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STRONG CONVERGENCE OF CERTAIN MEANS WITH RESPECT TO THE WALSH—FOURIER SERIES

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1. Introduction. It is known [1] that the Walsh—Paley system is not a Schauder basis in $L^1[0, 1]$. Moreover, there exists a function in the (dyadic) Hardy space $H^1[0, 1]$, the partial sums of which are not bounded in $L^1[0, 1]$. In this article we shall prove that some means of the L^1-norms of these partial sums can be convergent for all elements of $H^1[0, 1]$. For the trigonometric analogue of this statement see the work of B. Smith [5]. (In the proof we follow his method.) The sharpness of our theorem is also investigated.

2. We recall briefly some notations and definitions. First of all denote w_n $(n=0, 1, \ldots)$ the n-th Walsh—Paley function, i.e. let

$$w_1(t) := \begin{cases} 1 & (0 \leqq t < 1/2) \\ -1 & (1/2 \leqq t < 1), \end{cases} \quad w_1(t) = w_1(t+1) \quad \text{(for all real } t\text{)}$$

and

$$w_{2^n}(t) := w_1(2^n t) \quad (0 \leqq t \leqq 1, \ n = 0, 1, \ldots).$$

If $n = \sum_{i=0}^{\infty} n_i 2^i$ $(n_i = 0, 1)$ is the dyadic representation of $n = 0, 1, \ldots$ then let

$$w_n := \prod_{k=0}^{\infty} w_{2^k}^{n_k}.$$

It is well-known that $(w_n, n=0, 1, \ldots)$ is a complete orthonormal system. (For more details see e.g. [1].) For $f \in L^1 := L^1[0, 1]$ let $\hat{f}(n)$ be the n-th Walsh—Fourier coefficient of f, i.e.

$$\hat{f}(n) := \int_0^1 f w_n \quad (n = 0, 1, \ldots).$$

Furthermore, we denote by D_n the n-th Dirichlet kernel with respect to $(w_n, n= =0, 1, \ldots)$:

$$D_n := \sum_{k=0}^{n-1} w_k \quad (n = 0, 1, \ldots).$$

Later we shall often use the following assertions (see [4]):

(1) $$D_n = w_n \sum_{k=0}^{\infty} n_k w_{2^k} D_{2^k} \quad (n = \sum_{k=0}^{\infty} n_k 2^k = 0, 1, \ldots),$$

(2) $$D_{2^k}(t) = \begin{cases} 2^k & (0 \leqq t < 2^{-k}) \\ 0 & (2^{-k} \leqq t < 1) \end{cases} \quad (k = 0, 1, \ldots).$$

The so-called Hardy space plays an important part in the further investigations. Let $Q(f)$ be the quadratic variation of $f \in L^1$, i.e.

$$Q(f) := \Big(\sum_{n=0}^{\infty} (S_{2^{n+1}}(f) - S_{2^n}(f))^2\Big)^{1/2},$$

where

$$S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k) w_k \quad (n = 1, 2, \ldots).$$

Then the space $H^1 := H^1[0, 1]$ is defined by $H^1 := \{f \in L^1 \colon Q(f) \in L^1\}$. It is well-known (see [2]) that the elements of H^1 can be represented as linear combinations of so-called atoms. A function $a \in L^\infty[0, 1]$ is called an atom, if either $a=1$ or $\int_0^1 a = 0$ and there is a dyadic interval $I_a \subset [0, 1]$ such that supp $a \subset I_a$ and $|a| \leqq |I_a|^{-1}$ ($|I_a|$ is the length of I_a.) Then $f \in L^1$ belongs to H^1 if and only if there exist real coefficients α_i and atoms a_i $(i=0, 1, \ldots)$ so that

$$\sum_{i=0}^{\infty} |\alpha_i| < +\infty \quad \text{and} \quad f = \sum_{i=0}^{\infty} \alpha_i a_i.$$

3. It is known in the Walsh—Fourier analysis (see [1]) that the system $(w_n, n=0, 1, \ldots)$ is not a Schauder basis in L^1. Moreover, there exists a function $f \in H^1$ such that the L^1-norms of the partial sums $S_n(f)$ $(n=1, 2, \ldots)$ are not bounded. However, the following theorem shows that certain means of the $\|S_n(f)\|_1$'s can be convergent for all $f \in H^1$.

