river pollution, once a 2-D state model has been validated.

To keep the paper within an acceptable size, we found it impossible to give a detailed account of unidimensional continuous time Streeter-Phelps models. Thus we only selected from the current mass of literature some references [Rinaldi, Soncini, Sessa, Stehfest and Tamura 1979; Fair and Geyer 1965] that seem well suited for our modelling purposes.

Relevant features of 2-D systems are outlined in Section 2, but the development of 2-D theory has been carried out only to the extent necessary for the subsequent sections. Thus many important topics had to be omitted and the reader inclined to pursue the subject further is referred to [Bose 1985; Fornasini and Marchesini 1983; Eising 1979; Bose 1990; Bisiacco, Fornasini and Marchesini 1989], which contain a large bibliography up to 1989. Section 3 is devoted to a fairly detailed analysis of the problem of representing pollution dynamics via 2-D state space models, when longitudinal dispersion can be neglected. Finally, in Section 4 a number of 2-D models that incorporate the diffusion process are discussed.

2. 2-D State Space Models

The first contributions [Attasi 1973; Roesser 1975; Fornasini and Marchesini 1976] that discussed the problem of defining dynamical systems with input, output, and state functions depending on two independent variables appeared nearly 15 years ago.

From the beginning, deep and substantial differences from the theory of dynamical systems in one variable have been evidentiated. These are due to the mathematical tools to be used and, above all, to the structure of state updating equations. In this case, there is no *canonical* algebraic construction that provides an intrinsic meaning to a finite dimensional state. Thus several state models have been introduced, with different recursive structures, although they are generated by the same same underlying idea that a recursive computation is made possible by a finite dimensional *local state* and that the complete information on the past is kept by an infinite sequence of local states, called *global state.*

The support of a 2-D dynamics is constituted by the discrete plane $\mathbf{Z} \times \mathbf{Z}$. Usually, a partial order is introduced in it, by taking the product of the orderings of the coordinate axes.

Remark. Sometimes we found it convenient to refer to different coordinates in $\mathbf{Z} \times \mathbf{Z}$, using a transformation as

$$\begin{bmatrix} a \\ b \end{bmatrix} = M \begin{bmatrix} h \\ k \end{bmatrix}, \tag{2.1}$$

where M is a unimodular matrix in $\mathbf{Z}^{2\times 2}$. In these cases the partial order in $\mathbf{Z} \times \mathbf{Z}$ often consists of the product of the orderings of the new coordinate axes (see figure 1).

We associate with each point (h, k) in $\mathbf{Z} \times \mathbf{Z}$ a *local state* $x(h, k) \in \mathbf{R}^n$, that determines the output value $y(h, k)$. The local states updating is given by a linear recursive

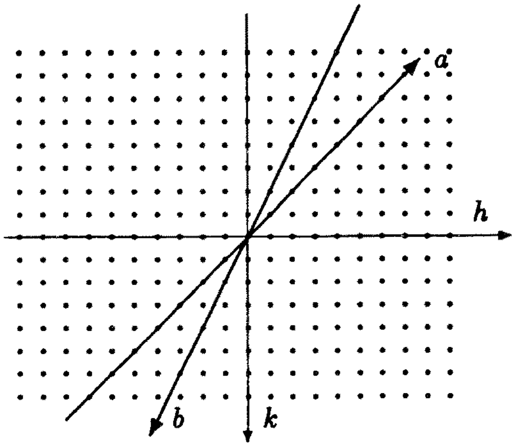

Figure 1.

equation, that involves local states and input values at some points that precede (h, k), according to the partial order.

Depending on the delay structure of the updating equations, there are essentially two different kinds of 2-D state space models.

First order models (that include also Roesser's models) are characterized by the following state space equations:

$$\begin{aligned} x(h+1, k+1) &= A_1 x(h, k+1) + A_2 x(h+1, k) \\ &\quad + B_1 u(h, k+1) + B_2 u(h+1, k) \\ y(h, k) &= Cx(h, k) \end{aligned} \tag{2.2}$$

and *second order models* (that include also Attasi's models) by the equations:

$$\begin{aligned} x(h+1, k+1) &= A_1 x(h, k+1) + A_2 x(h+1, k) \\ &\quad + A_0 x(h, k) + Bu(h, k) \\ y(h, k) &= Cx(h, k) \end{aligned} \tag{2.3}$$

Slight modifications are sometimes useful, as in a couple of models in the next sections, where we deal with a first order state updating structure and the input-state map has second order:

$$\begin{aligned} x(h+1, k+1) &= A_1 x(h, k+1) + A_2 x(h+1, k) \\ &\quad + Bu(h, k) \\ y(h, k) &= Cx(h, k) \end{aligned} \tag{2.4}$$

Note that, however, (2.4) can be also viewed as a particular case of (2.3), with $A_0 = 0$.

The local state in the above equations does not exhibit the *separation property*, in the sense that the knowledge of a single local state at (h, k) is not sufficient for computing the local states that follow $x(h, k)$ according to the partial ordering. Actually, obtaining the whole evolution of a 2-D system requires to know all local states that below to a suitable infinite subset (separation set) of $\mathbf{Z} \times \mathbf{Z}$. Some examples of separation sets for systems having equations (2.2) are shown in figure 2.

The linear structure of equations (2.2)–(2.4) allows to compute the state dynamics in the *future* of a separation set as the superposition of the free evolution, induced by the local states assignment on a separation set, and the forced evolution, induced by the input values on a separation set and in its future.

Whatever model we refer to, both evolutions involve, via linear combinations and two-dimensional convolutions, the knowledge of the system response to a single local state. As an example, assuming $x(0, 0) = \bar{x}$ and $u(\cdot, \cdot) \equiv 0$ in model (2.2), the local states in the positive orthant are given by

$$x(h, k) = A_1{}^h \sqcup\!\sqcup^k A_2 \bar{x} \tag{2.5}$$

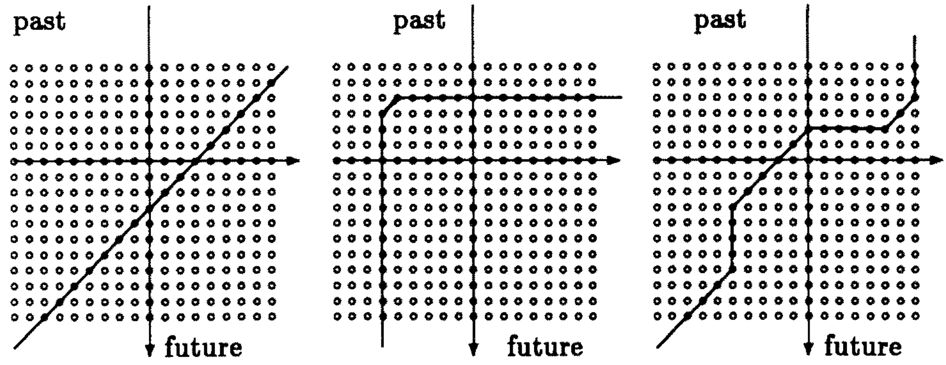

Figure 2.

where the matrices $A_1{}^h \sqcup\!\sqcup^h A_2 \in \mathbf{R}^{n \times n}$ are defined recursively as follows

$$
\begin{aligned}
&A_1^h \sqcup\!\sqcup^0 A_2 := A_1^h \\
&A_1^0 \sqcup\!\sqcup^k A_2 := A_2^k \\
&A_1^h \sqcup\!\sqcup^k A_2 := A_1(A_1^{h-1} \sqcup\!\sqcup^k A_2) + A_2(A_1^h \sqcup\!\sqcup^{k-1} A_2), \text{ if } h > 0,\ k > 0
\end{aligned}
\tag{2.5}
$$

When convolutions are involved, we found it useful to use the two-dimensional formal z-transform, that associates with a 2-D sequence $\{s_{h,k}\}$ the formal power series

$$S(z_1, z_2) = \sum_{h,k} s_{h,k} z_1^h z_2^k.$$

The convolution of two sequences in the space/time domain corresponds to the Cauchy product of the corresponding power series in the transforms domain. If one of the sequences is the impulse reponse of a 2-D system, the associated power series is (the expansion of)

$$C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) \tag{2.6}$$

in case of model (2.2) and

$$C(I - A_1 z_1 - A_2 z_2 - A_0 z_1 z_2)^{-1} B z_1 z_2 \tag{2.7}$$

in case of model (2.3).

3. 2-D Streeter-Phelps Model

In this section we aim to introduce and discuss 2-D state space models that describe the natural self-purification process of a river. The underlying biochemical hypotheses are the same as in the classical continuous Streeter-Phelps model: modifications only account for the discretization of both space and time variables.

Most river quality problems are generated by pollutants which are discharged into the river as a consequence of human activities. Discharged matters and river organisms, such as bacteria, algae and fish, interact in a very intricated system of nutritional relations between the species. The food compounds included in polluting materials are thereby oxidized and eventually converted into abiotic substances (like carbon dioxide, nitrate, etc.) and heat.

Obviously, one of the first steps in building a mathematical model of the above process is the selection of the variables relevant to the problem. The only variable which occurs naturally in the selfpurification models is the dissolved oxygen (DO) concentration, which also provides an important criterion for water quality. Besides that, it is clear that one cannot introduce a state variable for each pollutant and each living species. So the problem arises as to what extent one has to proceed in aggregating variables in the model.

The simplest approach is to reduce the variety of compounds to one class of oxidizable substances, and to measure the concentration of these somewhat fictitious reactants by the amount of oxygen needed for their complete biochemical oxidation (BOD = biological oxygen demand). Differently form *ecological models*, where living organisms are lumped together in a number of compartments, associated with specific state variables, here we postulate only the existence of a chemical reaction, induced by living organisms, between dissolved oxygen and oxidizable matter, without worrying about an explicit description of the organisms. Consequently, the only variables taken into account remain DO and BOD concentrations.

