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ON THE DEGREE OF APPROXIMATION OF A CLASS OF FUNCTIONS BY MEANS OF FOURIER SERIES

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1. Definitions and notations. Let f be 2π-periodic and L-integrable on $[-\pi, \pi]$. The Fourier series associated with f at the point x, is given by

$$\frac{1}{2} a_0 + \sum_{n=1}^{\alpha} (a_n \cos nx + b_n \sin nx). \tag{1.1}$$

A function $f \in \mathrm{Lip}\, \alpha$ $(\alpha > 0)$ if

$$f(x+h) - f(x) = O(|h|^\alpha) \quad (h \to 0) \tag{1.2}$$

and if f is defined on $[-\pi, \pi]$ then the expression

$$\omega(\delta) = \omega(\delta, f) = \sup_{x_1, x_2} |f(x_1) - f(x_2)|, \quad |x_1 - x_2| \leqq \delta \tag{1.3}$$

is called the modulus of continuity of f (Zygmund [5], p. 42).

Let $A = (a_{n,k})$ $(k, n = 0, 1, \ldots)$ be a lower-triangular infinite matrix of real numbers. We denote by $T_n(f)$ the A-transform of the Fourier series of f given by

$$T_n(f;\, x) = \sum_{k=0}^{n} a_{nk} s_k(x) \quad (n = 0, 1, \ldots), \tag{1.4}$$

where $s_n(x)$ is the n-th partial sum of the series (1.1).

Suppose $A = (a_{nk})$ is defined as follows:

$$a_{nk} = \begin{cases} p_k / P_n; & 0 \leqq k \leqq n \\ 0; & k > n, \end{cases} \tag{1.5}$$

where (p_k) is non-negative and that $P_n = p_0 + p_1 + \ldots + p_n \neq 0$ $(n \geqq 0)$. Then the matrix is called Riesz matrix and the means are called Riesz-means or (R, p_n)-means. In this case we write $R_n(f; x)$ for $T_n(f; x)$. Also if

$$a_{nk} = \begin{cases} p_{n-k} / P_n; & 0 \leqq k \leqq n \\ 0; & k > n, \end{cases} \tag{1.6}$$

The matrix (a_{nk}) is called Nörlund matrix and in this case we write $N_n(f; x)$ for $T_n(f; x)$. Throughout (a_{nk}) will denote a lower triangular infinite matrix.

We use the following notations in this paper:

(1.7) $$\Phi_x(t) = \frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\},$$

(1.8) $$b_{nk} = \sum_{r=0}^{k} a_{nr}; \quad b_{nk} = b_n(k),$$

(1.9) $\tau = [\pi/t]$, the integral part of k/t in $0 < t \leqq \pi$,

(1.10) $C^*[0, 2\pi]$, the space of all 2π-periodic continuous functions defined on $[0, 2\pi]$.

Throughout, the norm $\|\cdot\|$ will be the sup norm on $0 \leqq x \leqq 2\pi$ and $\omega(t)$ will be the modulus of continuity of $f \in C^*[0, 2\pi]$.

2. Introduction. By employing Riesz matrix, we [1] obtained the following result concerning the degree of approximation:

THEOREM A. *Let $f \in C^*[0, 2\pi]$ and let $f \in \mathrm{Lip}\,\alpha$ $(0<\alpha\leqq 1)$. Then the degree of approximation of f by (R, p_n)-means of its Fourier series is given by*

$$\|R_n(f)-f\| = \begin{cases} O\{(p_n/P_n)^\alpha\}; & 0<\alpha<1 \\ O\{(p_n/P_n)\log(P_n/p_n)\}; & \alpha = 1, \end{cases}$$

where (p_n) is positive and non-decreasing with $n \geqq n_0$.

Recently this result was extended to the lower triangular matrix in the Hölder metric (see [4]).