THEOREM. *If* $f \in H^1$, *then*

(3) $$\lim_{n \to +\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \|S_k(f)\|_1 = \|f\|_1.$$

Let n be a natural number for which $2^{N-1} \leqq n < 2^N$ $(N=2, 3, \ldots)$ holds. Then

$$\left|\frac{1}{\log n} \sum_{k=1}^{n-1} k^{-1} \|S_k(f)\|_1 - \|f\|_1\right| = \left|\frac{1}{\log n} \sum_{k=1}^{n-1} k^{-1} (\|S_k(f)\|_1 - \|f\|_1) + \right.$$

$$\left. + \|f\|_1 \left(\frac{1}{\log n} \sum_{k=1}^{n-1} k^{-1} - 1\right)\right| \leqq \frac{1}{\log n} \sum_{k=1}^{n-1} k^{-1} \|f - S_k(f)\|_1 + o(1) \quad (n \to +\infty),$$

i.e.

(4) $$\lim_{n \to +\infty} \frac{1}{\log n} \sum_{k=1}^{n-1} k^{-1} \|f - S_k(f)\|_1 = 0$$

implies our statement. On the other hand

$$\frac{1}{\log n}\sum_{k=1}^{n-1} k^{-1}\|f-S_k(f)\|_1 \leqq 2N^{-1}\sum_{k=0}^{N-1}\sum_{j=0}^{2^k-1}(2^k+j)^{-1}\|f-S_{2^k+j}(f)\|_1 \leqq$$

$$\leqq 2N^{-1}\sum_{k=0}^{N-1}2^{-k}\sum_{j=0}^{2^k-1}\|f-S_{2^k+j}(f)\|_1 =: 2N^{-1}\sum_{k=0}^{N-1} d_k(f).$$

If we denote by $E_n(f)$ the arithmetic mean of $d_{n+1}(f), \ldots, d_{2n}(f)$, i.e.

$$E_n(f) := n^{-1}\sum_{k=n+1}^{2n} d_k(f) \quad (n = 1, 2, \ldots),$$

then it is clear that

$$\lim_{n\to+\infty} E_n(f) = 0 \tag{5}$$

is sufficient for (4) to hold.

The statement of the theorem cannot hold for all $f\in L^1$. To this end let $(\alpha_k, k=1, 2, \ldots)$ be a sequence of real numbers of bounded variation, i.e.

$$\sum_{k=1}^{\infty}|\alpha_k-\alpha_{k+1}| < +\infty,$$

and take the function f defined by

$$f := \alpha_k(D_{2^{k+1}}-D_{2^k}). \tag{6}$$

Since $\|D_{2^k}\|_1=1$ $(k=0, 1, \ldots)$ (see (2)), by means of Abel transformation it follows that $f\in L^1$. On the other hand if $n=1, 2, \ldots$ and $j=0, \ldots, 2^n-1$, then

$$S_{2^n+j}(f) = \sum_{k=1}^{n-1}\alpha_k(D_{2^{k+1}}-D_{2^k})+\alpha_n(D_{2^n+j}-D_{2^n}) = \sum_{k=1}^{n-1}\alpha_k(D_{2^{k+1}}-D_{2^k})+\alpha_n w_{2^n}D_j,$$

from which

$$\|S_{2^n+j}(f)\|_1 \geqq |\alpha_n|\,\|D_j\|_1 - \|\sum_{k=1}^{n-1}\alpha_k(D_{2^{k+1}}-D_{2^k})\|_1 \geqq$$

$$\geqq |\alpha_n|\,\|D_j\|_1 - \left\|\sum_{k=2}^{n-1}(\alpha_{k-1}-\alpha_k)D_{2^k}-\alpha_1 D_2+\alpha_{n-1}D_{2^n}\right\|_1 \geqq$$

$$\geqq |\alpha_n|\,\|D_j\|_1 - \left(\sum_{k=2}^{n-1}|\alpha_k-\alpha_{k-1}|+|\alpha_1|+|\alpha_{n-1}|\right) = |\alpha_n|\,\|D_j\|_1+O(1) \quad (n\to+\infty)$$

follows. This leads to

$$2^{-n}\sum_{j=0}^{2^n-1}\|S_{2^n+j}(f)\|_1 \geqq 2^{-n}|\alpha_n|\sum_{j=0}^{2^n-1}\|D_j\|_1+O(1) \quad (n\to+\infty).$$