We shall assume throughout that the variations of BOD and DO concentrations on river cross sections are much less important than the longitudinal ones. So we may confine ourselves to *one-dimensional* river models. One further hypothesis, that will be relaxed later on in this section, is that hydrological variables, and in particular the stream velocity v, are constant all over the river stretch.

Finally, longitudinal diffusion and dispersion will be neglected in this section. Models where these processes are taken into account will be discussed in Section 4.

3.1. Model Structure

The first stage in constructing a 2-D model is to divide the river into *elementary reaches* of length Δl. The time step Δt and the elementary reach Δl are connected through the stream velocity v

$$\Delta t = \frac{\Delta l}{v},$$

so that the water element centered in l at time t will be centered in $l + \Delta l$ at time $t + \Delta t$.

Let $\beta(t, l)$ and $\delta(t, l)$ denote BOD concentration and DO deficit (w.r. to the saturation level) that exist in the elementary river reach centered in l at time t. BOD and DO values at $(t + \Delta t, l + \Delta l)$ are computed on the basis of a discretized balanced equation accounting for

- the self-purification process, due to the degradation of the originally discharged pollutants by bacteria. We assume that BOD concentration is decreased by the same amount

$$a_1\beta(t, l)\Delta t$$

the DO deficit is increased

- the reaeration process, taking place at the water/atmosphere interface. The simplest hypothesis we may assume is that DO deficit is reduced of an amount given by

$$a_2\delta(t, l)\Delta t$$

- BOD sources (effluents, local runoff, etc.) and, possibly, reoxygenation plants, denoted by $\text{in}_\beta(\cdot, \cdot)$ and $\text{in}_\delta(\cdot, \cdot)$ respectively.

When using intensive variables, like $\beta(\cdot, \cdot)$ and $\delta(\cdot, \cdot)$, the addition and extraction of water require to keep also track of hydrological variables, such as the water volume of the elementary reaches, in order to be able to update the resulting concentration of dissolved materials. In this paper, however, our discussion is confined to inputs of BOD and DO that do not involve variations in the flow rate of the river.

Since longitudinal diffusion and dispersion are not taken into account, the values of the variables at the point $(\bar{h}\Delta t, \bar{k}\Delta l)$ of the discrete plane

$$\{(h\Delta t, k\Delta l) \mid (h, k) \in \mathbf{Z} \times \mathbf{Z}\}$$

only affect the values at

$$\{((\bar{h} + i)\Delta t, (\bar{k} + i)\Delta l) \mid i \in \mathbf{Z}_+\},$$

i.e., along the diagonal line passing through $(\bar{h}\Delta t, \bar{k}\Delta l)$.

The balance equations are easily obtained and have the following structure

$$\beta((h + 1)\Delta t, (k + 1)\Delta l) = (1 - a_1\Delta t)\,[\beta(h\Delta t, k\Delta l) + M\,\text{in}_\beta(h\Delta t, k\Delta l)] \tag{3.1}$$

$$\delta((h + 1)\Delta t, (k + 1)\Delta l) = a_1\Delta t\beta(h\Delta t, k\Delta l) + (1 - a_2\Delta t)[\delta(h\Delta t, k\Delta l) - N\text{in}_\delta(h\Delta t, k\Delta l)] \tag{3.2}$$

Letting

$$x(h, k) := \begin{bmatrix} \beta(h\Delta t, k\Delta l) \\ \delta(h\Delta t, k\Delta l) \end{bmatrix}, \quad u(h, k) := \begin{bmatrix} u_\beta(h, k) \\ u_\delta(h, k) \end{bmatrix} = \begin{bmatrix} \text{in}_\beta(h\Delta t, k\Delta l) \\ \text{in}_\delta(h\Delta t, k\Delta l) \end{bmatrix},$$

equations (3.2) are easily rewritten as a 2-D second order model

$$x(h + 1, k + 1) = \begin{bmatrix} 1 - a_1\Delta t & 0 \\ a_1\Delta t & 1 - a_2\Delta t \end{bmatrix} x(h, k)$$

$$+ \begin{bmatrix} (1 - a_1\Delta t)M & 0 \\ 0 & -(1 - a_2\Delta t)N \end{bmatrix} u(h, k) \tag{3.3}$$

$$= A_0 x(h, k) + B_0 u(h, k)$$

Remark. The above 2-D model can be thought of as the juxtaposition of infinitely many copies of the same 1-D system, each copy being associated with a different diagonal of the discrete plane. The elementary volume of water that at time 0 is in the position $k\Delta l$ is characterized by a state

$$\xi(0) := \begin{bmatrix} \beta(0, k\Delta l) \\ \delta(0, k\Delta l) \end{bmatrix} = x(0, k)$$

At time $i\Delta t$ its position along the river is $(k + i)\Delta l$. Once the corresponding state has been written as

$$\xi(i) := \begin{bmatrix} \beta(i\Delta t, (k + i)\Delta l) \\ \delta(i\Delta t, (k + i)\Delta l) \end{bmatrix} = x(i, k + i)$$

and the forcing input as

$$\eta(i) := \begin{bmatrix} u_\beta(i, k + i) \\ u_\delta(i, k + i) \end{bmatrix},$$

BOD concentration and DO deficit, as seen by an observer that moves along with the elementary volume of water, are modeled by a 1-D system of the following form

$$\xi(i + 1) = A_0\xi(i) + B_0\eta(i). \tag{3.4}$$

If a 2-D first order model is adopted, it is necessary to increase the dimension of the state space. This is easily seen, since the impulse response support of a 2-D system of dimension one is either the whole positive orthant or one of the coordinate axes, while the BOD and DO impulse responses exhibit a diagonal support. Therefore two components are needed in the local state vector for representing the dynamical behavior of just one single variable.

Consider first the BOD evolution, and let

$$x_\beta(h, k) := \begin{bmatrix} \beta(h\Delta t, k\Delta l) \\ \beta(h\Delta t, (k + 1)\Delta l) \end{bmatrix} \tag{3.5}$$

be the local state vector at (h, k). Using (3.1), one gets

$$x_\beta(h+1, k+1) = \begin{bmatrix} 0 & 0 \\ 1 - a_1\Delta t & 0 \end{bmatrix} x_\beta(h, k+1)$$

$$+ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_\beta(h+1, k) + \begin{bmatrix} 0 \\ (1 - a_1\Delta t)M \end{bmatrix} u_\beta(h, k) \tag{3.6}$$

$$= A_{1\beta} x_\beta(h, k+1) + A_{2\beta} x_\beta(h+1, k) + B_\beta u_\beta(h, k),$$

where a second order delay appears in the input/state map.

Next, assuming

$$x_\delta(h, k) := \begin{bmatrix} \delta(h\Delta t, k\Delta l) \\ \delta(h\Delta t, (k+1)\Delta l) \end{bmatrix}, \tag{3.7}$$

we obtain

$$x_\delta(h+1, k+1) = \begin{bmatrix} 0 & 0 \\ 1 - a_2\Delta t & 0 \end{bmatrix} x_\delta(h, k+1) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_\delta(h+1, k)$$

$$+ \begin{bmatrix} 0 & 0 \\ a_1\Delta t & 0 \end{bmatrix} x_\beta(h, k+1) + \begin{bmatrix} 0 \\ -N(1 - a_2\Delta t) \end{bmatrix} u_\delta(h, k) \tag{3.8}$$

$$= A_{1\delta} x_\delta(h, k+1) + A_{2\delta} x_\delta(h+1, k) + A_{\beta\delta} x_\beta(h, k+1) + B_\delta u_\delta(h, k)$$

Tying together (3.7) and (3.8) we get the following model

$$\begin{bmatrix} x_\beta(h+1, k+1) \\ x_\delta(h+1, k+1) \end{bmatrix} = \overbrace{\begin{bmatrix} A_{1\beta} & 0 \\ A_{\beta\delta} & A_{1\delta} \end{bmatrix}}^{A_1} \begin{bmatrix} x_\beta(h, k+1) \\ x_\delta(h, k+1) \end{bmatrix}$$

$$+ \overbrace{\begin{bmatrix} A_{2\beta} & 0 \\ 0 & A_{2\delta} \end{bmatrix}}^{A_2} \begin{bmatrix} x_\beta(h+1), k) \\ x_\delta(h+1), k) \end{bmatrix} + \overbrace{\begin{bmatrix} B_\beta & 0 \\ 0 & B_\delta \end{bmatrix}}^{B} \begin{bmatrix} u_\beta(h, k) \\ u_\delta(h, k) \end{bmatrix} \tag{3.9}$$

Both matrices A_1 and A_2 are nilpotent, with nilpotency index 2. Thus

$$A_1^i \sqcup\!\sqcup^j A_2 = 0 \text{ if } |i - j| > 1,$$

which in turn implies that the evolution of system (3.9) takes place along *discretized diagonal lines*, as shown in figure 3.

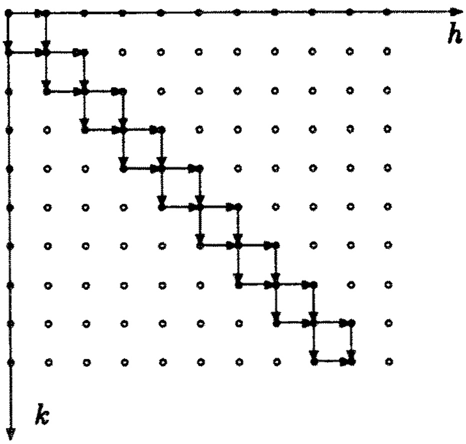

Figure 3.

There are other methods by which one can proceed to build up a 2-D state space model. It would be tedious and unnecessary to discuss here all of them; we confine ourselves to a first order model, with local states of dimension 2, which illustrates some advantages one gets when the coordinates (h, k) of the points in the discrete plane are not directly identified with time and space values in the physical model. This approach will prove to be fruitful in next section, where diffusion will be taken into account.

