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ADDENDUM TO "A PASCAL THEOREM APPLIED TO MINKOWSKI GEOMETRY"

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The results of the author's previous paper imply that the class of all (B*S)-geometries of Benz (see Benz, Leissner, and Schaeffer [2])coincides, up to isomorphisms, with the class of all plane Minkowski geometries over commutative fields.

In [1] the author proved that the validity of π_X in a plane based on a set of axioms M1 through M7 made the plane into a plane Minkowski geometry over a field. One of the axioms (M3) was the well known tangency theorem (Berührsatz). In [2] a geometry based on axioms equivalent to those, except for the absence of the tangency postulate, was called a B*-geometry, and furthermore a theorem S, equivalent to the validity of π_X for all points X of the plane, was discussed. The paper [2] came to the author's attention only after the completion of [1]. Using [1], we will now show that a B*-geometry in which S holds (called a B*S-geometry in [2]) is a plane Minkowski geometry over a field. All that is required is a proof of the tangency theorem.

With the notation as in [1], our assumptions will be the following (cf. [2]).

(1) Given two points A and B, there exists a unique point C with $A \|_U C \|_V B$.

(2) Given a point P and a circle c, there are unique points Q and R on c such that $Q \|_U P \|_V R$.

(3) There are three points, no two of which are parallel.
(4) Through three points, no two of which are parallel, there is a unique circle.
S. π_X holds for each circle and for all points X not on this circle.

If P, Q, R, and T are points, c a circle containing Q and R, and if $(P,T) \parallel (Q,R)$, we denote $T = P^c$.

LEMMA 1. *If P_1, P_2, P_3, and X are distinct concircular points, $(P_1,P_3) \parallel (C,D)$ and $(P_1,P_2) \parallel (A,B)$, then* $ABX \cap CDX = \{X\}$.
Proof. This is π^*_{UV} of [1, p.6].

LEMMA 2. *If two circles j and j' are such that* $j \cap j' = \{P\}$, *and if $X \nparallel P$, $X \notin j,j'$, then X, X^j, $X^{j'}$, and P are concircular.*
Proof. Suppose $P \notin X^j X^{j'} X = c$, say. By S, P^c lies on j and j'. But $j \cap j' = \{P\}$, hence $P = P^c$, a contradiction.

THEOREM. *Let k be a circle, P and Q two nonparallel points, P on c, Q not on c. Then there is a unique circle j through P and Q such that* $j \cap k = \{P\}$.
Proof. (i) Existence. Let $A \in k$, $A \parallel_U Q$. Choose B so that $B \parallel_V A$, $B \nparallel P$, $B \neq Q^k$. This is possible in every plane of order >2; in the case of order 2 the theorem is trivial. Let c be the circle through the pairwise nonparallel points B, P, B^k. Since $B^k \neq Q$, Q is not on c. By Lemma 1, the circle through Q, P, and Q^c satisfies the requirements for j.
(ii) Uniqueness. Suppose there is a second circle $j' \neq j$ such that $j' \cap k = \{P\}$. Let Y be a point not on k, j, j', and not parallel to P or Q. Obviously $Y^j \neq Y^{j'}$. By Lemma 2, Y^j and $Y^{j'}$ lie on the circle YPY^k. Let $(A,A^j) \parallel (P,Q)$. By S, Y^j and $Y^{j'}$ lie on the circle YAA^j. Since this circle contains the distinct points Y, Y^j, $Y^{j'}$, it is the same as YPY^k. This implies $A = P = A^j$, a contradiction to $P \neq Q$.

Remark: The uniqueness proof required only π_X for one fixed point X. In the existence proof the full postulate S was used, and the author does not know whether or not this proof too could be based on π_X only.

References

[1] ARTZY, R.: A Pascal theorem applied to Minkowski geometry. J. Geometry 3 (1973) 95-102

[2] BENZ, W.; LEISSNER, W.; SCHAEFFER, H.: Kreise, Zykel, Ketten. Zur Geometrie der Algebren. Ein Bericht. Jber. Deutsch. Math.-Ver. 74 (1972) 107-122.

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