Since there exists an absolute constant $C>0$ such that (see [3])

$$n^{-1}\sum_{j=0}^{n-1}\|D_j\|_1 \geqq C\log n \quad (n\to+\infty), \tag{7}$$

therefore

$$2^{-n}\sum_{j=0}^{2^n-1}\|S_{2^n+j}(f)\|_1 \geqq C\,n|\alpha_n|+O(1)\quad (n\to+\infty).$$

Hence, if $\lim_{+\infty} n\cdot|\alpha_n|=+\infty$, then it is easy to prove that

$$\lim_{n\to+\infty}(\log n)^{-1}\sum_{k=1}^{n}k^{-1}\|S_k(f)\|_1=+\infty.$$

For example the sequence $\alpha_n:=n^{-1/2}$ $(n=1,2,\ldots)$ satisfies the conditions required above.

On the other hand there exists a function $f\in L^1\setminus H^1$ such that (3) holds. Indeed, if we take in (6)

$$\alpha_k:=(k\cdot\log k)^{-1}\quad (k=2,3,\ldots)\quad\text{and}\quad \alpha_1:=0,$$

then $f\in L^1$ and

$$Q(f)\|_1=\sum_{n=0}^{\infty}\int_{2^{-n-1}}^{2^{-n}}Q(f)\geqq\sum_{n=2}^{\infty}\int_{2^{-n-1}}^{2^{-n}}|S_{2^{n+1}}(f)|-S_{2^n}(f)|=$$

$$=\sum_{n=2}^{\infty}\alpha_n\int_{2^{-n-1}}^{2^{-n}}|D_{2^{n+1}}-D_{2^n}|=1/2\sum_{n=2}^{\infty}(n\log n)^{-1}=+\infty,$$

i.e. $f\notin H^1$. Fruthermore, if $n=1,2,\ldots$ and $j=0,\ldots,2^n-1$, then

$$\|f-S_{2^n+j}(f)\|_1=\Big\|\sum_{k=n}^{\infty}\alpha_k(D_{2^{k+1}}-D_{2^k})-\alpha_n w_{2^n}D_j\Big\|_1\leqq$$

$$\leqq\Big\|\sum_{k=n+1}^{\infty}(\alpha_{k-1}-\alpha_k)D_{2^k}-\alpha_n D_{2^n}\Big\|_1+\alpha_n\|D_j\|_1\leqq\sum_{k=n+1}^{\infty}|\alpha_{k-1}-\alpha_k|+\alpha_n+O(n\cdot\alpha_n)=$$

$$=o(1)\quad (n\to+\infty).$$

(Here we used the fact (see [3]) that $\|D_j\|_1=O(\log j)$ $(j\to+\infty)$.) From this (4) follows evidently.

Finally, we remark that $\lim_{k\to+\infty}d_k(f)=0$ cannot be true for all $f\in H^1$. Indeed,

$$f:=\sum_{k=1}^{\infty}k^{-2}(D_{2^{k^3}+1}-D_{2^{k^3}})\in H^1$$

and

$$d_{k^3}(f)=2^{-k^3}\sum_{j=0}^{2^{k^3}-1}\|S_{2^{k^3}+j}(f)-f\|_1\geqq$$

$$\geqq 2^{-k^3}\sum_{j=0}^{2^{k^3}-1}\|S_{2^{k^3}+j}(f)-S_{2^{k^3}}(f)\|_1-\|S_{2^{k^3}}(f)-f\|_1.$$