We assume that the pair $(h\Delta t, k\Delta l)$ is associated with the point $(a, b) \in \mathbf{Z} \times \mathbf{Z}$ that satisfies

$$a = h - k, \; b = k \tag{3.10}$$

So, the points of the separation set

$$C_{\bar{h}} := \{(a, b) \mid a + b = \bar{h}\}$$

represent locations $k\Delta l$ along the river stretch at the same time instant $\bar{h}\Delta t$. On the other hand, the points of the set

$$\{(a, b) \mid b = \bar{k}\} = \{(a, \bar{k})\}$$

represent time instants $h\Delta t = (a - \bar{k})\Delta t$ at the same location $\bar{k}\Delta l$.

Letting

$$\begin{bmatrix} \beta(h\Delta t, k\Delta l) \\ \delta(h\Delta t, k\Delta l) \end{bmatrix} := x(h - k, k) = x(a, b)$$

$$\begin{bmatrix} \text{in}_\beta(h\Delta t, k\Delta l) \\ \text{in}_\delta(h\Delta t, k\Delta l) \end{bmatrix} := u(h - k, k) = u(a, b)$$

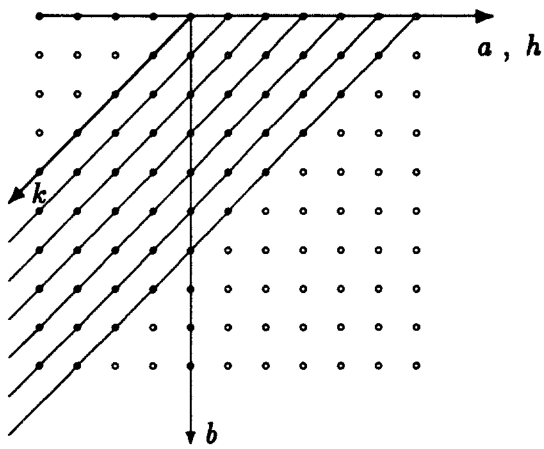

Figure 4.

equations (3.1) and (3.2) give

$$x(h-k,k+1) = \begin{bmatrix} 1-a_1\Delta t & 0 \\ a_1\Delta t & 1-a_2\Delta t \end{bmatrix} x(h-k,k) + \begin{bmatrix} (1-a_1\Delta t)M \\ -(1-a_2\Delta t)N \end{bmatrix} u(h-k,k)$$

or, equivalently

$$x(a,b+1) = \begin{bmatrix} 1-a_1\Delta t & 0 \\ a_1\Delta t & 1-a_2\Delta t \end{bmatrix} x(a,b) + \begin{bmatrix} (1-a_1\Delta t)M \\ -(1-a_2\Delta t)N \end{bmatrix} u(a,b) \quad (3.11)$$

In figure 4, the *characteristic lines* of the system are the vertical axes a = const.

3.2. *Initial Conditions*

Since model (3.3) is the juxtaposition of infinitely many independent 1-D systems evolving along the diagonals of $\mathbf{Z} \times \mathbf{Z}$, the most general structure of initial conditions consists in assigning exactly one local state on each diagonal line of the discrete plane. Any one of the above sets of conditions is *reachable*, since it can be thought of as produced by the application of suitable space/time distributions of BOD and DO sources.

The assignment of initial conditions in model (3.9) deserves a more detailed investigation. First of all, the local state components specify the values of BOD concentration and DO deficit at the same time instant in two consecutive spatial locations. Therefore, when initial conditions are given on some straight line

$$\{(h, \bar{k}) \mid h \in \mathbf{Z}\}$$

or along the boundary of the positive orthant, the second and the fourth component of $x(h, k)$ must coincide with the first and the third component of $x(h, k + 1)$, respectively. This amounts to say that the physical meaning of local states allows to consider only reachable arrays of admissible conditions.

One more aspect of the dynamical structure of the system, however, must be considered if the assignment of the initial states is to be meaningful. Namely, the state updating operation must not modify the original values of the initial conditions on the boundary. When some boundary points are in the future of some others, it is patently inconsistent to compute the free state evolution by superposing local state values, as determined by rule

$$x(h, k) = A_1^h \sqcup\!\sqcup^k A_2 x(0, 0) \tag{3.12}$$

In fact, this would possibly modify the boundary values themselves. In this connnection we shall compute here the formal power series associated with the doubly indexed sequence of states in two cases, seemingly the most significant ones.

Suppose first that the initial conditions have been assigned on the boundary

$$\{(h, 0) \mid h \in \mathbf{Z}_+\} \cup \{(0, k) \mid k \in \mathbf{Z}_+\} \tag{3.13}$$

and the input values on

$$\{(h, k) \mid h \geq 0, k \geq 0, h + k > 0\}) \tag{3.14}$$

Due to the recursive structure of (3.9), the computation of $x(\bar{h}, \bar{k})$, $\bar{h} > 0$, $\bar{k} > 0$ only involves the initial local states $\{x(h, 0), 0 < h \leq \bar{h}\} \cup \{x(0, k), 0 < k \leq \bar{k}\}$ and the input values $\{u(h, k), 0 \leq h < \bar{h}, 0 \leq k < \bar{k}, h + k > 0\}$, as shown in figure 5.

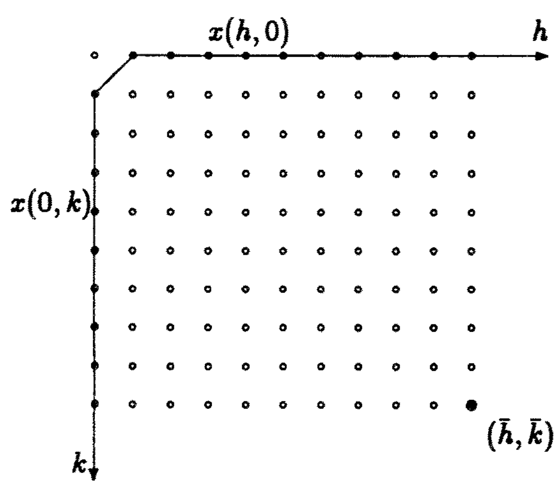

Figure 5.

Consider the formal power series

$$X(z_1, z_2) := \sum_{h>0,k>0} x(h, k)z_1^h z_2^k \tag{3.15}$$

associated to the double indexed array of local states $\{x(h, k)\}_{h,k>0}$ and let $X_\ell(z_1, z_2)$ be the corresponding free evolution induced by the assignment of local states (3.14) on the boundary (3.13). $X_\ell(z_1, z_2)$ can be computed according to

$$\begin{aligned} X_\ell(z_1, z_2) &= \sum_{h,k>0} x(h, k)z_1^h z_2^k \\ &= \sum_{h,k>0} \{A_1 x(h-1, k) + A_2 x(h, k-1)\} z_1^h z_2^k \\ &= \sum_{\substack{j\geq 0 \\ k>0}} A_1 x(i, k) z_1^{i+1} z_2^k + \sum_{\substack{h>0 \\ j\geq 0}} A_2 x(h, j) z_1^h z_2^{j+1} \qquad (3.16) \\ &= (I - A_1 z_1 - A_2 z_2)^{-1} \left[z_1 A_1 \sum_{i>0} x(i, 0) z_1^i + z_2 A_2 \sum_{j>0} x(0, j) z_2^j \right] \end{aligned}$$

On the other hand, forced evolution is easily obtained as

$$X_f(z_1, z_2) = (I - A_1 z_1 - A_2 z_2)^{-1} B z_1 z_2 U(z_1, z_2) \tag{3.17}$$

where

$$U(z_1, z_2) := \sum_{h,k\geq 0} u(h, k) z_1^h z_2^k \tag{3.18}$$

is the formal power series associated with the input sequence.

The second case we discuss is the discrete analogue of assigning BOD and DO values at some point of the river (e.g. at $l = 0$) for all t in $\mathbf{R}$. This corresponds to specifying in model (3.9) local states on the line

$$\{(h, 0) \mid h \in \mathbf{Z}\}$$

and output values on the half plane

$$\{(h, k) \mid k \geq 0\},$$

and in computing $x(h, k)$ on the half plane

$$\{(h, k) \mid k > 0\}.$$

An obvious role of the nilpotency of A_1 and A_2 is to guarantee that a single local state $x(h, k)$ does not influence local states on the diagonal lines that do not intersect the set $\{(h, k), (h-1, k), (h+1, k)\}$. The following equations reveal the importance of this property in determining the free evolution of the system (see figure 6):

$$\begin{aligned}
x(h, 1) &= A_1 x(h-1, 1) + A_2 x(h, 0) \\
&= A_1 A_2 x(h-1, 0) + A_2 x(h, 0) \\
x(h, 2) &= A_1 A_2 x(h-1, 1) + A_2 x(h, 1) \\
&= A_1 A_2 A_1 A_2 x(h-2, 0) + A_2 A_1 A_2 x(h-1, 0) \\
&\dots \\
x(h, k) &= \underbrace{A_1 A_2 \dots A_1 A_2}_{2k \text{ terms}} x(h-k, 0) + \underbrace{A_2 A_1 A_2 \dots A_2}_{2k-1 \text{ terms}} x(h-k+1, 0) \\
&= (A_1^k \sqcup\!\sqcup^{k-1} A_2) A_2 x(h-k, 0) + (A_1^{k-1} \sqcup\!\sqcup^{k-1} A_2) A_2 x(h-k+1, 0)
\end{aligned} \tag{3.19}$$