In this paper we first extend Theorem A by using the modulus of continuity of *f in the following form:*

THEOREM 1. *Let (a_{nk}) satisfy the following conditions:*

(2.1) $$a_{nk} \geqq 0 \quad (n, k = 0, 1, \ldots), \quad \sum_{k=0}^{n} a_{nk} = 1,$$

(2.2) $$a_{nk} \leqq a_{n,k+1} \quad (k = 0, 1, \ldots, n-1, n = 0, 1, \ldots).$$

Suppose $\omega(t)$ is such that

(2.3) $$\int_u^\pi t^{-2}\omega(t)\,dt = O(H(u)) \quad (u \to 0+),$$

where $H \geqq 0$ and that

(2.4) $$tH(t) = o(1) \quad (t \to 0+)$$

and

(2.5) $$\int_0^t H(u)\,du = O\{tH(t)\} \quad (t \to 0+).$$

Then

$$\|T_n(f)-f\| = O\{a_{nn}H(a_{nn})\}. \tag{2.6}$$

We also prove

THEOREM 2. *Let* (a_{nk}) *satisfy* (2.1) *and* (2.2) *and let* $\omega(t)$ *satisfy* (2.3). *Then*

$$\|T_n(f)-f\| = O\{\omega(\pi/n)\}+O\{a_{nn}H(\pi/n)\}, \tag{2.7}$$

where H is non-negative. If, in addition to (2.3), $\omega(t)$ *satisfies* (2.5) *then*

$$\|T_n(f)-f\| = O\{a_{nn}H(\pi/n)\}. \tag{2.8}$$

Lastly, we intend to investigate some results, one of which is analogous to Theorem A in the case when (p_n) is non-negative and non-increasing. In fact, we first obtain general results for a triangular matrix by using the modulus of continuity of f from which the desired results may be obtained. Precisely, we prove the following:

THEOREM 3. *Let* (a_{nk}) *satisfy* (2.1) *and let*

$$a_{nk} \geqq a_{n,k+1} \quad (k = 0, 1, \ldots, n-1,\ n = 0, 1, \ldots). \tag{2.9}$$

Then

$$\|T_n(f)-f\| = O\left\{\omega(\pi/n)+\sum_{k=1}^{n} k^{-1}\omega(\pi/k)\, b_n(k+1)\right\}. \tag{2.10}$$

THEOREM 4. *Let* (a_{nk}) *satisfy* (2.1) *and* (2.9) *and let* $\omega(t)$ *satisfy* (2.3), (2.4) *and* (2.5). *Then*

$$\|T_n(f)-f\| = O\{a_{n0}H(a_{n0})\}. \tag{2.11}$$

3. We shall use the following lemmas in the proof of the theorems:

LEMMA 1. *Let* $\omega(t)$ *satisfy* (2.3), (2.4) *and* (2.5). *Then*

$$\int_0^r t^{-1}\omega(t)\,dt = O\{rH(r)\} \quad (r \to 0+).$$

PROOF. Integrating by parts, we have

$$\int_0^r t^{-1}\omega(t)\,dt = \left[-t\int_t^\pi u^{-2}\omega(u)\,du\right]_0^r+\int_0^r dt\int_t^\pi u^{-2}\omega(u)\,du =$$

$$= O\{rH(r)\}+O(1)\int_0^r H(t)\,dt = O\{rH(r)\},$$

by (2.3), (2.4) and (2.5).

This completes the proof of the lemma.

LEMMA 2. *Let* (a_{nk}) *satisfy* (2.9) *and let* $a_{nk}\geqq 0$ $(n, k=0, 1, \ldots)$. *Then, uniformly in* $0<t\leqq\pi$,

$$\sum_{k=0}^{n} a_{nk}\sin\left(k+\frac{1}{2}\right)t = O\{b_n(\tau)\}.$$

PROOF. Since $a_{nk} \geqq 0$, we have by Abel's lemma

$$\left|\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t\right| \leqq \sum_{k=0}^{\tau} a_{nk} + \left|\sum_{\tau}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t\right| =$$

$$= b_n(\tau) + O\{\tau a_{n\tau}\} = O\{b_n(\tau)\},$$

by (2.9). This completes the proof of the lemma.