Since $\|f-S_{2^k}(f)\|_1=o(1)$ $(k\to+\infty)$ (see [3]) thus applying (7) we get

$$d_{k^3}(f)\geqq k^{-2}\cdot 2^{-k^3}\sum_{j=0}^{2^{k^3}-1}\|D_{2^{k^3}+j}-D_{2^{k^3}}\|_1+o(1)=$$

$$=k^{-2}\cdot 2^{-k^3}\sum_{j=0}^{2^{k^3}-1}\|D_j\|_1+o(1)\geqq C\cdot k+o(1)\quad (k\to+\infty).$$

4. For the proof of the theorem we need the following

LEMMA. *Let $a\in H^1$ be an atom. Then for all $n=1,2,\dots$ we have*

$$\|S_n(a)-a\|_1\leqq 12|\hat{a}(n)|\log|I_a|^{-1}+e_n(a),$$

where $\lim\limits_{n\to+\infty}e_n(a)=0$ *and* $|e_n(a)|\leqq 2.$

PROOF. Since for $a=1$ the lemma is trivial, we may suppose that $a\neq 1$. Let $I_a=[k2^{-m},(k+1)2^{-m})$ $(m=0,1,\dots$ and $k=0,\dots,2^m-1)$ and $x\in[0,1]\setminus I_a$. Then by (1) and (2)

$$S_n(a)(x)-a(x)=S_n(a)(x)=\int_0^1 a(t)w_n(x\dotplus t)\sum_{j=0}^{\infty}n_jw_{2^j}(x\dotplus t)D_{2^j}(x\dotplus t)\,dt=$$

$$=w_n(x)\sum_{j=0}^{m-1}n_jw_{2^j}(x\dotplus k2^{-m})\int_{I_a}a(t)w_n(t)D_{2^j}(x\dotplus t)\,dt=$$

$$=w_n(x)\sum_{j=0}^{j(x)}n_jw_{2^j}(x\dotplus k2^{-m})2^j\int_{I_a}aw_n,$$

where $j(x)$ denotes the maximum of indices $j=0,\dots,m-1$ such that $D_{2^j}(x\dotplus t)=2^j$ $(t\in I_a)$. ($\dotplus$ stands for the dyadic addition.) If $x\in[s2^{-m},(s+1)2^{-m})$ $(s=0,\dots,2^m-1,$ $s\neq k)$, then $2^{j(x)}\leqq 2^m|s-k|^{-1}$, therefore

$$\int_{[0,1]\setminus I_a}|S_n(a)-a|=\sum_{\substack{s=0\\ s\neq k}}^{2^m-1}\int_{s2^{-m}}^{(s+1)2^{-m}}|S_n(a)-a|\leqq$$

$$=2\sum_{\substack{s=0\\ s\neq k}}^{2^m-1}|\hat{a}(n)||s-k|^{-1}\leqq 4|\hat{a}(n)|\sum_{j=1}^{2^m}j^{-1}<12|\hat{a}(n)|\,m.$$

On the other hand

$$\int_{I_a}|S_n(a)-a|\leqq|I_a|^{1/2}\|S_n(a)-a\|_2=:e_n(a)=o(1)\quad(n\to+\infty)$$

and

$$e_n(a)\leqq 2^{-m/2}(\|S_n(a)\|_2+\|a\|_2)\leqq 2^{1-m/2}\|a\|_2\leqq 2.$$

This completes the proof of Lemma.

PROOF OF THE THEOREM. Let $f\in H^1$ be an arbitrary function and consider an atomic decomposition of f:

$$f=\sum_{i=0}^{\infty}\alpha_i a_i,\quad \sum_{i=0}^{\infty}|\alpha_i|<+\infty.$$

Furthermore, introduce the notation $I_i:=I_{a_i}$ $(i=0,1,\ldots)$ and rearrange the above decomposition of f as follows:

$$f=\sum_{s=0}^{\infty}\sum_{|I_i|=2^{-s}}\alpha_i a_i.$$

If n is a natural number, then

$$E_n(f)=n^{-1}\sum_{k=n+1}^{2n}d_k(f)=n^{-1}\sum_{k=n+1}^{2n}2^{-k}\sum_{j=0}^{2^k-1}\Big\|\sum_{s=0}^{\infty}\sum_{|I_i|=2^{-s}}\alpha_i\big(S_{2^k+j}(a_i)-a_i\big)\Big\|_1\leqq$$