As a consequence, when we use the formal power series notation, we have

$$\begin{aligned}
X(z_1, z_2) &= \sum_{\substack{k \geq 1 \\ h \in \mathbf{Z}}} x(h, k) z_1^h z_2^k \\
&= \sum_{\substack{k \geq 1 \\ h \in \mathbf{Z}}} (A_1^k \sqcup\!\sqcup^{k-1} A_2) A_2 x(h-k, 0) z_1^h z_2^k \\
&\quad + \sum_{\substack{k \geq 1 \\ h \in \mathbf{Z}}} (A_1^{k-1} \sqcup\!\sqcup^{k-1} A_2) A_2 x(h-k+1, 0) z_1^h z_2^k \\
&= \sum_{k \geq 1} \left[A_1^k \sqcup\!\sqcup^{k-1} A_2 z_1^h z_2^k + A_1^{k-1} \sqcup\!\sqcup^{k-1} A_2 z_1^{k-1} z_2^k \right] A_2 \sum_{h \in \mathbf{Z}} x(h, 0) z_1^h \\
&= \left[\sum_{\nu \geq 0} [A_1^{\nu+1} \sqcup\!\sqcup^{\nu} A_2 z_1 z_2 + A_1^{\nu} \sqcup\!\sqcup^{\nu} A_2 z_2] A_2 z_1^{\nu} z_2^{\nu} \right] \sum_{h \in \mathbf{Z}} x(h, 0) z_1^h
\end{aligned} \tag{3.20}$$

Making the assumption that the BOD and DO levels on the 0-th river stretch are independent of time, that is

$$x(h, 0) = \bar{x}, \qquad \forall h \in \mathbf{Z},$$

it is straightforward to obtain from (3.20) at steady state solution, given by

$$X_\ell(z_1, z_2) = \sum_{\substack{\nu \geq 1 \\ h \in \mathbf{Z}}} [A_1^{\nu+1} \sqcup\!\sqcup^{\nu} A_2 z_1 z_2 + A_1^{\nu} \sqcup\!\sqcup^{\nu} A_2 z_2] A_2 \bar{x} z_1^h z_2^{\nu} \tag{3.21}$$

The state vector in the k-th river stretch is the coefficient of any monomial $z_1^* z_2^k$ in (3.21), i.e.

$$x(h, k) = [A_1^{k-1} \sqcup\!\sqcup^{k-1} A_2 + A_1^k \sqcup\!\sqcup^{k-1} A_2] A_2 \bar{x} \tag{3.22}$$

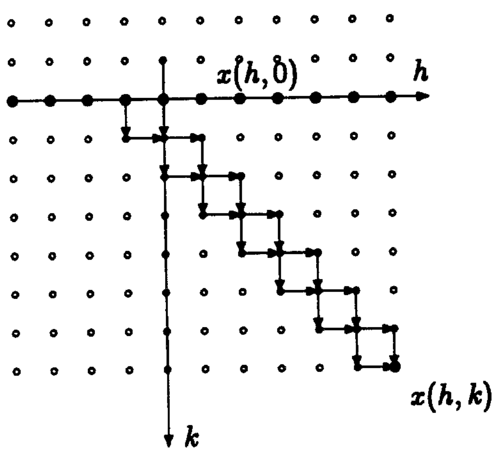

Figure 6.

The assignment of initial states and the formal power series description of local states dynamics in model (3.11) are similar to those for the model (3.9), but simpler because no constraints are needed among initial states. We leave the details to the reader.

Remark. As an alternative way for introducing a reachable set of boundary conditions, we may use an input sequence that forces the boundary conditions in a 2-D system originally at rest (i.e. on a river that is perfectly clean and aereated). Substituting forced dynamics for boundary conditions is a matter of taste and computational convenience.

3.3. Space-Dependent Dynamics

Our original assumption in this section was that all river parameters do not depend on the space abscissa l. It is often the case, however, that certain parameters of the one-dimensional model are strongly influenced by the geometrical and physical attributes of the underlying three-dimensional real model. Relaxing that assumption can certainly enhance our capability of modelling river phenomena. So in the remaining part of this section we suppose that the river velocity v as well as the coefficients a_1 and a_2 possibly depend on l.

It is not difficult to figure out situations where a dependence on l may arise. Apart from the obvious ones, that refer to velocity variations, the dependence of a_1 on l may be adscribed to an inhomogeneous bacterial oxidation (due, e.g., to thermal variations or to some bacterial species that locally prevail on some others), while the dependence of a_2 may be connected with turbulences, falls etc., that induce some variations on the reaeration process.

While the time quantization interval Δt is kept constant, the length Δl of the elementary reaches will vary so as to satisfy in all cases the condition

$$\Delta t = \frac{\Delta l}{v(l)}$$

More precisely, the river stretch will be divided into elementary reaches $\Delta l_k = [l_k, l_{k+1}]$, with

$$\Delta l_k = v(l_k)\Delta t \tag{3.23}$$

so that an elementary volume of water in position l_k at time t will be in position l_{k+1} at time $t + \Delta t$. After introducing the families of l_k-dependent coefficients $a_1(l_k)$ and $a_2(l_k)$, we are in a position to rewrite model (3.3) as follows

$$\begin{aligned} x(h+1, k+1) &= \begin{bmatrix} 1 - a_1(k)\Delta t & 0 \\ a_1(k)\Delta t & 1 - a_2(k)\Delta t \end{bmatrix} x(h, k) \\ &+ \begin{bmatrix} M[1 - a_1(k)\Delta t] & 0 \\ 0 & -N[1 - a_2(k)\Delta t] \end{bmatrix} \begin{bmatrix} u_\beta(h, k) \\ u_\delta(h, k) \end{bmatrix} \\ &= A_0(k)x(h, k) + B_0(k)u(h, k) \end{aligned} \tag{3.24}$$

where the local state vector is defined by

$$x(h, k) := \begin{bmatrix} \beta(h\Delta t, l_k) \\ \delta(h\Delta t, l_k) \end{bmatrix}.$$

Note that $-a_1(k)\Delta t\beta(h\Delta t, l_k)$ and $-a_2\Delta t\delta(h\Delta t, l_k)$ represent the decrement of BOD concentration and DO deficit when crossing the elementary reach $[l_k, l_{k+1}]$.

The 1-D model (3.4) we associated with (3.3) becomes now

$$\xi(i+1) = \begin{bmatrix} 1 - a_1(i)\Delta t & 0 \\ a_1(i)\Delta t & 1 - a_2(i)\Delta t \end{bmatrix} \xi(i)$$

$$+ \begin{bmatrix} M[1 - a_1(i)\Delta t]M & 0 \\ 0 & -N[1 - a_2(i)\Delta t] \end{bmatrix} \eta(i) \tag{3.25}$$

$$= A_0(i)\xi(i) + B_0(i)\eta(i)$$

Given any time instant $h\Delta t$, $\xi(i)$ is the state vector at abscissa l_i and time $(h + i)\Delta t$, produced by the assignment of a state vector $\xi(0)$ at abscissa l_0 and by the input values $\eta(j) = u((h + j)\Delta t, l_j)$, $j = 0, 1 \ldots$

Free evolution of $\xi(\cdot)$ satisfies

$$\xi(i+1) = A_0(i)A_0(i-1) \ldots A_0(1)A_0(0)\xi(0) = \Phi(i)\xi(0) \tag{3.26}$$

with

$$\Phi(i) := \begin{bmatrix} \prod_{\nu=0}^{i} [1 - a_1(\nu)\Delta t] & 0 \\ \sum_{\ell=0}^{i} \prod_{\mu=\ell+1}^{i} [1 - a_2(\mu)\Delta t] a_1(\ell)\Delta t \prod_{\nu=0}^{\ell-1} [1 - a_1(\nu)\Delta t] & \prod_{\nu=0}^{i} [1 - a_2(\nu)\Delta t] \end{bmatrix} \tag{3.27}$$

The asymptotic behavior of (3.26) can be deduced from the absolute convergence criterion for infinite products [Knopp 1956]. Actually, because of the inequalities

$$0 \leq a_1(\nu)\Delta t < 1, \qquad 0 \leq a_2(\nu)\Delta t < 1$$

a necessary and sufficient condition for having

$$\lim_{i \to +\infty} \prod_{\nu=0}^{i} [1 - a_1(\nu)\Delta t] = 0 \quad \text{and} \quad \lim_{t \to +\infty} \prod_{\nu=0}^{i} [1 - a_2(\nu)\Delta t] = 0 \tag{3.28}$$

is that both the following series

$$\sum_{\nu=0}^{+\infty} a_1(\nu) \quad \text{and} \quad \sum_{\nu=0}^{+\infty} a_2(\nu) \tag{3.29}$$

diverge.

Therefore, the divergent character of (3.29) constitutes a criterion for guaranteeing

1. a complete bacterial oxidation of any BOD load injected at the l_0-section
2. a complete reareation of a deoxigenated river, if the BOD load is assumed to be zero.

We shall prove now that, when (3.29) diverge, the term in position (2, 1) in the transition matrix $\Phi(i)$ converges to zero as $i \to \infty$. This shows that the divergence of both series (3.29) constitutes a necessary and sufficient condition for the selfpurification of the river.

First of all, note that $A_0(\nu)$ can be viewed as the 2×2 left top diagonal block of the 3×3 stochastic matrix.

$$A^{(a)}(\nu) = \begin{bmatrix} 1 - a_1(\nu)\Delta t & 0 & 0 \\ a_1(\nu)\Delta t & 1 - a_2(\nu)\Delta t & 0 \\ 0 & a_2(\nu)\Delta t & 1 \end{bmatrix} \tag{3.30}$$

Therefore

$$\Phi^{(a)}(i) := A^{(a)}(i)A^{(a)}(i-1) \ldots A^{(a)}(1)A^{(a)}(0) \tag{3.31}$$

$$= \begin{bmatrix} \multicolumn{2}{c}{\Phi(i)} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \phi_{31}^{(a)}(i) & \phi_{32}^{(a)}(1) & 1 \end{bmatrix} = \begin{bmatrix} \phi_{11}(i) & 0 & 0 \\ \phi_{21}(i) & \phi_{22}(i) & 0 \\ \phi_{31}^{(a)} & \phi_{32}^{(a)}(i) & 1 \end{bmatrix}$$

is a stochastic matrix for all $i \in \mathbf{Z}_+$.