4. In this section, we shall prove the theorems mentioned in Section 2.

PROOF OF THEOREM 1. We have

$$T_n(f;\ x) - f(x) = \sum_{k=0}^{n} a_{nk} s_k(x) - f(x) =$$

$$= \frac{2}{\pi} \int_0^{\pi} \left\{\{\Phi_x(t)/\left(2 \sin \frac{1}{2} t\right)\right\} \left\{\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right) t\right\} dt,$$

by (2.1). Now we observe that $\|\Phi(t)\| \leqq \omega(t)$, therefore

$$\|T_n(f-f)\| \leqq \frac{2}{\pi} \int_0^{\pi} \frac{\omega(t)}{2 \sin \frac{1}{2} t} \left|\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right) t\right| dt = \tag{4.1}$$

$$= \frac{2}{\pi} \left(\int_0^{a_{nn}} + \int_{a_{nn}}^{\pi}\right) = I_1 + I_2, \quad \text{say.}$$

However, by (2.1), the sum in the integral does not exceed 1 and hence

$$I_1 = O(1) \int_0^{a_{nn}} t^{-1} \omega(t)\, dt = O\{a_{nn} H(a_{nn})\},$$

by Lemma 1. Also, by (2.2) and Abel's lemma

$$I_2 = O(a_{nn}) \int_{a_{nn}}^{\pi} t^{-2} \omega(t) dt = O\{a_{nn} H(a_{nn})\},$$

by (2.3).

Combining I_1 and I_2, we get (2.6) and this completes the proof of the theorem.

PROOF OF THEOREM 2. We have from (4.1)

$$\|T_n(f) - f\| \leqq \frac{2}{\pi} \int_0^{\pi} \frac{\omega(t)}{2 \sin \frac{1}{2} t} \left|\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right) t\right| dt =$$

$$= \frac{2}{\pi} \left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right) = I_1 + I_2, \quad \text{say.}$$

Using the inequality $\left|\sin\left(k+\frac{1}{2}\right)t\right| \leqq \left(k+\frac{1}{2}\right)t$ and (2.1), we get

$$I_1 = O(n)\int_0^{\pi/n} \omega(t)dt = O\{\omega(\pi/n)\}.$$

Also, by (2.2) and Abel's lemma

$$I_2 = O\{a_{nn}H(\pi/n)\}.$$

Combining I_1 and I_2, we get (2.7).

For the estimate (2.8), we first observe that

$$I_1 = O(n)\int_0^{\pi/n} \omega(t)dt.$$

Now integrating by parts and using (2.3), (2.5) we get

$$\int_0^{\pi/n} \omega(t)dt = \left[-t^2\int_t^{\pi} (\omega(u)/u^2)du\right]_0^{\pi/n} + \int_0^{\pi/n} 2t\,dt\int_t^{\pi} u^{-2}\omega(u)du =$$

$$= O\left\{n^{-2}H(\pi/n) + \int_0^{\pi/n} tH(t)\,dt\right\} = O\{n^{-2}H(\pi/n)\}.$$

Hence

$$I_1 = O\{n^{-1}H(\pi/n)\}.$$

And proceeding as in I_2 above, we get

$$I_2 = O\{a_{nn}H(\pi/n)\}.$$

However, by (2.2), $\{a_{nk}\}_{k=0}^{n}$ is non-decreasing and hence

$$(n+1)a_{nn} \geqq \sum_{k=0}^{n} a_{nk} = 1,$$

by (2.1). Thus using the inequality $n^{-1} = O(a_{nn})$ in I_1 and combining it with I_2 we get (2.8).

This completes the proof of Theorem 2.

PROOF OF THEOREM 3. Proceeding as in Theorem 2, we get

$$\|T_n(f)-f\| \leqq I_1 + I_2,$$

where

$$I_1 = O\{\omega(\pi/n)\}$$

and

$$I_2 = \frac{2}{\pi}\int_{\pi/n}^{\pi} \frac{\omega((t)}{2\sin\frac{1}{2}t}\left|\sum_{k=0}^{n} a_{nk}\sin\left(k+\frac{1}{2}\right)t\right| dt.$$

By Lemma 2, we get

$$I_2 = O(1) \int_{\pi/n}^{\pi} t^{-1}\omega(t) b_n([\pi/t])\,dt = O(1) \sum_{k=1}^{n-1} \int_{\pi/(k+1)}^{\pi/k} t^{-1}\omega(t) b_n([\pi/t])\,dt =$$

$$= O(1) \sum_{k=1}^{n-1} \omega(\pi/k) \int_{\pi/(k+1)}^{\pi/k} t^{-1} b_n([\pi/t])\,dt = O(1) \sum_{k=1}^{n-1} \omega(\pi/k) b_n(k+1) k^{-1}.$$

Thus combining I_1 and I_2, we get the required result and hence the proof of the theorem is complete.