$$\leqq\sum_{s=0}^{\infty}n^{-1}\sum_{k=n+1}^{2n}\sum_{|I_i|=2^{-s}}|\alpha_i|2^{-k}\sum_{j=0}^{2^k-1}\big\|S_{2^k+j}(a_i)-a_i\big\|_1=$$

$$=\sum_{s=0}^{2n}+\sum_{s=2n+1}^{\infty}=:A_n+B_n.$$

To the estimation of B_n we remark that if $a\in H^1$ is an atom, then $\hat{a}(m)=0$ for all $m=0,1,\ldots,|I_a|^{-1}-1$. Hence, all of the partial sums of the a_i's in B_n are equal to zero, therefore

$$B_n\leqq\sum_{s=2n+1}^{\infty}\sum_{|I_i|=2^{-s}}|\alpha_i|=o(1)\quad(n\to+\infty).$$

Let us decompose A_n into two further parts as

$$A_n=\sum_{s=0}^{n}+\sum_{s=n+1}^{2n}=:A_{n1}+A_{n2}.$$

Then applying the lemma and Cauchy—Schwarz inequality we get

$$A_{n1}\leqq 12\sum_{s=0}^{n}sn^{-1}\sum_{k=n+1}^{2n}\sum_{|I_i|=2^{-s}}|\alpha_i|2^{-k}\sum_{j=0}^{2^k-1}|\hat{a}_i(2^k+j)|+o(1)\leqq$$

$$\leqq 12\sum_{s=0}^{2n}sn^{-1}\sum_{k=n+1}^{2n}\sum_{|I_i|=2^{-s}}|\alpha_i|2^{-k/2}\|a_i\|_2+o(1)\leqq$$

$$\leqq 12\sum_{s=0}^{n}sn^{-1}\sum_{k=n+1}^{2n}2^{(s-k)/2}\sum_{|I_i|=2^{-s}}|\alpha_i|+o(1)=o(1)\quad(n\leftarrow+\infty).$$

Furthermore,

$$A_{n2}=\sum_{s=n+1}^{2n}n^{-1}\sum_{k=n+1}^{s}\sum_{|I_i|=2^{-s}}|\alpha_i|2^{-k}\sum_{j=0}^{2^k-1}\|S_{2^k+j}(a_i)-a_i\|_1+$$

$$+\sum_{s=n+1}^{2n}n^{-1}\sum_{k=s+1}^{2n}\sum_{|I_i|=2^{-s}}|\alpha_i|2^{-k}\sum_{j=0}^{2^k-1}\|S_{2^k+j}(a_i)-a_i\|_1=:A_{n2}^1+A_{n2}^2$$

and A_{n2}^1, A_{n2}^2 can be estimated by the same method as above. Thus

$$A_{n2}^1 \leqq \sum_{s=n+1}^{2n} n^{-1} \sum_{k=n+1}^{s} \sum_{|I_i|=2^{-s}} |\alpha_i| = n^{-1} \sum_{s=n+1}^{2n} (s-n) \sum_{|I_i|=2^{-s}} |\alpha_i| \leqq$$

$$\leqq \sum_{s=n+1}^{\infty} \sum_{|I_i|=2^{-s}} |\alpha_i| = o(1) \quad (n \to +\infty)$$

and

$$A_{n2}^2 \leqq 12 \sum_{s=n+1}^{2n} sn^{-1} \sum_{k=s+1}^{2n} \sum_{|I_i|=2^{-s}} |\alpha_i| \, 2^{-k} \sum_{j=0}^{2^k-1} |\hat{a}_i(2^k+j)| + o(1) \leqq$$

$$\leqq 12 \sum_{s=n+1}^{2n} sn^{-1} \sum_{k=s+1}^{2n} \sum_{|I_i|=2^{-s}} |\alpha_i| \, 2^{(s-k)/2} + o(1) \leqq$$

$$\leqq 12 \sum_{s=n+1}^{2n} sn^{-1} \sum_{|I|=2^{-s}}^{2n} |\alpha_i| + o(1) = o(1) \quad (n \to +\infty).$$

This completes the proof of the theorem.

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