Next, apply the recursive equation

$$\phi_{31}^{(a)}(i+1) = a_2(i+1)\Delta t\phi_{21}(i) + \phi_{31}^{(a)}(i) \tag{3.32}$$

to obtain the following identity

$$\phi_{31}^{(a)}(i+1) = a_2(i+1)\Delta t\phi_{21}(i) + a_2(i)\Delta t\phi_{21}(i-1) + \ldots + a_2(1)\Delta t\phi_{21}(0) \tag{3.33}$$

Because of (3.33), the sequence $\{\phi_{31}^{(a)}\}$ monotonically increases. Moreover, the stochastic character of $\Phi^{(a)}(\nu)$ implies

$$\phi_{31}^{(a)}(\nu) \leq 1, \qquad \forall \nu \in \mathbf{Z}_+.$$

This shows that the above sequence converges to a limit $\bar{\phi}_{31} \in [0, 1]$:

$$\bar{\phi}_{31} = \lim_{\nu \to +\infty} \phi_{31}^{(a)}(\nu) \tag{3.34}$$

Now, taking the limit as $\nu \to +\infty$ on the right side of

$$1 = \phi_{11}(\nu) + \phi_{21}(\nu) + \phi_{31}^{(a)}(\nu) \tag{3.35}$$

and recalling that the sequence $\{\phi_{11}(\nu)\}$ converges to 0, we see that the sequence $\{\phi_{21}(\nu)\}$ converges to $\bar{\phi}_{21} = 1 - \bar{\phi}_{31}$.

Finally, we have to prove that $\bar{\phi}_{21} = 0$. Assume, by contradiction, $\bar{\phi}_{21} > 0$. Then there exists an integer ν_0 such that

$$\phi_{21}(\nu) > \frac{\bar{\phi}_{21}}{2}, \forall i \geq \nu_0$$

and therefore

$$\begin{aligned} \phi_{31}^{(a)}(i + 1 + \nu_0) &\geq \sum_{\nu=\nu_0}^{\nu_0+i} a_2(\nu + 1)\Delta t \phi_{21}(\nu) \\ &\geq \frac{\bar{\phi}_{21}}{2} \Delta t \sum_{\nu=\nu_0}^{\nu_0+1} a_2(\nu + 1) \end{aligned} \tag{3.35}$$

Taking into account that the series $\Sigma_\nu a_2(\nu)$ diverges, we see that the sequence $\{\phi_{31}^{(a)}(\nu)\}$ would diverge too, which is a contradiction, since $\bar{\phi}_{31}$ was finite.

We therefore have

$$\bar{\phi}_{21} = 0 \quad \text{and} \quad \Phi(i) \rightarrow 0 \text{ as } i \rightarrow \infty$$

4. 2-D Diffusion Models

The 2-D models considered in Section 3 do not incorporate longitudinal diffusion and/or dispersion phenomena. As well known, introducing diffusion in the Streeter-Phelps equations gives rise to partial differential equations that include the second derivative w.r. to the space coordinate. 2-D analogs of continuous diffusion models may be obtained using a suitable discretization procedure; however, we prefer to start here directly from a discrete representation of the diffusion mechanism and to set up a *first principles* derivation of 2-D models.

4.1. Building an Elementary Model

Perhaps the simplest representation of the diffusion mechanism can be obtained by introducing in model (3.1) additional terms that account for BOD and DO diffusion between two contiguous elementary reaches.

Diffusion is therefore modeled by assuming that the BOD content of the elementary water volume centered on l at time t undergoes variations in Δt that are proportional to the differences

$$\beta(t, l - \Delta l) - \beta(t, l)$$
$$\beta(t, l + \Delta l) - \beta(t, l)$$

Therefore, equation (3.1) has to be modified as follows:

$$\begin{aligned}\beta((h + 1)\Delta t, (k + 1)\Delta l) &= [1 - a_1\Delta t]\beta(h\Delta t, k\Delta l)\\ &\quad + [1 - a_1\Delta t]M\mathrm{in}_\beta(h\Delta t, k\Delta l)\\ &\quad + D_\beta[\beta(h\Delta t, (k - 1)\Delta l) - \beta(h\Delta t, k\Delta t)]\Delta t\\ &\quad + D_\beta[\beta(h\Delta t, (k + 1)\Delta l) - \beta(h\Delta t, k\Delta l)]\Delta t\end{aligned} \tag{4.1}$$

Similarly equation (3.2) becomes now

$$\begin{aligned}\delta((h + 1)\Delta t, (k + 1)\Delta l) &= a_1\Delta t\beta(h\Delta t, k\Delta l)\\ &\quad + [1 - a_2\Delta t]\delta(h\Delta t, k\Delta l)\\ &\quad - [1 - a_2\Delta t]N\mathrm{in}_\delta(h\Delta t, k\Delta l)\\ &\quad + D_\delta[\delta(h\Delta t, (k - 1)\Delta l) - \delta(h\Delta t, k\Delta l)]\Delta t\\ &\quad + D_\delta[\delta(h\Delta t, (k + 1)\Delta l) - \delta(h\Delta t, k\Delta l)]\Delta t\end{aligned} \tag{4.2}$$

Letting

$$x_\beta(h, k) = \begin{bmatrix} \beta(h\Delta t, (k - 1)\Delta l) \\ \beta(h\Delta t, k\Delta l) \end{bmatrix} \qquad x_\delta\delta(h, k) = \begin{bmatrix} \delta(h\Delta t, (k - 1)\Delta l) \\ \delta(h\Delta t, k\Delta l) \end{bmatrix}$$

and assuming

$$x(h, k) = x_\beta(h, k) \oplus x_\delta(h, k)$$

one gets

$$x(h + 1, k + 1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ D_\beta\Delta t & 1 - \bar{a}_1\Delta t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_1\Delta t & D_\delta\Delta t & 1 - \bar{a}_2\Delta t \end{bmatrix} x(h, k)$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & D_\beta \Delta t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_\delta \Delta t \end{bmatrix} x(h, k+1) + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(h+1, k) \tag{4.3}$$

$$+ \begin{bmatrix} 0 & 0 \\ M(1 - a_1 \Delta t) & 0 \\ 0 & 0 \\ 0 & -N(1 - a_2 \Delta t) \end{bmatrix} \begin{bmatrix} u_\beta(h, k) \\ u_\delta(h, k) \end{bmatrix}$$

with

$$\bar{a}_1 = a_1 + 2D_\beta, \qquad \bar{a}_2 = a_2 + 2D_\delta, \tag{4.4}$$

which is a 2-D system in form (2.3).

Assume now that a unitary BOD pulse at (0, 0) constitutes the forcing input to a river that is perfectly clean and aerated. This gives rise to a spatially symmetric distribution of BOD, that extends at time $h\Delta t$ from the abscissa Δl up to the abscissa $(2h - 1)\Delta l$, having $k\Delta t$ as a center of symmetry.

The general local state response can be described by a formal power series

$$X^{(\beta)}(z_1, z_2) = \sum_{h,k} x_\beta z_1^h z_2^k$$

$$= (I - A_1^{(\beta)} z_1 - A_2^{(\beta)} z_2 - A_0^{(\beta)} z_1 z_2)^{-1} B^{(\beta)} z_1 z_2$$

with

$$A_0^{(\beta)} = \begin{bmatrix} 0 & 0 \\ D_\beta \Delta t & 1 - \bar{a}_1 \Delta t \end{bmatrix} \qquad A_1^{(\beta)} = \begin{bmatrix} 0 & 0 \\ 0 & D_\beta \Delta t \end{bmatrix}$$

$$A_2^{(\beta)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B^{(\beta)} = \begin{bmatrix} 0 \\ M(1 - \bar{a}_1 \Delta t) \end{bmatrix}$$

Letting

$$\begin{aligned} L_\beta &= M[1 - a_1 \Delta t] \\ b(z_2) &= D_\beta \Delta t + [1 - \bar{a}_1 \Delta t] z_2 + D_\beta \Delta t z_2^2 \end{aligned} \tag{4.5}$$

the BOD impulse response, provided by the second component of the vector $X^{(\beta)}(z_1, z_2)$, can be rewritten as

$$X_2^{(\beta)}(z_1, z_2) = \frac{z_1 z_2 L_\beta}{1 - z_1 b(z_2)} = L_\beta z_1 z_2 \sum_{h=0}^{+\infty} z_1^h b(z_2)^h \tag{4.6}$$

The distribution of the BOD concentration at time $h\Delta t$ can be found by considering the polynomial in $\mathbf{R}[z_2]$ that constitutes the coefficient of the monomial z_1^h in the above series, i.e.

$$L_\beta z_2 b(z_2)^{h-1}$$

The power series representing the DO deficit distribution can be obtained along the same lines. Letting

$$A_0^{(\delta)} = \begin{bmatrix} 0 & 0 \\ D_\delta \Delta t & 1 - \bar{a}_1 \Delta t \end{bmatrix} \qquad A_1^{(\delta)} = \begin{bmatrix} 0 & 0 \\ 0 & D_\delta \Delta t \end{bmatrix}$$

$$A_2^{(\delta)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A^{(\beta\delta)} = \begin{bmatrix} 0 & 0 \\ 0 & a_1 \Delta t \end{bmatrix}$$

one gets from (4.3)

$$\begin{aligned} X^{(\delta)}(z_1, z_2) &= z_1 z_2 (I - A_1^{(\delta)} z_1 - A_2^{(\delta)} z_2 - A_0^{(\delta)} z_1 z_2)^{-1} A^{(\beta\delta)} X^{(\beta)}(z_1, z_2) \\ &= a_1 \Delta t z_1 z_2 \, (I - A_1^{(\delta)} z_1 - A_2^{(\delta)} z_2 - A_0^{(\delta)} z_1 z_2)^{-1} \begin{bmatrix} 0 \\ X_2^{(\beta)}(z_1, z_2) \end{bmatrix} \end{aligned}$$