PROOF OF THEOREM 4. Splitting up the integral in (4.1) into the sub-integrals $\int_0^{a_{n0}}$ and $\int_{a_{n0}}^{\pi}$ and proceeding as in Theorem 1, the proof of the theorem may be completed.

5. In this section, we specialize the matrix $A=(a_{nk})$ to obtain corollaries of the theorems.

By (1.5), we get the following corollary from Theorem 1:

COROLLARY 1. *Let* $\omega(t)$ *satisfy* (2.3), (2.4) *and* (2.5) *and let* (p_n) *be non-negative and non-decreasing. Then*

$$\|R_n(f)-f\| = O\{(p_n/P_n) H(p_n/P_n)\}.$$

If $f\in \mathrm{Lip}\,\alpha$ $(0<\alpha\leqq 1)$, then $\omega(t)=O(t^{\alpha})$ $(0<\alpha\leqq 1)$ and

$$H(u) = \begin{cases} \log(\pi/u) & \alpha = 1 \\ u^{\alpha-1} & 0 < \alpha < 1. \end{cases}$$

Hence Theorem A is a particular case of Corollary 1.

It is interesting to note that one can get the estimate of Corollary 1 by using Nörlund matrix (see (1.6)), in place of Riesz matrix. On setting $a_{nk}=p_{n-k}/P_n$ in Theorem 4, we get

COROLLARY 2. *Let* $\omega(t)$ *and* (p_n) *be as defined in Corollary* 1. *Then*

$$\|N_n(f)-f\| = O\{(p_n/P_n) H(p_n/P_n)\}.$$

Now we give the following corollary from Theorem 3:

COROLLARY 3. *The degree of approximation of* $f\in C^*[0, 2\pi]$ *by the* (R, p_n)-*means of Fourier series of* f *is given by*

$$\|R_n(f)-f\| = O\Big\{(P_n)^{-1} \sum_{k=1}^{n} k^{-1} P_k \omega(\pi/k)\Big\}, \tag{5.1}$$

where (p_n) *is non-negative and non-increasing.*

PROOF. We have, by (1.5),

$$b_n(k+1) = \sum_{r=0}^{k+1} a_{nr} = P_{k+1}/P_n.$$

However (p_n) is non-increasing therefore

$$b_n(k+1) = O(P_k/P_n)$$

and $(k^{-1}P_k)$ is non-increasing and hence

$$\omega(\pi/n) \leqq (P_n)^{-1} \sum_{k=1}^{n} \omega(\pi/k)\, k^{-1} P_k.$$

Using these estimates in (2.10), we get the required result.

It is interesting to note that the estimate in (5.1) was earlier obtained in [3] by using Nörlund matrix as defined by (1.6), where (p_n) is defined as in Corollary 3.

Since $f \in \mathrm{Lip}\, \alpha$ implies that $\omega(t) = O(t^\alpha)$, we deduce the following corollary from Corollary 3:

COROLLARY 4. *Let $f \in C^*[0, 2\pi]$ and let $f \in \mathrm{Lip}\, \alpha$ $(0 < \alpha \leqq 1)$. Then the degree of approximation of f by (R, p_n)-means of its Fourier series is given by*

$$\|R_n(f) - f\| = O\Big\{(1/P_n) \sum_{k=1}^{n} k^{-1-\alpha} P_k\Big\},$$

where (p_n) is non-increasing and non-negative.

Once again, the estimate in Corollary 4 was earlier obtained in [2] in the case of Nörlund matrix generated by non-negative and non-increasing sequence (p_n).

References

[1] P. Chandra, On the degree of approximation of functions belonging to the Lipschitz class, *Nanta Math.*, **8** (1975), 88—91.

[2] D. S. Goel and B. N. Sahney, On the degree of approximation of continuous functions, *Ranchi University Math. Jour.*, **4** (1973), 50—53.

[3] A. B. S. Holland, B. N. Sahney and J. Tzimbalario, On degree of approximation of a class of functions by means of Fourier series, *Acta Sci. Math. (Szeged)*, **38** (1976), 69—72,

[4] R. N. Mohapatra and P. Chandra, Degree of approximation of functions in the Hölder metric, *Acta Math. Hung.*, **41** (1983), 67—76.

[5] A. Zygmund, *Trigonometric series*, Vol. I, Cambridge University Press (Cambridge, 1968).

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