After introducing the following shorthand notations

$$\begin{aligned} d(z_2) &= D_\delta \Delta t + (1 - \bar{a}_2 \Delta t) z_2 + D_\delta \Delta t z_2^2, \\ L_\delta &= (1 - a_1 \Delta t) M a_1 \Delta t \end{aligned} \tag{4.7}$$

the distribution of the DO deficit is represented by

$$X_2^{(\delta)}(z_1, z_2) = \frac{z_1 z_2 a_1 \Delta t}{1 - z_1 d(z_2)} X_2^{(\beta)}(z_1, z_2) = \frac{z_1 z_2 L_\delta}{(1 - z_1 d(z_2))(1 - z_1 b(z_2))} \tag{4.8}$$

4.2. Steady-State Distributions

Suppose that a time constant unitary input of BOD is applied at the 0-th reach of the river. Using the superposition principle we represent the BOD distribution as given by a power series in z_1, with coefficients in the ring $\mathbf{R}[[z_2]]$ off formal power series in z_2:

$$X_2^{(\beta)}(z_1, z_2) = \frac{z_2 L_\beta}{1 - z_1 b(z_2)} \sum_{h=-\infty}^{+\infty} z_1^h = L_\beta \sum_{h=-\infty}^{+\infty} z_1^h \sum_{i=0}^{+\infty} z_2 b(z_2)^i$$

Accordingly, it can be inferred that a stationary BOD distribution settles down along the river, represented by the series

$$L_\beta z_2 \sum_{i=0}^{+\infty} b(z_2)^i = \frac{L_\beta z_2}{1 - b(z_2)} \tag{4.9}$$

The same reasonings also show that the space/time distribution of the DO deficit is represented by the formal power series

$$X_2^{(\delta)}(z_1, z_2) = \frac{z_2^2 L_\delta}{(1 - z_1 b(z_2))(1 - z_1 d(z_2))} \sum_{h=-\infty}^{+\infty} z_1^h$$

$$= L_\delta z_2^2 (1 + [z_1(b(z_2) + d(z_2)) - z_1^2 b(z_2) d(z_2)] + [\cdots\cdots]^2 + \cdots) \sum_{h=-\infty}^{+\infty} z_1^h$$

and the steady state DO distribution by the series expansion of the rational function

$$\frac{L_\delta z_2^2}{(1 - b(z_2))(1 - d(z_2))} \tag{4.10}$$

The long term behavior of the above steady-state distributions is determined by the root locations of the polynomials $1 - b(z_2)$ and $1 - d(z_2)$. Stability issues, in particular, are connected with root locations w.r. to the unitary complex circle. These roots are therefore of special interest to the analyst.

The roots of

$$1 - b(z_2) = (1 - D_\beta \Delta t) - (1 - \bar{a}_1 \Delta t) z_2 - D_\beta \Delta t z_2^2$$

are clearly the same as of

$$1 - b_1 z_2 - b_2 z_2^2 := 1 - \frac{1 - \bar{a}_1 \Delta t}{1 - D_\beta \Delta t} z_2 - \frac{D_\beta \Delta\, t}{1 - D_\beta \Delta t} z_2^2 \tag{4.11}$$

It is well known [Mullis and Roberts 1987] that $1 - b(z_2)$ is a stable polynomial if and only if

$$1 + b_2 > 0 \quad \text{and} \quad -(1 - b_2) < -b_1 < 1 - b_2$$

Since $D_\beta \Delta t$ is negligible w.r. to 1, the first condition above

$$1 > \frac{D_\beta \Delta t}{1 - D_\beta \Delta t}$$

is clearly fulfilled. The second condition reduces to

$$-1 + 2D_\beta \Delta t < -1 + \bar{a}_1 \Delta t < 1 - 2D_\beta \Delta t \tag{4.12}$$

The right hand inequality above holds, since $\bar{a}_1 \Delta t \ll 1$. On the other hand, rewriting the left hand side of (4.12) as follows

$$2D_\beta \Delta t < a_1 \Delta t + 2D_\beta \Delta t$$

one sees that the inequality holds true, because a_1 is positive. Therefore $1 - b(z_2)$ and, by similar reasonings, $1 - d(z_2)$ are stable polynomials. We conclude that, according to our physical intuition, stationary distributions of BOD concentration and DO deficit converge to zero as l goes to infinity.

We aim here to make a comparison of BOD and DO steady-state regimes, with and without diffusion. In order to get a detailed information on the shapes of BOD and DO distributions along the river stretch, it is convenient to introduce first a partial fraction expansion of (4.9) and (4.10), and then to expand each fraction into a geometric power series.

4.2.1. BOD Distribution. Rewrite (4.9) as follows

$$X_2^{(\beta)}(z_2) = \frac{G_\beta z_2}{1 - b_1 z_2 - b_2 z_2^2} \tag{4.13}$$

with

$$G_\beta = \frac{L_\beta}{1 - D_\beta \Delta t}, \quad b_1 = \frac{1 - \bar{a}_1 \Delta t}{1 - D_\beta \Delta t}, \quad b_2 = \frac{D_\beta \Delta t}{1 - D_\beta \Delta t} \tag{4.14}$$

Moreover, let

$$\mu_1 := 1 - a_1 \Delta t,$$

$$\bar{\mu}_1 := \frac{2}{\frac{b_1}{b_2}\left[-1 + \sqrt{1 + 4\frac{b_2}{b_1^2}}\right]}, \quad \bar{\nu}_1 := \frac{2}{\frac{b_1}{b_2}\left[1 + \sqrt{1 + 4\frac{b_2}{b_1^2}}\right]} \tag{4.15}$$

If $b_2 = 0$, and therefore polluting materials do not undergo diffusion, the denominator of (4.13) is the first order polynomial $1 - \mu z_2$. Otherwise, the denominator of (4.13) factorizes as $(1 - \bar{\mu}_1 z_2)(1 - \bar{\nu}_1 z_2)$.

Our first concern is to show that, if D_β is small enough, then μ_1 is smaller than $\bar{\mu}_1$. Using the binomial series expansion, one gets form (4.15)

$$\frac{1}{\bar{\mu}_1} = \frac{1}{b_1} - 2\frac{b_2}{b_1^3} + 4\frac{b_2^3}{b_1^6} + \dots$$

On the other hand we have

$$\frac{1}{b_1} = \frac{1 - D_\beta \Delta t}{\mu_1\left[1 - 2\dfrac{D_\beta \Delta t}{\mu_1}\right]} = \frac{1 - D_\beta \Delta t}{\mu_1}\left[1 + 2\frac{D_\beta \Delta t}{\mu_1} + 4\frac{(D_\beta \Delta t)^2}{\mu_1^2} + \dots\right]$$

so that

$$\frac{1}{\mu_1} = \frac{1}{b_1} - \frac{1}{\mu_1}\left[D_\beta \Delta t\left(\frac{2}{\mu_1} - 1\right) + \text{h.o.terms}\right]$$

Since μ_1, b_1, $1 - D_\beta \Delta t$ are very close to 1, the difference

$$\frac{1}{\mu_1} - \frac{1}{\bar{\mu}_1} = D_\beta \Delta t\left[-\frac{2}{\mu_1^2} + \frac{1}{\mu_1} + \frac{2}{1 - D_\beta \Delta t}\frac{1}{b_1^3}\right] + \text{h.o.terms} \tag{4.17}$$

is positive if D_β is small enough.

Consider now the partial fraction expansion of (4.13)

$$X_2^{(\beta)}(z_2) = z_2\left(\frac{G_{\beta\mu}}{1 - z_2\bar{\mu}_1} + \frac{G_{\beta\nu}}{1 + z_2\bar{\nu}_1}\right) \tag{4.18}$$

with

$$G_{\beta\mu} = \frac{G_\beta}{1 + \dfrac{\bar{\nu}_1}{\bar{\mu}_1}}, \quad G_{\beta\nu} = \frac{G_\beta}{1 + \dfrac{\bar{\mu}_1}{\bar{\nu}_1}}$$

Geometric series expansions give

$$X_2^{(\beta)} = G_{\beta\mu} z_2 \sum_{i=0}^{+\infty} z_2^i \bar{\mu}_1^i + G_{\beta\nu} z_2 \sum_{i=0}^{+\infty} (-1)^i z_2^i \bar{\nu}_1^i$$

so that the BOD concentration at the abscissa $(i + 1)\Delta l$ is

$$\beta(\cdot, (i + 1)\Delta l) = G_{\beta\mu}\, \bar{\mu}_1^i + G_{\beta\nu}(-1)^i \bar{\nu}_1^i \tag{4.18}$$

which reduces to

$$\beta(\cdot, (i + 1)\Delta l) = G_{\beta}\mu_1^i \tag{4.19}$$

when BOD does not undergo diffusion phenomena.

Some interesting conclusion can be drawn from these simple calculations. First, since $G_{\beta\mu} \gg G_{\beta\nu}$, the BOD regime with diffusion can be viewed as represented by a decreasing geometric sequence with ratio $\bar{\mu}_1$, perturbed by an oscillatory term, whose amplitude, infinitesimal as i goes to infinity, is everywhere negligibly small.

Second, when the model does not incorporate diffusion, the above geometric sequence converges to zero more rapidly (since $\mu_1 < \bar{\mu}_1$) and the oscillatory perturbation disappears.

4.2.2. DO Distribution. The comparison of the DO regimes with and without diffusion requires to introduce some more notations. Rewrite first (4.10) as

$$X_2^{(\delta)}(z_2) = \frac{G_\delta z_2^2}{(1 - b_1 z_2 - b_2 z_2^2)(1 - d_1 z_2 - d_2 z_2^2)} \tag{4.20}$$

with

$$G_\delta = \frac{L_\delta}{(1 - D_\beta \Delta t)(1 - D_\delta \Delta t)}, \quad d_1 = \frac{1 - \bar{a}_2 \Delta t}{1 - D_\delta \Delta t}, \quad d_2 = \frac{D_\delta \Delta t}{1 - D_\delta \Delta t} \tag{4.21}$$

and let

$$\mu_2 := 1 - a_2 \Delta t,$$

$$\bar{\mu}_2 := \frac{2}{\dfrac{d_1}{d_2}\left[-1 + \sqrt{1 + 4\dfrac{d_2}{d_2^1}}\right]}, \quad \bar{\nu}_2 := \frac{2}{\dfrac{d_1}{d_2}\left[1 + \sqrt{1 + 4\dfrac{d_2}{d_1^2}}\right]} \tag{4.22}$$

Suppose now that the diffusion phenomena are negligible. Then we have $b_2 = d_2 = 0$ and the denominator of (4.20) is a second order polynomial

$$(1 - \mu_1 z_2)(1 - \mu_2 z_2) \tag{4.23}$$

In that case the partial fraction expansion of (4.20) is

$$X_2^{(\delta)}(z_2) = \frac{G_\delta z_2^2}{\mu_1 - \mu_2} \left(\frac{\mu_1}{1 - z_2\mu_1} - \frac{\mu_2}{1 - z_2\mu_2} \right) \tag{4.24}$$

and the DO deficit at the abscissa $(i + 2)\Delta l$ is:

$$\delta(\cdot, (i + 2)\Delta l) = G_\delta \frac{\mu_1^{i+1} - \mu_2^{i+1}}{\mu_1 - \mu_2} \tag{4.25}$$

If (4.25) is viewed as a function of the real variable i, its maximum value is attained when

$$\mu_1^{i+l} \ln \mu_1 = \mu_2^{i+1} \ln \mu_2$$

Denoting by $\lfloor x \rfloor$ the integer part of the real number x, the maximum value of the sequence (4.25) is attained either at

$$i_M = \left\lfloor \ln \left(\frac{\ln \mu_2}{\ln \mu_1} \right) \frac{1}{\ln \frac{\mu_1}{\mu_2}} \right\rfloor \tag{4.26}$$

or at $i_M + 1$. Note that i_M is a nonnegative number, as we may expect from the physical assumption that the river is perfectly aerated at the abscissa $l = 0$ and therefore any pollutants injection will increase the DO deficit at the subsequent river reaches, until the bacterial oxidative processes are balanced by the natural rearation. This leads to the important qualitative conclusion that the DO behavior given by (4.25) is a discrete analogue of the sag profile of DO concentration in the continuous Street-Phelps model.

Finally, suppose that diffusion is not negligible, so that the denominator of (4.20) factorizes into the following factors:

$$(1 - \bar{\mu}_1 z_2)(1 + \bar{\nu}_1 z_2)(1 - \bar{\mu}_2 z_2)(1 + \bar{\nu}_2 z_2) \tag{4.27}$$

where $\bar{\mu}_1 < \mu_2$ and $\bar{\mu}_2 < \mu_2$ if the values of D_β and D_δ are small enough. In the partial fraction expansion of (4.20) several cases should be considered, depending on the possibility of multiple roots of the denominators. Here we confine ourselves to the case when there exist only simple roots, i.e. when

$$\frac{b_1}{b_2} - \frac{d_1}{d_1} \neq \pm \left[\frac{b_1}{b_2} \sqrt{1 + 4\frac{b_2}{b_1^2}} - \frac{d_1}{d_2} \sqrt{1 + 4\frac{d_2}{d_1^2}} \right] \tag{4.28}$$

Keeping this restriction in force, we have

$$X_2^{(\delta)}(z_2) = z_2^2 \left[\frac{G_{\delta\mu_1}\bar{\mu}_1}{1 - z_2\bar{\mu}_1} - \frac{G_{\delta\mu_2}\bar{\mu}_2}{1 - z_2\bar{\mu}_2} + \frac{G_{\delta\nu_1}\bar{\nu}_1^3}{1 + z_2\bar{\nu}_1} - \frac{G_{\delta\nu_2}\bar{\nu}_2}{1 + z_2\bar{\nu}_2} \right] \tag{4.29}$$

with

$$G_{\delta\mu_1} = \frac{G_\delta}{(\bar{\mu}_1 - \bar{\mu}_2)\left(1 + \dfrac{\bar{\nu}_1}{\bar{\mu}_1}\right)\left(1 + \dfrac{\bar{\nu}_2}{\bar{\mu}_1}\right)},$$

$$G_{\delta\mu_2} = \frac{G_\delta}{(\bar{\mu}_1 - \bar{\mu}_2)\left(1 + \dfrac{\bar{\nu}_1}{\bar{\mu}_2}\right)\left(1 + \dfrac{\bar{\nu}_2}{\bar{\mu}_2}\right)}$$

$$G_{\delta\nu_1} = \frac{-G_\delta}{(\bar{\nu}_1 - \bar{\nu}_2)\left(1 + \dfrac{\bar{\nu}_1}{\bar{\mu}_1}\right)\left(1 + \dfrac{\bar{\nu}_2}{\bar{\mu}_1}\right)} \frac{1}{\bar{\mu}_1\bar{\mu}_2},$$

$$G_{\delta\nu_2} = \frac{-G_\delta}{(\bar{\nu}_1 - \bar{\nu}_2)\left(1 + \dfrac{\bar{\nu}_2}{\bar{\mu}_1}\right)\left(1 + \dfrac{\bar{\nu}_2}{\bar{\mu}_2}\right)} \frac{1}{\bar{\mu}_1\bar{\mu}_2}$$

Since $\bar{\nu}_i/\bar{\mu}_j$ are negligible w.r. to 1, we introduce the approximations

$$G_{\delta\mu_1} \approx G_{\delta\mu_2} = \frac{G_\delta}{\bar{\mu}_1 - \bar{\mu}_2}, \quad G_{\delta\nu_1} \approx G_{\delta\nu_2} \approx -\frac{G_\delta}{\bar{\nu}_1 - \bar{\nu}_2} \frac{1}{\bar{\mu}_1\bar{\mu}_2}$$

that give the partial fraction expansion (4.29) a simpler structure

$$\begin{aligned} X_2^{(\delta)}(z_2) \approx z_2^2 \Bigg[& \frac{G_\delta}{\bar{\mu}_1 - \bar{\mu}_2}\left(\frac{\bar{\mu}_1}{1 - z_2\bar{\mu}_1} - \frac{\bar{\mu}_2}{1 - z_2\bar{\mu}_2}\right) \\ & - \frac{G_\delta}{\bar{\mu}_1\bar{\mu}_2(\bar{\nu}_1 - \bar{\nu}_2)}\left(\frac{\bar{\nu}_1^3}{1 + z_2\bar{\nu}_1} - \frac{\bar{\nu}_2^3}{1 + z_2\bar{\nu}_2}\right)\Bigg] \end{aligned} \tag{4.30}$$

and express the DO deficit at the abscissa $(i + 2)\Delta l$ as

$$\delta(\cdot, (i + 2)\Delta l) = G_\delta \frac{\bar{\mu}_1^{i+1} - \bar{\mu}_2^{i+1}}{\bar{\mu}_1 - \bar{\mu}_2} + (-1)^{i+1} \frac{G_\delta}{\bar{\mu}_1\bar{\mu}_2} \frac{\bar{\nu}_1^{i+3} - \bar{\nu}_2^{i+3}}{\bar{\nu}_1 - \bar{\nu}_2} \tag{4.31}$$

Eqn. (4.31) is very convenient for a quick discussion of the steady-state DO profile. First of all, the second term constitutes and oscillatory perturbation of the first, that goes to zero a $i \to \infty$. The negligibility of this perturbation is due to the fact that $\bar{\mu}_1$ and $\bar{\mu}_2$ are approximately 1; so for all i's we have

$$\frac{G_\delta}{\bar\mu_1\bar\mu_2}\frac{\bar\nu_1^{i+3}-\bar\nu_2^{i+3}}{\bar\nu_1-\bar\nu_2} \approx G_\delta\left[\bar\nu_1^{i+2}+\bar\nu_1^{i+1}\bar\nu_2+\ldots+\bar\nu_1\bar\nu_2^{i+1}+\bar\nu_2^{i+2}\right]$$
$$\leqslant G_\delta\frac{\bar\mu_1^{i+1}-\bar\mu_2^{i+1}}{\bar\mu_1-\bar\mu_2} \approx G_\delta\left[\bar\mu_1^{i}+\bar\mu_1^{i-1}\bar\mu_2+\ldots+\bar\mu_1\bar\mu_2^{i-1}+\bar\mu_2^{i}\right] \tag{4.32}$$

The first term in (4.31) has the same structure of the DO profile in models without diffusion. However, the DO deficit dies out more slowly, which agrees with a parallel result on BOD behavior.

4.3. Modelling Diffusion Velocity

An implicit assumption in equations (4.1)–(4.2) as well as in the corresponding state model (4.3) was that the diffusion velocity is equal to the velocity of the riverstsream.

This is clearly seen from the support of the impulse response shown in Figure 7. A BOD injection at time $t = 0$ on the origin of the space coordinates gives rise, for any t, to a spatially symmetric distribution, in which the maximum point lies on the diagonal of the first orthant (i.e. on the support of the impulse response when diffusion is neglected) and the initial point is steadily at the abscissa Δl. This implies that the backward propagation of the diffusion wavefront exactly balances the advection velocity of the river.

There are several ways for introducing 2-D models where diffusion and advection velocity do not coincide. Here we shall only sketch a sampling of these methods, and outline two conceptually different approaches to the problem.

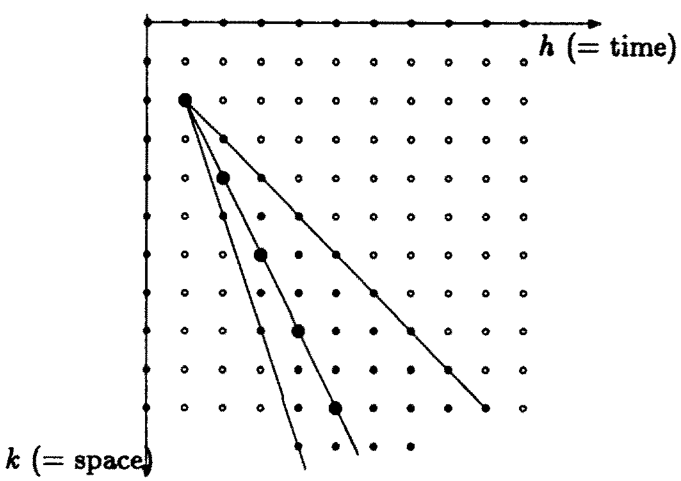

Figure 7.

The first approach is based on the intuitive assumption that more complex dynamics require in general higher order systems to be represented; the second one exploits a suitable reinterpretation of the integers grid $\mathbf{Z} \times \mathbf{Z}$, along the same lines we followed in the third state model of Section 3.1.

For sake of simplicity, we deal here only with BOD diffusion equation. Assume that the elementary volume of water, centered on the abscissa l at time t, attains the abscissa $l + 2\Delta l$ at time $t + \Delta t$, so that the advection velocity is

$$v = \frac{2\Delta l}{\Delta t}.$$

We still keep in force the BOD degradation scheme considered at the beginning of this section, assuming in particular that diffusion in Δt only affects contiguous elementary reaches. Thus BOD updating takes the explicit form

$$\begin{aligned} \beta((h+1)\Delta t, (k+2)\Delta l) &= [1 - a_1\Delta t]\beta(h\Delta t, k\Delta l) \\ &+ D_\beta(\beta(h\Delta t, (k-1)\Delta l) - \beta(h\Delta t, k\Delta l))\Delta t + [1 - a_1\Delta t]M\mathrm{in}_\beta(h\Delta t, k\Delta l) \\ &+ D_\beta(\beta(h\Delta t, (k+1)\Delta l) - \beta(h\Delta t, k\Delta l))\Delta t \end{aligned} \tag{4.33}$$

After introducing the local state vector

$$x_\beta(h, k) = \begin{bmatrix} \beta(h\Delta t, (k-1)\Delta l \\ \beta(h\Delta t, k\Delta l) \\ \beta(h\Delta t, (k+1)\Delta l \end{bmatrix}$$

one gets a second order model with structure (2.3)

$$\begin{aligned} x_\beta(h+1, k+1) = A_0^{(\beta)} x_\beta(h, k) + A_1^{(\beta)} x_\beta(h, k+1) \\ + A_2^{(\beta)} x_\beta(h+1, k) + B^{(\beta)} u_\beta(h, k) \end{aligned} \tag{4.34}$$

with

$$u_\beta(h, k) = \mathrm{in}_\beta(h\Delta t, k\Delta l)$$

$$A_0^{(\beta)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D_\beta\Delta t & 1 - a_1\Delta t - 2D_\beta\Delta t & D_\beta\Delta t \end{bmatrix}$$

$$A_1^{(\beta)} = 0_3 \quad A_2^{(\beta)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Assuming as system output the BOD concentration, i.e.

$$y(h, k) = \beta(h\Delta t, k\Delta l) = [0 \quad 1 \quad 0]\, x_\beta(h, k),$$

the impulse reponse is given by the power series expansion of the following transfer function

$$\begin{aligned} W(z_1, z_2) &= C(I - A_1^{(\beta)} z_1 - A_2^{(\beta)} z_2 - A_0^{(\beta)} z_1 z_2)^{-1} B z_1 z_2 \\ &= \frac{L_\beta z_1 z_2^2}{1 - z_1 z_2 b(z_2)} = L_\beta z_1 z_2^2 \,[1 + z_1 z_2 b(z_2) + z_1^2 z_2^2 b(z_2)^2 + \ldots] \end{aligned} \tag{4.35}$$

where $b(z_2)$ and L_β have been defined in (4.5). The support of (4.35) is represented in figure 8, which clearly shows that the diffusion wavefront progresses with a velocity which is different from (actually, smaller than) the river advection velocity.

The second approach is reminiscent of the philosophy that underlies model (3.11). The interpretation of the grid $\mathbf{Z} \times \mathbf{Z}$ given in figure 4 with reference to model (3.11) is well suited also for representing the diffusion model (4.1). In fact, letting

$$\beta(h\Delta t, k\Delta l) = x_\beta(h - k, k) = x_\beta(a, b)$$

$$\text{in}_\beta(h\Delta t, k\Delta l) = u_\beta(h - k, k) = u_\beta(a, b)$$

equation (4.1) becomes

$$\begin{aligned} x_\beta(h - k, k + 1) &= [1 - \bar{a}_1\Delta t]\, x_\beta(h - k, k) + [1 - a_1\Delta t] M u_\beta(h - k, k) \\ &+ D_\beta \Delta t x_\beta(h - k + 1, k - 1) + D_\beta \Delta t x_\beta(h - k - 1, k + 1) \end{aligned}$$

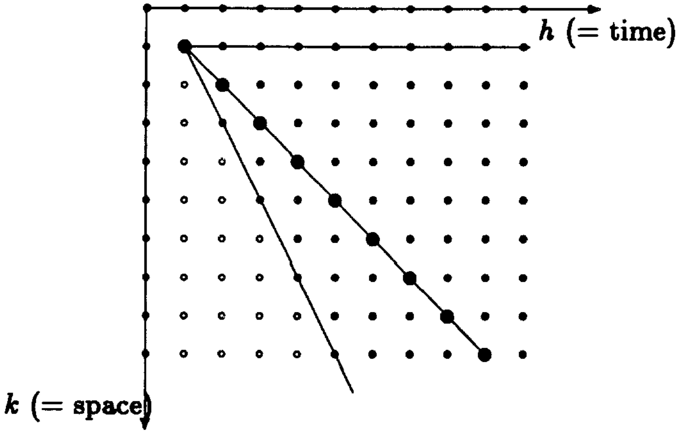

Figure 8.

or equivalently

$$x_\beta(a, b+1) = [1 - \bar{a}_1\Delta t]x_\beta(a, b) + D_\beta\Delta t x_\beta(a+1, b-1) + D_\beta\Delta t x_\beta(a-1, b+1) + [1 - a_1\Delta t]Mu_\beta(a, b)$$

In figure 9 we dashed the causality cone of the point $(a+1, b)$, i.e. the set of points of the discrete plane that contribute to the BOD concentration at $(a+1, b)$.

Consider now the equation (4.33) and associate with the space/time pair $(h\Delta t, k\,\Delta l)$ the point $(a, b) \in \mathbf{Z} \times \mathbf{Z}$, whose (integer) coordinates satisfy

$$a = 2h - k, \qquad b = k - h \tag{4.36}$$

In this way the points of the separation set

$$C_{\bar{h}} = \{(a, b) \mid a + b = \bar{h}\}$$

represent different locations along the river stretch at the same time instant $\bar{h}\Delta t$, while the points of the set

$$\tau_{\bar{k}} = \{(a, b) \mid a + 2b = \bar{k}\}$$

correspond to the location $\bar{k}\Delta l$ and to different time instants.

Letting

$$\begin{aligned}\beta(h\Delta t, k\Delta l) &= x_\beta(2h - k, k - h) = x_\beta(a, b)\\ u_\beta(h\Delta t, k\Delta l) &= u_\beta(2h - k, k - h = u_\beta(a, b)\end{aligned} \tag{4.37}$$

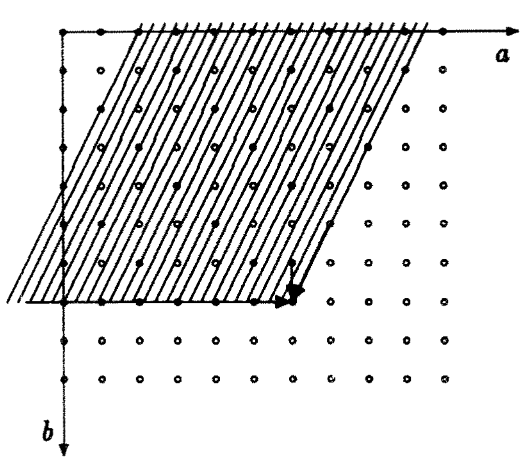

Figure 9.

(4.33) becomes now

$$x_\beta(2h - k, k - h + 1) = [1 - \bar{a}_1\Delta t]x_\beta(2h - k, k - h) + D_\beta\Delta t x_\beta(2h - k + 1, k - h - 1) + D_\beta\Delta t x_\beta(2h - k - 1, k - h + 1) + [1 - a_1\Delta t]Mu_\beta(2h - k, k - h)$$

or, equivalently

$$x_\beta(a, b + 1) = [1 - \bar{a}_1\Delta t]x_\beta(a, b) + D_\beta\Delta t x_\beta(a + 1, b + 1) + D_\beta\Delta t x_\beta(a - 1, b + 1) + [1 - a_1\Delta t]Mu_\beta(a, b) \quad (4.38)$$

Clearly system (4.38) only constitutes a reformulation of (4.33), where the maximum delay associated with the elementary updating steps amounts to 3, instead of 4, and initial conditions can be arbitrarily assigned along a diagonal separation set $C_{\bar{h}}$ instead of a vertical line h = const. A possible advantage of (4.38) is that the equations provide a 2-D weakly causal system in *standard* form (i.e. the local state at any point depends on a finite number of local states along the closest diagonal line that precedes that point).

Causal state space models like (4.34), or first order models with structure (2.4) that could be obtained thereout, cannot be derived without paying some price in terms of state space dimension. On the other hand, using causal state models makes it available the whole body of standard 2-D theory in control and state estimation.

5. Conclusions

This paper makes a first attempt to introduce 2-D systems methods in the analysis of the selfpurification process of a river. Several models have been considered, that take into account physical phaenomena of increasing complexity; it is clear, however, that our investigation is still far from complete.

Further investigations should take into account more refined models and their capability of coping with a larger class of phaenomena, such as sedimentation, photosynthesis, etc. Nonetheless, 2-D theory already provides a fairly large amount of results on state reconstruction and feedback control. It is to be hoped that the application of these results to pollution monitoring and control leads eventually to new interesting problems in 2-D systems and indicates directions of further research also in the theoretical field.

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